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ON A CLASS OF UNBOUNDED OPERATOR ALGEBRAS III

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In this paper we continue our study of unbounded operator algebras begun in previous papers. In particular, the unbounded Hilbert algebras are studied. The primary purpose of this paper is to give necessary and sufficient conditions under which an unbounded Hilbert algebra is pure.

1. Introduction. In the previous paper [6] we began our study of unbounded Hilbert algebras and raised the following problem.

Problem. Let $\mathscr{D}_0$ be a maximal Hilbert algebra in a Hilbert space $\mathfrak{H}$. Does there exist a pure unbounded Hilbert algebra over $\mathscr{D}_0$ in $\mathfrak{H}$?

In this paper we find that if $\mathscr{D}_0 \neq \mathfrak{H}$ then the answer is affirmative. That is, if $\mathscr{D}_0 \neq \mathfrak{H}$, then the maximal unbounded Hilbert algebra $L_2^\omega(\mathscr{D}_0)$ is a pure unbounded Hilbert algebra over $\mathscr{D}_0$ in $\mathfrak{H}$. It therefore seems that our study of a class of unbounded operator algebras called $EW^\#$-algebras is significant. For, from ([6] Theorem 3.10) if $\mathscr{D}_0 \neq \mathfrak{H}$ then there necessarily exist pure $EW^\#$-algebras over the left von Neumann algebra $\mathscr{U}_0(\mathscr{D}_0)$ of $\mathscr{D}_0$ and if $\mathfrak{A}_0$ is a semifinite von Neumann algebra with a faithful normal semifinite trace φ_0 on $\mathfrak{A}_0^+$ and $L^2(\varphi_0) \neq \mathfrak{A}_0 \cap L^2(\varphi_0)$, then there exist pure EW^*-algebras over $\mathfrak{A}_0$ such that are isomorphic to standard $EW^\#$-algebras.

2. Basic theory for unbounded Hilbert algebras. We give here only the basic definitions and facts needed. For a more complete discussion of the basic properties of unbounded Hilbert algebras the reader is referred to [6, 7].

Let $\mathscr{D}$ be a pre-Hilbert space with an inner product $[\,|\,]$ and be a $*$-algebra. Let $\mathfrak{H}$ be the completion of $\mathscr{D}$. Suppose that $\mathscr{D}$ satisfies;

$$(1) \qquad (\xi|\eta) = (\eta^*|\xi^*)\,, \quad \xi, \eta \in \mathscr{D}\,,$$

$$(2) \qquad (\xi\eta|\zeta) = (\eta|\xi^*\zeta)\,, \quad \xi, \eta, \zeta \in \mathscr{D}\,.$$

Now, we define $\pi(\xi)$ and $\pi'(\xi)$ by;

$$\pi(\xi)\eta = \xi\eta \quad \text{and} \quad \pi'(\xi)\eta = \eta\xi\,, \quad \eta \in \mathscr{D}\,.$$

Then, by (2), we know that $\pi(\xi)$ and $\pi'(\xi)$ are closable operators on $\mathfrak{H}$ with the domain $\mathscr{D}$ and $\pi(\xi)^* \supset \pi(\xi^*)$, $\pi'(\xi)^* \supset \pi'(\xi^*)$.

DEFINITION 2.1. If $\mathscr{D}$ satisfies (1), (2) and (3) $\mathscr{D}_0^2$ is dense in $\mathfrak{H}$, where

$$\mathscr{D}_0 = \{\xi \in \mathscr{D};\ \pi(\xi) \text{ is continuous with respect to the pre-Hilbert space structure of } \mathscr{D}\},$$

then $\mathscr{D}$ is called an unbounded Hilbert algebra over $\mathscr{D}_0$ in $\mathfrak{H}$ and π(resp. π') is called the left (resp. right) regular representation of $\mathscr{D}$. In particular, if $\mathscr{D}_0 \neq \mathscr{D}$, then $\mathscr{D}$ is called pure.

Let $\mathscr{D}$ be an unbounded Hilbert algebra over $\mathscr{D}_0$ and let $\mathfrak{H}$ be the completion of $\mathscr{D}$. Clearly $\mathscr{D}_0$ is a Hilbert algebra and the completion of $\mathscr{D}_0$ is the Hilbert space $\mathfrak{H}$. Let π(resp. π') be the left (resp. right) regular representation of $\mathscr{D}$ and let π_0(resp. π_0') be the left (resp. right) regular representation of the Hilbert algebra $\mathscr{D}_0$.

Let $\mathfrak{A}$ be a family of closable operators on a Hilbert space. Then we denote by $\bar{A}$ the closure of $A \in \mathfrak{A}$ and put $\bar{\mathfrak{A}} = \{\bar{A};A \in \mathfrak{A}\}$.

For each $x \in \mathfrak{H}$ we denote $\pi_0(x)$ and $\pi_0'(x)$ by;

$$\pi_0(x)\xi = \overline{\pi_0'(\xi)}x\ , \quad \pi_0'(x)\xi = \overline{\pi_0(\xi)}x\ , \quad \xi \in \mathscr{D}_0\ .$$

Then $\pi_0(x)$ and $\pi_0'(x)$ are linear operators on $\mathfrak{H}$ with the domain $\mathscr{D}_0$. The involution on $\mathscr{D}$ is extended to an involution on $\mathfrak{H}$, which is also denoted by $*$. Then we have $\overline{\pi_0(x^*)} = \pi_0(x)^*$ and $\overline{\pi_0'(x^*)} = \pi_0'(x)^*$.

LEMMA 2.2. (1) *For each* $\xi \in \mathscr{D}$ *we have*

$$\overline{\pi(\xi)} = \overline{\pi_0(\xi)}\ , \quad \overline{\pi'(\xi)} = \overline{\pi_0'(\xi)}\ ,$$

$$\overline{\pi(\xi^*)} = \pi(\xi)^*\ , \quad \overline{\pi'(\xi^*)} = \pi'(\xi)^*\ .$$

(2) *For each* $\lambda \in \boldsymbol{C}$ (the field of complex numbers) *and* $\xi, \eta \in \mathscr{D}$ *we have*

$$\overline{\pi(\xi)} + \overline{\pi(\eta)} := \overline{\overline{\pi(\xi)} + \overline{\pi(\eta)}} = \overline{\pi(\xi + \eta)}\ ,$$

$$\overline{\pi(\xi)} \cdot \overline{\pi(\eta)} := \overline{\overline{\pi(\xi)}\,\overline{\pi(\eta)}} = \overline{\pi(\xi\eta)}\ ,$$

$$\lambda \cdot \overline{\pi(\xi)} := \begin{cases} \lambda\overline{\pi(\xi)}\ , & \text{if } \lambda \neq 0 \\ 0\ , & \text{if } \lambda = 0 \end{cases} = \overline{\pi(\lambda\xi)}\ , \quad \pi(\xi)^* = \overline{\pi(\xi^*)}\ .$$

Therefore $\overline{\pi(\mathscr{D})}$ *is a* $*$*-algebra of closed operators on* $\mathfrak{H}$ *under the operations of strong sum, strong product, adjoint and strong scalar multiplication. Similarly* $\overline{\pi'(\mathscr{D})}$ *is a* $*$*-algebra of closed operators on* $\mathfrak{H}$.

Proof. ([6] Lemma 2.1 and Proposition 2.3)

Let $\mathscr{U}_0(\mathscr{D}_0)$ (resp. $\mathscr{V}_0(\mathscr{D}_0)$) be the left (resp. right) von Neumann algebra of the Hilbert algebra $\mathscr{D}_0$ and let φ_0 be the natural trace on $\mathscr{U}_0(\mathscr{D}_0)^+$. Let $\mathfrak{B}(\mathfrak{H})$ be the set of all bounded linear operators on $\mathfrak{H}$. Putting

$$(\mathscr{D}_0)_b = \{x \in \mathfrak{H}; \overline{\pi_0(x)} \in \mathfrak{B}(\mathfrak{H})\} ,$$

$(\mathscr{D}_0)_b$ is a Hilbert algebra containing $\mathscr{D}_0$. If $\mathscr{D}_0 = (\mathscr{D}_0)_b$, then $\mathscr{D}_0$ is called a maximal Hilbert algebra in $\mathfrak{H}$.

Let $\mathfrak{M}$ be the set of all measurable operators on $\mathfrak{H}$ with respect to $\mathscr{U}_0(\mathscr{D}_0)$. For every $T \in \mathfrak{M}^+$ we put

$$\mu_0(T) = \sup [\varphi_0(\overline{\pi_0(\xi)}); 0 \leqq \overline{\pi_0(\xi)} \leqq T, \xi \in (\mathscr{D}_0)_b^2]$$

and

$$L^p(\varphi_0) = \{T \in \mathfrak{M}; \|T\|_p: = \mu_0(|T|^p)^{1/p} < \infty\} , \quad 1 \leqq p < \infty .$$

Then $\|T\|_p$ is called the L^p-norm of T in $L^p(\varphi_0)$ and μ_0 is called the integral on $L^1(\varphi_0)$. If $p = \infty$, we shall identify $\mathscr{U}_0(\mathscr{D}_0)$ with $L^\infty(\varphi_0)$ and we denote by $\|T\|$ or $\|T\|_\infty$ the operator norm of $T \in \mathscr{U}_0(\mathscr{D}_0)$.

DEFINITION 2.3. We define L^ω-spaces with respect to φ_0 and $\mathscr{D}_0$ as follows;

$$L^\omega(\varphi_0) = \bigcap_{1 \leqq p < \infty} L^p(\varphi_0) , \quad L_2^\omega(\varphi_0) = \bigcap_{2 \leqq p < \infty} L^p(\varphi_0) ,$$

and

$$L^\omega(\mathscr{D}_0) = \{x \in \mathfrak{H}; \overline{\pi_0(x)} \in L^\omega(\varphi_0)\} , \quad L_2^\omega(\mathscr{D}_0) = \{x \in \mathfrak{H}; \overline{\pi_0(x)} \in L_2^\omega(\varphi_0)\} ,$$

respectively. For $p \geqq 2$ we set

$$L_2^p(\mathscr{D}_0) = \{x \in \mathfrak{H}; \overline{\pi_0(x)} \in L^p(\varphi_0)\} ,$$

$$\|x\|_p = \|\overline{\pi_0(x)}\|_p , \quad x \in L_2^p(\mathscr{D}_0)$$

$$\|x\|_\infty = \|\overline{\pi_0(x)}\|_\infty , \quad x \in L_2^\infty(\mathscr{D}_0) = (\mathscr{D}_0)_b .$$

THEOREM 2.4. *$L_2^\omega(\mathscr{D}_0)$ (resp. $L^\omega(\mathscr{D}_0)$) is an unbounded Hilbert algebra over $(\mathscr{D}_0)_b$ (resp. $(\mathscr{D}_0)_b^2$) in $\mathfrak{H}$. If $\mathscr{D}$ is a pure unbounded Hilbert algebra, then $\mathscr{D}$ is a $*$-subalgebra of $L_2^\omega(\mathscr{D}_0)$. Hence $L_2^\omega(\mathscr{D}_0)$ is maximal among unbounded Hilbert algebras containing $\mathscr{D}_0$*

Proof. ([6] Theorem 3.9)

3. Necessary and sufficient conditions under which $L_2^\omega(\mathscr{D}_0)$ is pure. Let $\mathscr{D}_0$ be a Hilbert algebra in a Hilbert space $\mathfrak{H}$ and let φ_0 be the natural trace on $\mathscr{U}_0(\mathscr{D}_0)^+$.

LEMMA 3.1. *For $2 \leqq p < q$ we have*

$$L_2^2(\mathscr{D}_0) = \mathfrak{H} \supset L_2^p(\mathscr{D}_0) \supset L_2^q(\mathscr{D}_0) \supset L_2^\omega(\mathscr{D}_0) \supset L_2^\infty(\mathscr{D}_0) = (\mathscr{D}_0)_b$$

and

$$L_2^\omega(\mathscr{D}_0) = \bigcap_{2 \leqq n < \infty} L_2^n(\mathscr{D}_0)\ ,$$

where n is an integer.

Proof. For each $x \in L_2^q(\mathscr{D}_0)$ let $\overline{\pi_0(x)} = U|\overline{\pi_0(x)}|$ be the polar decomposition and let $|\overline{\pi_0(x)}| = \int_0^\infty \lambda dE(\lambda)$ be the spectral resolution of $|\overline{\pi_0(x)}|$. Then,

$$\begin{aligned} ||x||_p^p = ||\overline{\pi_0(x)}||_p^p &= -\int_0^\infty \lambda^p d\varphi_0(E(\lambda)^\perp) \\ &= -\int_0^1 \lambda^p d\varphi_0(E(\lambda)^\perp) - \int_1^\infty \lambda^p d\varphi_0(E(\lambda)^\perp) \\ &\leqq -\int_0^1 \lambda^2 d\varphi_0(E(\lambda)^\perp) - \int_1^\infty \lambda^q d\varphi_0(E(\lambda)^\perp) \\ &\leqq ||x||_2^2 + ||x||_q^q < \infty\ . \end{aligned}$$

Hence, $x \in L_2^p(\mathscr{D}_0)$. Consequently $L_2^p(\mathscr{D}_0) \supset L_2^q(\mathscr{D}_0)$, and so we can easily show that $L_2^\omega(\mathscr{D}_0) = \bigcap_{2 \leqq n < \infty} L_2^n(\mathscr{D}_0)$ (n; integer).

LEMMA 3.2. *If $L_2^p(\mathscr{D}_0) = L_2^q(\mathscr{D}_0)$ for some $q > p \geqq 2$, then $L_2^r(\mathscr{D}_0) = L_2^\omega(\mathscr{D}_0)$ for all $r \in [p, \infty)$.*

Proof. Let $x \in L_2^p(\mathscr{D}_0) = L_2^q(\mathscr{D}_0)$. Then, $|\overline{\pi_0(x)}|^{q/p} \in L^p(\varphi_0)$. Since $2 < 2q/p \leqq q$ and $L_2^2(\mathscr{D}_0) \supset L_2^{2q/p}(\mathscr{D}_0) \supset L_2^q(\mathscr{D}_0)$ (by Lemma 3.1), we get $x \in L_2^{2q/p}(\mathscr{D}_0)$, i.e., $|\overline{\pi_0(x)}|^{2q/p} \in L^1(\varphi_0)$. Hence, $|\overline{\pi_0(x)}|^{q/p} \in L^p(\varphi_0) \cap L^2(\varphi_0)$. Repeating the same argument, we get that $|\overline{\pi_0(x)}|^{(q/p)^n} \in L^p(\varphi_0) \cap L^2(\mathscr{D}_0)$ ($n = 1, 2, \cdots$). From $q/p > 1$ and Lemma 3.1, $x \in L_2^\omega(\mathscr{D}_0)$.

DEFINITION 3.3. An element e of $\mathscr{D}_0$ is called a projection if $e^2 = e = e^*$. Let $E(\mathscr{D}_0)$ denote the collection of all projections in $\mathscr{D}_0$.

THEOREM 3.4. *Let $\mathscr{D}_0$ be a Hilbert algebra in $\mathfrak{H}$. Then the following conditions are equivalent.*

(1) *$L_2^\omega(\mathscr{D}_0)$ is pure.*

(2) *$L^\omega(\mathscr{D}_0)$ is pure.*

(3) *There exists a sequence $\{e_n\}$ of nonzero mutually orthogonal projections in $(\mathscr{D}_0)_b$ such that $\sum_{n=1}^\infty ||e_n||_2^2 < \infty$.*

(4) *$\mathfrak{H}$ is not a Hilbert algebra,* i.e., *$(\mathscr{D}_0)_b \neq \mathfrak{H}$.*

(5) *$L_2^\omega(\mathscr{D}_0) \neq \mathfrak{H}$.*

(6) $L_2^p(\mathscr{D}_0) \neq L_2^q(\mathscr{D}_0)$ *for some* $2 \leqq p < q$.

(7) $L_2^p(\mathscr{D}_0) \neq L_2^2(\mathscr{D}_0)$ *for each* $p > 2$.

In particular, if $\mathscr{D}_0$ *has an identity, then* (1)~(7) *are eqvivalent to* (7)′;

(7)′ $L^p(\varphi_0) \neq L^q(\varphi_0)$ *for each* $q > p \geqq 1$.

Proof. From Lemma 3.1, for $2 \leqq p < q$

$$L_2^2 \supset L_2^p \supset L_2^q \supset L_2^\omega \supset (\mathscr{D}_0)_b .$$

Hence, (7) ⇒ (6) ⇒ (5) ⇒ (4) and (2) ⇒ (1) are easily showed.

(1) ⇒ (7); If $L_2^p(\mathscr{D}_0) = L_2^2(\mathscr{D}_0)$ for some $p > 2$, then from Lemma 3.2 we have $L_2^2(\mathscr{D}_0) = L_2^\omega(\mathscr{D}_0)$. Since $L_2^\omega(\mathscr{D}_0)$ is an algebra, for each $x \in L_2^\omega(\mathscr{D}_0)$, $\mathscr{D}(\overline{\pi_0(x)}) = \mathfrak{H}$, i.e., $\overline{\pi_0(x)} \in \mathfrak{B}(\mathfrak{H})$. Hence $L_2^\omega(\mathscr{D}_0)$ is a Hilbert algebra.

(4) ⇒ (3); Suppose that $x \in \mathfrak{H} - (\mathscr{D}_0)_b$. Let $|\overline{\pi_0(x)}| = \int_0^\infty \lambda dE(\lambda)$ be the spectral resolution of $|\overline{\pi_0(x)}|$. Since $|\overline{\pi_0(x)}| \notin \mathfrak{B}(\mathfrak{H})$, $E(n+1) - E(n) \neq 0$ for infinite many n, and so we may suppose that $E(n+1) - E(n) \neq 0$ $(n = 1, 2, \cdots)$. We shall show that $E(n+1) - E(n) \in L^\infty(\varphi_0) \cap L^2(\varphi_0)$. Clearly, $E(n+1) - E(n) \in L^\infty(\varphi_0) = \mathscr{U}_0(\mathscr{D}_0)$. Moreover, we have

$$\begin{aligned}\|E(n+1) - E(n)\|_2^2 = \varphi_0(E(n+1) - E(n)) &= -\int_n^{n+1} d\varphi_0(E(\lambda)^\perp) \\ &\leqq -\int_n^{n+1} \lambda^2 d\varphi_0(E(\lambda)^\perp) \leqq \|\overline{\pi_0(x)}\|_2^2 = \|x\|_2^2 .\end{aligned}$$

Hence, $E(n+1) - E(n) \in L^2(\varphi_0)$ $(n = 1, 2, \cdots)$, and so there exists $e_n \in (\mathscr{D}_0)_b$ such that $E(n+1) - E(n) = \overline{\pi_0(e_n)}$ $(n = 1, 2 \cdots)$. Clearly $\{e_n\}$ is a sequence of nonzero mutually orthogonal projections in $(\mathscr{D}_0)_b$. We shall show that $\sum_{n=1}^\infty \|e_n\|_2^2 < \infty$. In fact, for $m > n$

$$\begin{aligned}\sum_{k=n}^m \|e_k\|_2^2 = \sum_{k=n}^m \varphi_0(\overline{\pi_0(e_k)}) &= \sum_{k=n}^m \varphi_0(E(k+1) - E(k)) \\ &= \varphi_0(E(m+1) - E(n))\end{aligned}$$

and $\{E(m+1) - E(n)\}$ converges σ-weakly to $0(n, m \to \infty)$. Since φ_0 is σ-weakly continuous, we have

$$\lim_{m,n\to\infty} \sum_{k=n}^m \|e_k\|_2^2 = \lim_{m,n\to\infty} \varphi_0(E(m+1) - E(n)) = 0 .$$

Hence, $\sum_{n=1}^\infty \|e_n\|_2^2 < \infty$.

(3) ⇒ (2); For some positive integer k_0, $\sum_{n=k_0}^\infty \|e_n\|_2^2 < 1$. We set

$$a_0 = \sum_{n=k_0}^{\infty} \|e_n\|_2^2\ , \quad a_1 = \sum_{n=k_0+1}^{\infty} \|e_n\|_2^2, \cdots , \quad a_n = \sum_{k=k_0+n}^{\infty} \|e_k\|_2^2, \cdots ,$$

$$b_0 = |\log a_0|, \cdots , \quad b_n = |\log a_n|, \cdots$$

and

$$x = \sum_{n=0}^{\infty} b_n e_{k_0+n}\ .$$

We shall show that $x \in L^{\omega}(\mathscr{D}_0) - (\mathscr{D}_0)_b$. For every $p \in [1, \infty)$

$$\sum_{n=0}^{\infty} |b_n|^p \|e_{k_0+n}\|_2^2 < \int_0^1 |\log x|^p dx = p!\ ,$$

and so

$$\lim_{m,n\to\infty} \|\sum_{k=n}^{m} b_k e_{k_0+k}\|_2^2 = \lim_{m,n\to\infty} \sum_{k=n}^{m} |b_k|^2 \|e_{k_0+k}\|_2^2 = 0\ .$$

Hence, $x \in \mathfrak{H}$ and $\|x\|_2^2 = \sum_{n=0}^{\infty} |b_n|^2 \|e_{k_0+n}\|_2^2$. Similarly, for every $p \in [1, \infty)$, $x \in L_2^p(\mathscr{D}_0)$ and $\|x\|_p^p = \sum_{n=0}^{\infty} |b_n|^p \|e_{k_0+n}\|_2^2$. Therefore, $x \in L^{\omega}(\mathscr{D}_0)$. On the other hand, $\lim_{n\to\infty} b_n = \infty$ and $\|e_{k_0+n}\|_2^2 \neq 0$ $(n = 1, 2, \cdots)$, and so $\overline{\pi_0(x)} \notin \mathfrak{B}(\mathfrak{H})$. Hence, $x \in L^{\omega}(\mathscr{D}_0) - (\mathscr{D}_0)_b$. That is, $L^{\omega}(\mathscr{D}_0)$ is pure.

Suppose that $\mathscr{D}_0$ has an identity.

$(7)' \Rightarrow (7)$; Obvious.

$(7) \Rightarrow (7)'$; For $1 \leqq p < q$ we have

$$L^1(\varphi_0) \supset L^p(\varphi_0) \supset L^q(\varphi_0) \supset L^{\omega}(\varphi_0) \supset L^{\infty}(\varphi_0)\ .$$

Suppose that $L^p(\varphi_0) = L^q(\varphi_0)$ for $1 \leqq p < q$. Let $T \in L^1(\varphi_0)$. Then, $|T|^{1/p} \in L^p(\varphi_0) = L^q(\varphi_0)$. Hence, $|T|^{q/p} \in L^1(\varphi_0)$. Repeating the same argument, $|T|^{(q/p)^n} \in L^1(\varphi_0)(n = 1, 2, \cdots)$, and so $|T| \in L^{(q/p)^n}(\varphi_0)$ $(n = 1, 2, \cdots)$. From $q/p > 1$ and Lemma 3.1, $|T| \in L^{\omega}(\varphi_0)$, and so $T \in L^{\omega}(\varphi_0)$.

Let $\mathscr{D}_0$ be a Hilbert algebra in $\mathfrak{H}$. From Theorem 3.4, if $\mathfrak{H}$ is not a Hilbert algebra, i.e., $(\mathscr{D}_0)_b \neq \mathfrak{H}$, then $L_2^{\omega}(\mathscr{D}_0)$ becomes a pure unbounded Hilbert algebra over $(\mathscr{D}_0)_b$ in $\mathfrak{H}$. So, the previous problem is solved. If $L_2^{\omega}(\mathscr{D}_0)$ is a Hilbert algebra, then $\mathfrak{H}$ is a Hilbert algebra and $L_2^{\omega}(\mathscr{D}_0) = \mathfrak{H}$. Hence we can give some conditions for $L_2^{\omega}(\mathscr{D}_0)$ to be a Hilbert algebra.

COROLLARY 3.5. *Let $\mathscr{D}_0$ be a Hilbert algebra in $\mathfrak{H}$. Then the following conditions are equivalent.*

(1) *$\mathfrak{H}$ is a Hilbert algebra.*

(2) *$L_2^{\omega}(\mathscr{D}_0)$ is a Hilbert algebra.*

(3) $\mathfrak{H} = L_2^{\omega}(\mathscr{D}_0) = (\mathscr{D}_0)_b$.

(4) *Either* $E((\mathscr{D}_0)_b)$ *is a finite set or* $\sum_{n=1}^{\infty} \|e_n\|_2^2 = \infty$ *for each sequence* $\{e_n\}$ *of mutually orthogonal projections in* $E((\mathscr{D}_0)_b)$.

(5) *There exists* $C > 0$ *such that* $\|e\|_2 \geqq C$ *for all* $e \in E((\mathscr{D}_0)_b)$.

(6) $L_2^p(\mathscr{D}_0) = L_2^q(\mathscr{D}_0)$ *for each* $q > p \geqq 2$.

(7) $L_2^p(\mathscr{D}_0) = L_2^2(\mathscr{D}_0)$ *for some* $p > 2$.

In particular, if $\mathscr{D}_0$ *has an identity, then* (1) ~ (7) *are equivalent to* (7)′;

(7)′ $L_2^p(\mathscr{D}_0) = L_2^q(\mathscr{D}_0)$ *for some* $q > p \geqq 1$.

Proof. From Theorem 3.4 (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (7)′ are easily showed.

Let $E = \overline{\pi_0(e)}$ and $F = \overline{\pi_0(f)}$ for $e, f \in E((\mathscr{D}_0)_b)$. We denote by $E \cap F$ (resp. $E \cup F$) the projection onto $E\mathfrak{H} \cap F\mathfrak{H}$ (resp. $E\mathfrak{H} \cup F\mathfrak{H}$). Clearly, $E \cap F$ and $E \cup F$ in $L^{\infty}(\varphi_0) \cap L^2(\varphi_0)$. Hence there exist projections $e \cap f$ and $e \cup f$ in $(\mathscr{D}_0)_b$ such that $E \cap F = \overline{\pi_0(e \cap f)}$ and $E \cup F = \overline{\pi_0(e \cup f)}$.

If $E((\mathscr{D}_0)_b)$ is an infinite set, then there exists a sequence $\{e_n\}$ of mutually orthogonal projections in $E((\mathscr{D}_0)_b)$. In fact, the following two cases are considered.

(i) There exists a sequence $\{e_n\}$ of $E((\mathscr{D}_0)_b)$ such that

$$e_2 - (e_1 \cap e_2) \neq 0, \cdots, e_n - (e_1 \cup e_2 \cup \cdots \cup e_{n-1}) \cap e_n \neq 0, \cdots .$$

(ii) There exists a sequence $\{e_n\}$ of $E((\mathscr{D}_0)_b)$ such that $e_1 > e_n$ for all $n \geqq 2$.

(i); Obvious.

(ii); We set

$$p_1 = e_1, \cdots, \quad p_n = e_1 - \bigcup_{k=2}^{n} e_k, \cdots,$$

$$q_n = p_n - p_{n+1}, \quad n = 1, 2, \cdots .$$

If $q_n \neq 0$ for infinite many n, then $\{q_n\}$ is a sequence of mutually orthogonal projections in $E((\mathscr{D}_0)_b)$. If $q_n = 0$ for infinite many n, then $e_n > e_{n+1}$ for infinite many n. Putting $f_n = e_n - e_{n+1}$, $\{f_n\}$ is a sequence of mutually orthogonal projections in $E((\mathscr{D}_0)_b)$. From the above argument and Theorem 3.4, (2) $\Leftrightarrow$ (4) is easily showed.

(5) $\Rightarrow$ (4); Obvious.

(4) $\Rightarrow$ (5); Suppose that (5) is not satisfied. For each n there exists $e_n \in E((\mathscr{D}_0)_b)$ such that $\|e_n\|_2 < 1/n$. After a slight modification of the above, we can make a sequence $\{p_n\}$ of mutually orthogonal projections in $E((\mathscr{D}_0)_b)$ such that $\sum_{n=1}^{\infty} \|p_n\|_2^2 \leqq \sum_{n=1}^{\infty} \|e_n\|_2^2 \leqq \sum_{n=1}^{\infty} 1/n^2 < \infty$.

4. Standard $EW^{\#}$-algebras. From ([6] Theorem 3.10) if $\mathscr{D}$ is a pure unbounded Hilbert algebra over $\mathscr{D}_0$, then there exists the pure $EW^{\#}$-algebra $\mathscr{U}(\mathscr{D})$ on $L_2^{\omega}(\mathscr{D}_0)$ over $\mathscr{U}_0(\mathscr{D}_0)$. So, from Theorem 3.4, if $\mathscr{D}_0$ is a Hilbert algebra in $\mathfrak{H}$ and $(\mathscr{D}_0)_b \neq \mathfrak{H}$, then there necessarily exist pure $EW^{\#}$-algebras over $\mathscr{U}_0(\mathscr{D}_0)$. Hence it seems that our study of $EW^{\#}$-algebras is significant. For a more complete discussion of the above argument we give here the basic definitions and facts of $EW^{\#}$-algebras.

DEFINITION 4.1. Let $\mathfrak{D}$ be a pre-Hilbert space with an inner product $(\ |\ )$ and let $\mathfrak{H}$ be the completion of $\mathfrak{D}$. We denote the set of all linear operators on $\mathfrak{D}$ by $\mathfrak{L}(\mathfrak{D})$. A subalgebra $\mathfrak{A}$ of $\mathfrak{L}(\mathfrak{D})$ is called a $\#$-algebra on $\mathfrak{D}$ if there exists an involution on $\mathfrak{A}$; $A \to A^{\#}$ such that

$$(A\xi \mid \eta) = (\xi \mid A^{\#}\eta)\,, \quad A \in \mathfrak{A}\,, \quad \xi, \eta \in \mathfrak{D}\,.$$

We set

$$\mathfrak{A}_b = \{A \in \mathfrak{A};\ \bar{A} \in \mathfrak{B}(\mathfrak{H})\}\,.$$

Let $\mathfrak{A}$ be a $\#$-algebra on $\mathfrak{D}$ with an identity operator I. $\mathfrak{A}$ is called a symmetric $\#$-algebra on $\mathfrak{D}$ if $(I + A^{\#}A)^{-1}$ exists and lies in $\mathfrak{A}_b$ for every $A \in \mathfrak{A}$.

A symmetric $\#$-algebra $\mathfrak{A}$ on $\mathfrak{D}$ is said to be an $EW^{\#}$-algebra over $\overline{\mathfrak{A}_b}$ if $\overline{\mathfrak{A}_b}$ is a von Neumann algebra. If $\mathfrak{A} \neq \mathfrak{A}_b$, then $\mathfrak{A}$ is called a pure $EW^{\#}$-algebra.

Let $\mathfrak{A}$ be a set of densely-defined closed operators on $\mathfrak{H}$ which is *a* $*$-algebra under the operations of strong sum, strong product, adjoint and strong scalar multiplication. $\mathfrak{A}$ is said to be an EW^*-algebra over $\mathfrak{A}_b$ if $(I + T^*T)^{-1} \in \mathfrak{A}$ for every $T \in \mathfrak{A}$ and the sub-algebra $\mathfrak{A}_b$ of bounded operators in $\mathfrak{A}$ is a von Neumann algebra. If $\mathfrak{A} \neq \mathfrak{A}_b$, then $\mathfrak{A}$ is called a pure EW^*-algebra.

Clearly if $\mathfrak{A}$ is an (resp. pure) $EW^{\#}$-algebra, then $\overline{\mathfrak{A}}$ is an (resp. pure) EW^*-algebra.

Let $\mathscr{D}$ be an unbounded Hilbert algebra over $\mathscr{D}_0$ in a Hibert space $\mathfrak{H}$ and let φ_0 (resp. ψ_0) be the natural trace on $\mathscr{U}_0(\mathscr{D}_0)^+$ (resp. $\mathscr{V}_0(\mathscr{D}_0)^+$). For every $x \in \mathfrak{H}$ we see that

$$\overline{J\pi_0(x)J} = \overline{\pi_0'(x^*)} \quad \text{and} \quad \overline{J\pi_0'(x)J} = \overline{\pi_0(x^*)}\,,$$

where J denotes the involution $*$ on $\mathfrak{H}$. Hence we get that

$$JL^{\omega}(\varphi_0)J = L^{\omega}(\psi_0)\,, \quad JL_2^{\omega}(\varphi_0)J = L_2^{\omega}(\psi_0)\,,$$
$$L^{\omega}(\mathscr{D}_0) = \{x \in \mathfrak{H};\ \overline{\pi_0'(x)} \in L^{\omega}(\psi_0)\}\,, \quad L_2^{\omega}(\mathscr{D}_0) = \{x \in \mathfrak{H};\ \overline{\pi_0'(x)} \in L_2^{\omega}(\psi_0)\}$$

and

$$\mathscr{U}_0(\mathscr{D}_0)L_2^\omega(\mathscr{D}_0) \subset L_2^\omega(\mathscr{D}_0)\ , \quad \mathscr{V}_0(\mathscr{D}_0)L_2^\omega(\mathscr{D}_0) \subset L_2^\omega(\mathscr{D}_0)\ .$$

Let π_2^ω (resp. $(\pi')_2^\omega$) be the left (resp. right) regular representation of $L_2^\omega(\mathscr{D}_0)$ and let

$$\mathscr{U}_0(\mathscr{D}_0)/L_2^\omega(\mathscr{D}_0) = \{T/L_2^\omega(\mathscr{D}_0);\ T \in \mathscr{U}_0(\mathscr{D}_0)\}\ ,$$
$$\mathscr{V}_0(\mathscr{D}_0)/L_2^\omega(\mathscr{D}_0) = \{T'/L_2^\omega(\mathscr{D}_0);\ T' \in \mathscr{V}_0(\mathscr{D}_0)\}\ ,$$

where $T/L_2^\omega(\mathscr{D}_0)$ is the restriction of T onto $L_2^\omega(\mathscr{D}_0)$. Then $\pi_2^\omega(\mathscr{D})$, $(\pi')_2^\omega(\mathscr{D})$, $\mathscr{U}_0(\mathscr{D}_0)/L_2^\omega(\mathscr{D}_0)$ and $\mathscr{V}_0(\mathscr{D}_0)/L_2^\omega(\mathscr{D}_0)$ are $\#$-algebras on $L_2^\omega(\mathscr{D}_0)$ under $\pi_2^\omega(\xi)^\# = \pi_2^\omega(\xi^*)$, $(\pi')_2^\omega(\xi)^\# = (\pi')_2^\omega(\xi^*)$, $(T/L_2^\omega(\mathscr{D}_0))^\# = T^*/L_2^\omega(\mathscr{D}_0)$ and $(T'/L_2^\omega(\mathscr{D}_0))^\# = (T')^*/L_2^\omega(\mathscr{D}_0)$, respectively.

NOTATION. We denote by $\mathscr{U}(\mathscr{D})$ (resp. $\mathscr{V}(\mathscr{D})$) the $\#$-algebra on $L_2^\omega(\mathscr{D}_0)$ generated by $\pi_2^\omega(\mathscr{D})$ (resp. $(\pi')_2^\omega(\mathscr{D})$) and $\mathscr{U}_0(\mathscr{D}_0)/L_2^\omega(\mathscr{D}_0)$ (resp. $\mathscr{V}_0(\mathscr{D}_0)/L_2^\omega(\mathscr{D}_0)$).

THEOREM 4.2. *Let $\mathscr{D}$ be a pure unbounded Hilbert algebra over $\mathscr{D}_0$ in a Hilbert space $\mathfrak{H}$. Then $\mathscr{U}(\mathscr{D})$, $\mathscr{U}(L^\omega(\mathscr{D}_0))$ and $\mathscr{U}(L_2^\omega(\mathscr{D}_0))$* (resp. $\mathscr{V}(\mathscr{D})$, $\mathscr{V}(L^\omega(\mathscr{D}_0))$ and $\mathscr{V}(L_2^\omega(\mathscr{D}_0))$) *are pure $EW^\#$-algebras on $L_2^\omega(\mathscr{D}_0)$ over $\mathscr{U}_0(\mathscr{D}_0)$* (resp. $\mathscr{V}_0(\mathscr{D}_0)$). *Furthermore, we have*

$$\mathscr{U}(L_2^\omega(\mathscr{D}_0)) = \mathscr{U}(L^\omega(\mathscr{D}_0))\ , \quad \mathscr{V}(L_2^\omega(\mathscr{D}_0)) = \mathscr{V}(L^\omega(\mathscr{D}_0))$$

and

$$J\mathscr{U}(\mathscr{D})J = \mathscr{V}(\mathscr{D})\ , \quad J\mathscr{V}(\mathscr{D})J = \mathscr{U}(\mathscr{D})\ .$$

Proof. From ([6] Theorem 3.10) $\mathscr{U}(\mathscr{D})$, $\mathscr{U}(L^\omega(\mathscr{D}_0))$ and $\mathscr{U}(L_2^\omega(\mathscr{D}_0))$ are pure $EW^\#$-algebras on $L_2^\omega(\mathscr{D}_0)$ over $\mathscr{U}_0(\mathscr{D}_0)$. Similarly we can easily prove that $\mathscr{V}(\mathscr{D})$, $\mathscr{V}(L^\omega(\mathscr{D}_0))$ and $\mathscr{V}(L_2^\omega(\mathscr{D}_0))$ are pure $EW^\#$-algebras on $L_2^\omega(\mathscr{D}_0)$ over $\mathscr{V}_0(\mathscr{D}_0)$. We shall show that $\mathscr{U}(L^\omega(\mathscr{D}_0)) = \mathscr{U}(L_2^\omega(\mathscr{D}_0))$. Clearly, $\mathscr{U}(L^\omega(\mathscr{D}_0)) \subset \mathscr{U}(L_2^\omega(\mathscr{D}_0))$. Suppose that $x \in L_2^\omega(\mathscr{D}_0)$. Let $\overline{\pi_0(x)} = U|\overline{\pi_0(x)}|$ be the polar decomposition of $\overline{\pi_0(x)}$ and let $|\overline{\pi_0(x)}| = \int_0^\infty \lambda dE(\lambda)$ be the spectral resolution of $|\overline{\pi_0(x)}|$. Then, $|\overline{\pi_0(x)}| = U^*\overline{\pi_0(x)} = \overline{\pi_0(U^*x)} \in L_2^\omega(\varphi_0)$. Since $|\overline{\pi_0(x)}|$ is φ_0-restrictedly measurable, $E(\lambda_0)^\perp \in L^2(\varphi_0)$ for a positive number λ_0. Hence, $|\overline{\pi_0(x)}|E(\lambda_0)^\perp \in L_2^\omega(\varphi_0)(L^2(\varphi_0) \cap L^\infty(\varphi_0)) \subset L^\omega(\varphi_0)$. Therefore we have

$$|\overline{\pi_0(x)}| = \int_0^{\lambda_0} \lambda dE(\lambda) + |\overline{\pi_0(x)}|E(\lambda_0)^\perp$$
$$\in \mathscr{U}_0(\mathscr{D}_0) + L^\omega(\varphi_0)\ .$$

Hence, $\pi_2^\omega(U^*x) \in \mathscr{U}(L^\omega(\mathscr{D}_0))$, and so $\pi_2^\omega(x) \in \mathscr{U}(L^\omega(\mathscr{D}_0))$. Consequently $\mathscr{U}(L_2^\omega(\mathscr{D}_0)) = \mathscr{U}(L^\omega(\mathscr{D}_0))$. Similarly we can show that $\mathscr{V}(L_2^\omega(\mathscr{D}_0)) =$

$\mathscr{V}(L^\omega(\mathscr{D}_0))$. Since $J\mathscr{U}_0(\mathscr{D}_0)J = \mathscr{V}_0(\mathscr{D}_0)$ and $\overline{J\pi_0(x)J} = \overline{\pi_0'(x^*)}$ for every $x \in \mathscr{D}$, we see that $J\mathscr{U}(\mathscr{D})J = \mathscr{V}(\mathscr{D})$.

DEFINITION 4.3. $\mathscr{U}(\mathscr{D})$ (resp. $\mathscr{V}(\mathscr{D})$) is called the left (resp. right) $EW^\#$-algebra of $\mathscr{D}$.

THEOREM 4.4. *Let $\mathscr{D}_0$ be a Hilbert algebra in a Hilbert space $\mathfrak{H}$ and $(\mathscr{D}_0)_b \neq \mathfrak{H}$. Then $L_2^\omega(\mathscr{D}_0)$ is a pure unbounded Hilbert algebra, and $\mathscr{U}(L_2^\omega(\mathscr{D}_0))$ and $\mathscr{V}(L_2^\omega(\mathscr{D}_0))$ are pure $EW^\#$-algebras on $L_2^\omega(\mathscr{D}_0)$ over $\mathscr{U}_0(\mathscr{D}_0)$ and $\mathscr{V}_0(\mathscr{D}_0)$, respectively.*

Proof. Theorem 3.4 and Theorem 4.3.

DEFINITION 4.5. Let $\mathfrak{A}$ be an $EW^\#$-algebra. $\mathfrak{A}$ is called a standard $EW^\#$-algebra if there exists a pure unbounded Hilbert algebra $\mathscr{D}$ such that $\mathfrak{A} = \mathscr{U}(\mathscr{D})$.

Let $\mathfrak{A}_0$ be a semifinite von Neumann algebra on a Hilbert space $\mathfrak{H}$ and let φ_0 be a faithful normal semifinite trace on $\mathfrak{A}_0^+$. Let $\mathfrak{M}(\mathfrak{A}_0)$ denote the set of all measurable operators with respect to $\mathfrak{A}_0$. From ([4] Proposition 4.3) $\mathfrak{M}(\mathfrak{A}_0)$ is an EW^*-algebra over $\mathfrak{A}_0$. Let $\mathfrak{M}_{\varphi_0}$ be the maximal ideal associated with φ_0, i.e., $\mathfrak{M}_{\varphi_0} = \{T \in \mathfrak{A}_0; \varphi_0(|T|) < \infty\}$. For every $T \in \mathfrak{M}(\mathfrak{A}_0)^+$ we put

$$\mu(T) = \sup_{A \in \mathfrak{M}_{\varphi_0}^+ : A \leqq T} \varphi_0(A) ,$$

and

$$L^p(\varphi_0) = \{T \in \mathfrak{M}(\mathfrak{A}_0); \|T\|_p := \mu(|T|^p)^{1/p} < \infty\} , \quad 1 \leqq p < \infty ,$$
$$L^\infty(\varphi_0) = \mathfrak{A}_0 .$$

Then $L_2^\infty(\varphi_0) := L^\infty(\varphi_0) \cap L^2(\varphi_0)$ is a maximal Hilbert algebra in the Hilbert space $L^2(\varphi_0)$ under the inner product $(S|T) = \mu(T^* \cdot S)$ and $L_2^\omega(\varphi_0) := \bigcap_{2 \leqq p < \infty} L^p(\varphi_0)$ is a maximal unbounded Hilbert algebra over $L_2^\infty(\varphi_0)$. Let $\mathscr{D}(\varphi_0)$ be an unbounded Hilbert algebra in $L^2(\varphi_0)$ over $L_2^\infty(\varphi_0)$. Then $\mathscr{D}(\varphi_0)$ is regarded as a $*$-algebra on $\mathfrak{H}$ under the strong sum, strong product, adjoint and strong scalar multiplication. We denote by $\mathfrak{A}(\mathscr{D}(\varphi_0))$ the set of closed operators on $\mathfrak{H}$ which is the $*$-algebra generated by $\mathscr{D}(\varphi_0)$ and $\mathfrak{A}_0$. Then $\mathfrak{A}(\mathscr{D}(\varphi_0))$ is an EW^*-algebra over $\mathfrak{A}_0$ and it is isomorphic to the left $EW^\#$-algebra $\mathscr{U}(\mathscr{D}(\varphi_0))$.

THEOREM 4.5. *Let $\mathfrak{A}_0$ be a semifinite von Neumann algebra on a Hilbert space $\mathfrak{H}$ and let φ_0 be a faithful normal semifinite trace on*

$\mathfrak{A}_0^+$. *If* $L^2(\varphi_0)$ *is not a Hilbert algebra,* i.e., $L^2(\varphi_0) \neq L_2^\infty(\varphi_0)$, *then there exists a pure* EW^**-algebra* $\mathfrak{A}$ *over* $\mathfrak{A}_0$ *such that is isomorphic to a standard* $EW^\#$*-algebra. In particular, if* $\bigcap_{T \in L_2^\omega(\varphi_0)} \mathscr{D}(T)$ *is dense in* $\mathfrak{H}$, *then we may regard* $\mathfrak{A}$ *as a pure* $EW^\#$*-algebra over* $\mathfrak{A}_0$.

COROLLARY 4.6. *Let* $\mathfrak{A}_0$ *be a semifinite von Neumann algebra on a Hilbert space* $\mathfrak{H}$ *and let* φ_0 *be a faithful normal semifinite trace on* $\mathfrak{A}_0^+$. *If* $\mathfrak{A}$ *is a pure* EW^**-algebra over* $\mathfrak{A}_0$ *such that* $\mathfrak{A} \subset \mathfrak{A}(L_2^\omega(\varphi_0))$, *then* $\mathfrak{A}$ *is isomorphic to a standard* $EW^\#$*-algebra.*

Proof. We can easily prove that $\mathfrak{A} \cap L_2^\omega(\varphi_0)$ is a pure unbounded Hilbert algebra over $L_2^\infty(\varphi_0)$ and $\mathfrak{A}$ is isomorphic to $\mathscr{U}(\mathfrak{A} \cap L_2^\omega(\varphi_0))$.

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