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DECOMPOSITIONS FOR NONCLOSED PLANAR m-CONVEX SETS

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Let S be an m-convex set in the plane having the property that (int cl $S)\sim S$ contains no isolated points. If T is an m-convex subset of S having convex closure, then T is a union of $\sigma(m)$ or fewer convex sets, where

$$\sigma(m)=(m-1)[1+(2^{m-2}-1)2^{m-3}]\,.$$

Hence for $m\geq 3$, S is expressible as a union of $(m-1)^3 2^{m-3}\sigma(m)$ or fewer convex sets.

In case S is m-convex and (int cl $S)\sim S$ contains isolated points, an example shows that no such decomposition theorem is possible.

1. **Introduction.** For S a subset of Euclidean space, S is said to be m-convex, $m \geqq 2$, if and only if for every m distinct points of S, at least one of the line segments determined by these points lies in S. Several decomposition theorems have been proved for m-convex sets in the plane. A closed planar 3-convex set is expressible as a union of 3 or fewer convex sets (Valentine [4]), and an arbitrary planar 3-convex set is a union of 6 or fewer convex sets (Breen [1]). Concerning the general case, a recent study shows that for $m \geqq 3$, a closed planar m-convex set may be decomposed into $(m-1)^3 2^{m-3}$ or fewer convex sets (Kay and Breen [2]). This leads naturally to the problem considered here, that of determining whether such a bound exists for an arbitrary m-convex set $S \subseteqq R^2$: With the restriction that (int cl $S) \sim S$ contain no isolated points, a bound in terms of m is obtained; without this restriction, an example reveals that no bound is possible.

The following terminology will be used: For points x, y in S, we say *x sees y via S* if and only if the corresponding segment $[x, y]$ lies in S. Points $x_1, \cdots, x_n$ in S are *visually independent via S* if and only if for $1 \leqq i < j \leqq n$, x_i does not see x_j via S. Throughout the paper, conv S, bdry S, int S, and cl S will be used to denote the convex hull of S, the boundary of S, the interior of S and the closure of S, respectively.

2. **The decomposition theorem.** We shall be concerned with the proof of the following result, which yields the decomposition theorem as a corollary.

THEOREM. *Let T be an m-convex set in the plane having the property that* (int cl T) $\sim T$ *contains no isolated points. If* cl T *is convex, then T is a union of $\sigma(m)$ or fewer convex sets, where*

$$\sigma(m) = (m-1)[1 + (2^{m-2} - 1)2^{m-3}] .$$

The main steps in the proof will be accomplished by a sequence of lemmas. The first lemma, which generalizes [1, Theorem 5], will require the following result by Lawrence, Hare and Kenelly [3, Theorem 2].

Lawrence, Hare, Kenelly theorem. Let T be a subset of a linear space such that each finite subset $F \subseteqq T$ has a k-partition $\{F_i, \cdots, F_k\}$, where conv $F_i \subseteqq T$, $1 \leqq i \leqq k$. Then T is a union of k or fewer convex sets.

LEMMA 1. *Let T be an m-convex set in the plane, $m \geqq 3$, such that* cl T *is convex. If all points of* (cl T) $\sim T$ *are in* bdry(cl T), *then T is a union of* $\max(m-1, 3)$ *or fewer convex sets. The result is best possible.*

Proof. By the Lawrence, Hare, Kenelly theorem, it suffices to consider finite subsets of T, so without loss of generality we may assume that cl T is a convex polygon. Consider the collection of all intervals in cl T having endpoints in T and some relatively interior point not in T, and let $\mathscr{L}$ denote the collection of corresponding lines. Since (cl $T \sim T$) $\subseteqq$ bdry(cl T), each line L in $\mathscr{L}$ supports cl T along an edge, and by the m-convexity of T, $L \cap T$, has at most $m-1$ components. We will examine the components of $B = \cup\{L \cap T\colon L \text{ in } \mathscr{L}\}$.

Order the vertices of cl T in a clockwise direction along bdry-(cl T), letting p_i denote the ith vertex in our ordering, $1 \leqq i \leqq k$. If p_i lies in some component of B, let c_i denote this component. Otherwise, let $c_i = \emptyset$. Define sets A'_i, $1 \leqq i \leqq \max(3, m-1)$, each an appropriate collection of components of B: For i odd, $i < k$, assign c_i to A'_1; for i even, $i < k$, assign c_i to A'_2; assign c_k to A'_3. Now consider the remaining components of B. If the line $L(p_i, p_{i+1})$ determined by p_i and p_{i+1} is in $\mathscr{L}$, $1 \leqq i \leqq k$ (where $p_{k+1} = p_1$), assign each remaining component on this line to some A' set not containing $c_i \neq \emptyset$ or $c_{i+1} \neq \emptyset$, and assign at most one component to each A' set. Since there are at most $m-1$ components on each line, at most $m-1$ A' sets are required at each stage of the argument. Furthermore, no two components on any line will be assigned to the same A' set.

Finally, let $A_i \equiv T \sim \cup\{A'_j\colon j \neq i\}$, $1 \leqq i \leqq \max(m-1, 3)$. It is easy to show that the A_i sets are convex and that their union is T, completing the proof.

To see that the result in Lemma 1 is best possible, consider the following example.

EXAMPLE 1. Let T be a pentagonal region having exactly $m-2$ points deleted from the relative interior of each edge, $m \geqq 3$. Then T is m-convex and is not expressible as a union of fewer than max $(m-1, 3)$ convex sets.

Lemmas 2, 3 and 4 concern points in (int cl $S) \sim S$.

LEMMA 2. *Let S be an arbitrary set in the plane. If* (int cl $S) \sim S$ *contains at least r noncollinear segments, where $r = 2^n$, $n \geqq 0$, then S contains $n + 2$ visually independent points.*

Proof. The proof is by induction. If $n = 0$, then $r = 1$ and certainly S contains 2 visually independent points. Assume the theorem true for numbers less than n, $n \geqq 1$, to prove for n. Let L be the line determined by one of the 2^n (or more) noncollinear segments C in (int cl $S) \sim S$. Then at least half of the $2^n - 1$ remaining segments contain points in one of the open halfspaces H_1 determined by L. Hence $S' = S \cap H_1$ has the property that (int cl $S') \sim S'$ contains at least r' noncollinear segments, where $r' \geqq (2^n-1)/2 = 2^{n-1} - 1/2$. Since r' is an integer, $r' \geqq 2^{n-1}$, so by our induction hypothesis, S' contains $n + 1$ visually independent points $y_1, \cdots, y_{n+1}$. Letting H_2 denote the opposite open halfspace determined by L, select y_0 in $H_2 \cap S$ so that $[y_0, y_i]$ cuts C for $1 \leqq i \leqq n + 1$. Then $\{y_0, \cdots, y_{y+1}\}$ is a set of $n + 2$ visually independent points of S.

COROLLARY. *If S is planar and m-convex, then* (int cl $S) \sim S$ *contains at most $2^{m-2} - 1$ noncollinear segments.*

Proof. Assume that S contains $r \geqq 1$ noncollinear segments. Then $2^n \leqq r < 2^{n+1}$ for an appropriate $n \geqq 0$, and by the lemma, S contains $n + 2$ visually independent points. Since S is m-convex, we have $n + 2 \leqq m - 1$, so $r < 2^{m-2}$.

The author wishes to thank the referee for his conjecture of the following result.

LEMMA 3. *Let S be an m-convex set in the plane, $m \geqq 3$. If M is any line, then $M \cap$ [(int cl $S) \sim S$] has at most $m + [(m-3)/2]$ components. The result is best possible.*

Proof. Assume that $M \cap [(\text{int cl } S) \sim S] \neq \emptyset$, for otherwise there is nothing to prove. Since S is m-convex, it is easy to show that the set cl S is m-convex, so $M \cap \text{cl } S$ has at most $m - 1$ components M_i, $1 \leqq i \leqq m - 1$. There exist disjoint convex neighborhoods U_i of M_i, $1 \leqq i \leqq m - 1$, such that no point of $U_i \cap \text{cl } S$ sees any point of $U_j \cap \text{cl } S$ via cl S, $1 \leqq i < j \leqq m - 1$. Thus no point of $U_i \cap S$ sees any point of $U_j \cap S$ via S, $1 \leqq i < j \leqq m - 1$.

Note that if $M_i \cap [(\text{int cl } S) \sim S] \neq \emptyset$, there are at least two points in $U_i \cap S$ which are visually independent via S. Hence $M_i \cap [(\text{int cl } S) \sim S] \neq \emptyset$ for at most $[(m - 1)/2]$ of the M_i sets.

We use an inductive argument to prove the lemma. If S is 3-convex, then $M_1 \cap [(\text{int cl } S) \sim S] \neq \emptyset$ for at most one component M_1 of $M \cap \text{cl } S$, and it is easy to see that $M_1 \cap [(\text{int cl } S) \sim S]$ consists of at most three components. Assume that the result is true for j, $3 \leqq j < m$, to prove for m. For some component M_1 of $M \cap \text{cl } S$, assume that $M_1 \cap [(\text{int cl } S) \sim S]$ has k components. Then clearly $1 \leqq k \leqq m$. For the neighborhood U_1 defined above, there correspond at least $\max(2, k - 1)$ visually independent points of S in U_1. Examine the set $S' = \cup \{U_i \cap S : i \neq 1\}$. There are two cases to consider.

Case 1. If $k \geqq 3$, the set S' contains at most $m - k$ visually independent points, and S' is $(m - k + 1)$-convex. By our inductive assumption applied to S', $M \cap [(\text{int cl } S') \sim S']$ has at most $(m-k+1) + [(m - k + 1 - 3)/2]$ components. Then $M \cap [(\text{int cl } S) \sim S]$ has at most $k + (m - k + 1) + [(m - k - 2)/2] = m + [(m - k)/2]$ components. This number is maximal when $k = 3$, giving the desired result.

Case 2. If $1 \leqq k < 3$, then a similar argument shows that there are at most $2 + (m - 2) + [(m - 2 - 3)/2] = m + [(m - 5)/2] < m + [(m - 3)/2]$ components, finishing the proof of the lemma.

An inductive construction may be used to show that the result of Lemma 3 is best possible.

EXAMPLE 2. For $3 \leqq m \leqq 4$, remove m collinear segments appropriately from an open convex set to obtain an m-convex set having the required property. Inductively, for $m \geqq 5$ let S denote the union of an $(m - 2)$-convex set S_1 and a 3-convex set S_2, where (int cl S_i) $\sim S_i$ has the maximal number of collinear components, (int cl S_1) $\sim S_1$ and (int cl S_2) $\sim S_2$ are collinear, and $\text{cl } S_1 \cap \text{cl } S_2 = \emptyset$. By our inductive construction, the set $(\text{int cl } S) \sim S$ will have exactly $m - 2 + [(m - 5)/2] + 3 = m + [(m - 3)/2]$ collinear components.

LEMMA 4. *Let S be an m-convex set in the plane. If* $x \in$ (int

$\mathrm{cl}\, S) \sim S$ *and* x *is not an isolated point, then* x *lies in a segment in* $(\mathrm{int}\,\mathrm{cl}\, S) \sim S$.

Proof. Assume on the contrary that x is not in a segment in $(\mathrm{int}\,\mathrm{cl}\, S) \sim S$ to obtain a contradiction. By the corollary to Lemma 2, $(\mathrm{int}\,\mathrm{cl}\, S) \sim S$ contains at most $2^{m-2} - 1$ noncollinear segments. Also, by Lemma 3, for M any line determined by such a segment, $M \cap [(\mathrm{int}\,\mathrm{cl}\, S) \sim S]$ has at most $m + [(m-3)/2]$ components, so the segments in $(\mathrm{int}\,\mathrm{cl}\, S) \sim S$ may by written as a finite union of segments. Hence we may select an open disk N centered at x which is disjoint from each of these segments, with $N \subseteq \mathrm{int}\,\mathrm{cl}\, S$. Let N_0 be an open disk centered at x and properly contained in N. Let L be any line through x, and let C be any component of $(\mathrm{int}\,\mathrm{cl}\, S) \sim S$ containing x. Since x is not an isolated point, there are points of $C \cap N_0$ in at least one of the open halfspaces H_1 determined by L, and we let C_1 be a component of $C \cap H_1 \cap N_0$. Clearly C_1 is not a singleton set and cannot be collinear with x.

We assert that there is some point z_1 in $N \cap S$ and some neighborhood N_1 of x, $N_1 \subseteq N$, such that z_1 sees no point of $N_1 \cap S$ via S: Select points s, t in C_1 such that x, s, t are not collinear. Select $z_1 \in S$ in the open convex region bounded by the rays $R(x, s)$, $R(x, t)$ and in $N \sim N_0$ (where $R(x, s)$ denotes the ray emanating from x through s). Since $[x, z_1] \subseteq N$, each component of $[x, z_1] \sim S$ is a singleton point. Also, there are at most $m - 2$ such components, so there is some point q on $(x, z_1]$ such that $(x, q) \cap C_1 = \varnothing$.

Let line L_1 be parallel to L so that s, t, z_1 are on the same side of L_1 and so that L_1 contains some point $q_1 \in (x, q)$. Repeating an argument from the preceding paragraph, components of $C_1 \cap L_1$ are singleton sets. Hence there exist points v, w in $L_1 \cap N_0$, $v < q_1 < w$, with $(v, w) \cap C_1 = \varnothing$. Without loss of generality, assume that v and w are interior to the convex region determined by rays $R(z_1, s)$ and $R(z_1, t)$. Then for $v < y < w$, we see that $[z_i, y] \cap C_1 \neq \varnothing$: Otherwise, the path $\lambda = [z_1, y] \cup [y, q_1] \cup [q_1, x)$ would be disjoint from C_1, with s and t on opposite sides of λ. Since $z_1, x \notin H_1 \cap N_0$ and $C_1 \subseteq H_1 \cap N_0$, λ would separate C_1, impossible.

Finally, let N_1 be any open disk about x in the open convex region determined by $R(z_1, v)$ and $R(z_1, w)$ such that N_1 and z_1 are on opposite sides of L_1. Then for every y in N_1, $[z_1, y]$ intersects (v, w) and thus $[z_1, y]$ intersects C_1. Hence z_1 sees no point of $N_1 \cap S$ via S, the desired result.

Repeat the argument to obtain z_2 in $N_1 \cap S$ and $N_2 \subseteq N_1$ with z_2 seeing no point of $N_2 \cap S$ via S. By an obvious induction, we obtain $\{z_1, \cdots, z_m\}$ a set of m visually independent points in S. This contradicts the m-convexity of S, our original assumption is false, and x must lie in a segment in $(\operatorname{int} \operatorname{cl} S) \sim S$.

Finally, the following combinatorial result will be helpful.

LEMMA 5. *For each collection* $\mathscr{L}$ *of* $r \geqq 1$ *lines in the plane,* $R^2 \sim (\cup \mathscr{L})$ *consists of at most* $f(r) = 1 + \sum_{k=1}^{r} k$ *convex components.*

Proof. We use an inductive argument. If $r = 1$, the result is clear. Assume the result true for $r = n \geqq 1$ to prove for $n + 1$. For $\mathscr{L}$ consisting of $n + 1$ lines, select any member L of $\mathscr{L}$ and let $\mathscr{L}' = \mathscr{L} \sim \{L\}$. Then by our induction hypothesis, $R^2 \sim (\cup \mathscr{L}')$ consists of at most $f(n)$ convex components. The line L cuts each member of $\mathscr{L}'$ at most once, so there are at most n corresponding points of intersection. These n points in turn determine at most $n + 1$ intervals on L (two of which are unbounded), and each of these intervals cuts a component of $R^2 \sim (\cup \mathscr{L}')$, yielding two convex components where previously there was only one. Hence $R^2 \sim (\cup \mathscr{L})$ consists of at most $f(n) + n + 1 = f(n + 1)$ convex components.

THEOREM 1. *Let* T *be an* m*-convex set in the plane having the property that* $(\operatorname{int} \operatorname{cl} T) \sim T$ *contains no isolated points. If* $\operatorname{cl} T$ *is convex, then* T *is a union of* $\sigma(m)$ *or fewer convex sets, where*

$$\sigma(m) = (m - 1)[1 + (2^{m-2} - 1)2^{m-3}] .$$

Proof. If $m = 2$, the result is clear, so assume that $m \geqq 3$. By Lemma 4, $(\operatorname{int} \operatorname{cl} T) \sim T$ may be expressed as a union of segments, and by the corollary to Lemma 2, these segments determine a corresponding collection $\mathscr{L}$ of at most $r = 2^{m-2} - 1$ lines. Using Lemma 4, $R^2 \sim (\cup \mathscr{L})$ consists of at most $f(r)$ convex components C_i, $1 \leqq i \leqq f(r)$, where $f(r) = 1 + \sum_{k=1}^{r} k = 1 + (r(r + 1))/2 = 1 + (2^{m-2} - 1)(2^{m-3})$.

Let $T_i = (\operatorname{cl} C_i) \cap T$, $1 \leqq i \leqq f(r)$. Then clearly T_i is an m-convex set, $m \geqq 3$, such that $\operatorname{cl} T_i$ is convex and $(\operatorname{cl} T_i) \sim T_i \subseteqq \operatorname{bdry}(\operatorname{cl} T_i)$. Then by Lemma 1, T_i is a union of $\max(m - 1, 3)$ or fewer convex sets. Hence if $m \geqq 4$, T is a union of

$$\sigma(m) = (m - 1)[1 + (2^{m-2} - 1)2^{m-3}]$$

or fewer convex sets, the desired result.

In case $m = 3$, then by [1, Lemma 3], all points of $(\operatorname{int} \operatorname{cl} T) \sim T$ are collinear. If L is the corresponding line, $T \cap L$ contains at most two components L_1, L_2. Letting H_1, H_2 represent distinct open halfspaces determined by L, define $T_i = (H_i \cap T) \cup L_i$, $1 \leqq i \leqq 2$.

A proof similar to that of Lemma 1 shows that each T_i is a union of two or fewer convex sets, so T is a union of $\sigma(3) = 4$ or fewer convex sets, completing the proof of the theorem.

COROLLARY. *If S is an m-convex set in the plane, $m \geqq 3$, having the property that* $(\text{int cl } S) \sim S$ *contains no isolated points, then S is expressible as a union of* $(m-1)^3 2^{m-3}\sigma(m)$ *or fewer convex sets.*

Proof. It is easy to show that the set $\text{cl } S$ is m-convex, and by [2, Theorem 6], $\text{cl } S$ may be decomposed into $(m-1)^3 2^{m-3}$ or fewer closed convex sets. If C is one of these convex sets, let $T = C \cap S$. Clearly T is m-convex. There are two cases to consider.

Case 1. If C is contained in a line, then T contains at most $m - 1 < \sigma(m)$ convex components.

Case 2. If C is not contained in a line, then it is easy to show that $\text{cl } T = C$: First pick p in C. Since $C \subseteqq \text{cl } S$, every neighborhood of p contains points of S. If p is in $\text{int } C$, then points of S contained in small discs centered at p necessarily belong to $C \cap S = T$. Thus we conclude that $p \in \text{cl } T$. On the other hand, if $p \in \text{bdry } C$, then every neighborhood of p contains points of $\text{int } C$. By our previous remarks, $\text{int } C \subseteqq \text{cl } T$, so $p \in \text{cl}(\text{cl } T) = \text{cl } T$. Hence $C \subseteqq \text{cl } T$. The reverse inclusion is obvious, so $C = \text{cl } T$ and $\text{cl } T$ is convex. Certainly $(\text{int cl } T) \sim T$ contains no isolated points, so by the theorem, T is a union of $\sigma(m)$ or fewer convex sets. Thus S is a union of $(m-1)^3 2^{m-3}\sigma(m)$ or fewer convex sets.

3. An example.

The following example shows that no decomposition theorem is possible in case S is an m-convex set having isolated points as components of $(\text{int cl } S) \sim S$.

EXAMPLE 3. Let k be an arbitrary integer and let P be a regular polygon having $2k$ vertices $p_1, \cdots, p_{2k}$, *Let* $v_1, \cdots, v_{2k}$ be vertices of a regular polygon interior to P, where for $1 \leqq i \leqq 2k$, v_i is sufficiently close to p_i that the following holds: If x and y are visually independent points of $P' \equiv P \sim \{v_1, \cdots, v_{2k}\}$, then for every i, j, $1 \leqq i, j \leqq 2k$, either $(R(x, v_i) \sim [x, v_i]) \cap (R(y, v_j) \sim [y, v_j]) \cap P = \emptyset$ or x, v_i, y, v_j are collinear. Hence three points x, y, z are visually independent via P' only if they are collinear with a pair of distinct points v_i and v_j, and P' is 4-convex.

However, P' is not expressible as a union of fewer than $k + 2$

convex sets. (If the vertices v_i are ordered in a clockwise direction, $1 \leqq i \leqq 2k$, consider the $k+1$ subsets $P_1, \cdots, P_{k+1}$ of P' bounded by and disjoint from the k lines $L(v_1, v_{2k})$, $L(v_2, v_{2k-1}), \cdots, L(v_k, v_{k+1})$. Let $P_{k+2} = \text{conv}(\cup \{(v_i, v_{2k+1-i}): 1 \leqq i \leqq k\})$. Assign each remaining segment of $P' \cap L(v_i, v_{2k+1-i})$ to one of the adjacent regions P_i or P_{i+1}, $1 \leqq i \leqq k$, in the obvious manner. This yields a $(k+2)$-member decomposition of P'. The number $k+2$ is best possible.)

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