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A NOTE ON QUASISIMILARITY II

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Let $\mathscr{H}$ denote a separable, infinite dimensional complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. An operator X in $\mathscr{L}(\mathscr{H})$ is a *quasiaffinity* (or a *quasi-invertible* operator) if X is injective and has dense range. An operator A on $\mathscr{H}$ is a *quasiaffine transform* of operator B if there exists a quasiaffinity such that $BX = XA$. A and B are *quasisimilar* if they are quasiaffine transforms of one another. The purpose of this note is to study the quasisimilarity orbits of certain subsets of $\mathscr{L}(\mathscr{H})$ containing quasinilpotent, spectral, and compact operators.

In response to a question of E. Azoff **[5]**, we show in §2 that each countable direct sum of spectral operators is quasisimilar to a spectral operator. This result is used in §5 to give a sufficient condition for an operator with a family of spectral parts to be quasisimilar to a spectral operator. In §3 we give two examples concerning the problem of characterizing membership in $\mathscr{Q}_{qs}$, the quasisimilarity orbit of the set of all quasinilpotent operators in $\mathscr{L}(\mathscr{H})$ (cf. **[11]**). Let T and S denote quasisimilar operators. In **[21]**, Sz.-Nagy and Foiaş proved that if S is unitary, then $\mathrm{Lat}_h(T)$, the lattice of hyperinvariant subspaces of T, contains a sublattice that is lattice isomorphic to $\mathrm{Lat}_h(S)$. In §4 we give a generalization of this result to arbitrary operators from which the result of Sz.-Nagy and Foiaş is easily recovered. For the case when S is spectral, we obtain an analogous result involving the lattice of spectral subspaces of S. We show that in the general case $\mathrm{Lat}_h(T)$ always contains a sublattice that is lattice isomorphic to the Riesz lattice of S. In §5, as another application, we determine the quasisimilarity orbit of a class of compact operators.

For T in $\mathscr{L}(\mathscr{H})$, $\sigma(T)$, $\rho(T)$, and $r(T)$ denote, respectively, the spectrum, resolvent, and spectral radius of T. We will use basic facts about essential spectra from **[12]** and about quasitriangularity from **[7]**.

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2. Quasisimilarity and direct sums of spectral operators. In **[15]** F. Gilfeather proved that each direct integral of quasinilpotent operators is unitarily equivalent to a countable direct sum of quasinilpotent operators, and in **[11]** it was proved that each such direct sum is in $\mathcal{Q}_{qs}$. E. Azoff **[4]** extended Gilfeather's result by proving that each direct integral of spectral operators is unitarily equivalent to a countable direct sum of spectral operators. The following examples of such direct sums were pointed out in **[4]**. Each operator in a finite type I von Neumann algebra is a direct integral of operators acting on finite dimensional spaces, and is thus a direct sum of spectral operators by Azoff's result. In **[14]** it was proved that each root of an abelian analytic operator-valued function is unitarily equivalent to a countable direct sum of spectral operators (see **[14]** for terminology and details). We now show that each direct sum of spectral operators is quasisimilar to a spectral operator.

Recall that by a theorem of Dunford (see, for example, [**18**, §2]) an operator S in $\mathcal{L}(\mathcal{H})$ is spectral if and only if $S = R^{-1}NR + Q$, where R is invertible, N is normal, Q is quasinilpotent, and Q commutes with $R^{-1}NR$. This decomposition is unique and is called the canonical decomposition of S; moreover, if S is spectral, then $\sigma(S) = \sigma(N)$.

THEOREM 2.1. *The countable direct sum of spectral operators is quasisimilar to a spectral operator.*

Proof. For each $i \geqq 1$, let $S_i = R_i^{-1}N_iR_i + Q_i$ denote the canonical decomposition of the spectral operator S_i acting on the complex Hilbert space $\mathcal{H}_i$; the norm $\| \ \|_i$ on $\mathcal{H}_i$ is induced by the inner product $(\, , \,)_i$. We will prove that if $\{\|S_i\|\}$ is a bounded sequence, then the operator $S = \Sigma \bigoplus S_i$ acting on $\mathcal{H} = \Sigma \bigoplus \mathcal{H}_i$ is quasisimilar to a spectral operator.

For each $i \geqq 1$, let T_i denote the quasinilpotent operator $iR_iQ_iR_i^{-1}$. We define a new inner product $(\, , \,)_{i,0}$ on $\mathcal{H}_i$ by $(x, y)_{i,0} = \Sigma_{n=0}^{\infty}(T_i^n x, T_i^n y)_i$ for each x and y in $\mathcal{H}_i$. It is shown in [**17**, page 278] that the preceding series converges and induces a complete norm $\| \ \|_{i,0}$ on $\mathcal{H}_i$ that is equivalent to $\| \ \|_i$. Let $\mathcal{H}_{i,0}$ denote $\mathcal{H}_i$ equipped with this new norm and let $J_i \colon \mathcal{H}_i \to \mathcal{H}_{i,0}$ be defined by $J_i x = x$ for each x in $\mathcal{H}_i$. It is shown in **[17]** that J_i is bounded and that $\|J_iT_iJ_i^{-1}\|_{i,0} < 1$ (where $\| \ \|_{i,0}$ now also denotes the operator norm on $\mathcal{L}(\mathcal{H}_{i,0})$).

Since N_i is a normal operator that commutes with T_i, Fuglede's Theorem [**19**, Cor. 1.18, page 20] implies that T_i commutes with N_i^*. Thus, for each x and y in $\mathcal{H}_{i,0}$, we have

$$\begin{aligned}(J_iN_iJ_i^{-1}x, y)_{i,0} &= (N_ix, y)_{i,0} = \Sigma\,(T_i^nN_ix, T_i^ny)_i = \Sigma\,(N_iT_i^nx, T_i^ny)_i \\ &= \Sigma\,(T_i^nx, N_i^*T_i^ny)_i = \Sigma\,(T_i^nx, T_i^nN_i^*y)_i = (x, N_i^*y)_{i,0} \\ &= (x, J_iN_i^*J_i^{-1}y)_{i,0}.\end{aligned}$$

This identity implies that $(J_iN_iJ_i^{-1})^* = J_iN_i^*J_i^{-1}$, and thus $J_iN_iJ_i^{-1}$ is a normal operator in $\mathscr{L}(\mathscr{H}_{i,0})$. In particular, we have

$$\|J_iN_iJ_i^{-1}\|_{i,0} = r(J_iN_iJ_i^{-1}) = r(N_i) = r(S_i) \leqq \|S_i\|,$$

so we may define the normal operator $N = \Sigma \oplus J_iN_iJ_i^{-1}$ in $\mathscr{L}(\mathscr{H}_0)$, where $\mathscr{H}_0 = \Sigma \oplus \mathscr{H}_{i,0}$.

If $A_i = J_iR_iQ_iR_i^{-1}J_i^{-1}$, then $\|A_i\|_{i,0} = \|J_i(1/i)T_iJ_i^{-1}\|_{i,0} < 1/i$, and it follows that $Q = \Sigma \oplus A_i$ is quasinilpotent. Indeed, for $z \in \mathbf{C} - \{0\}$, let n be a positive integer such that $1/n < |z|$. For $i > n$, $\|A_i\|_{i,0} \leqq 1/i < 1/n < |z|$ and thus

$$\|(A_i - z)^{-1}\|_{i,0} \leqq (|z| - \|A_i\|_{i,0})^{-1} < (|z| - 1/i)^{-1} < (|z| - 1/n)^{-1}.$$

Now

$$\sup_{i \geqq 1} \|(A_i - z)^{-1}\|_{i,0} \leqq \max\left(\sup_{1 \leqq i \leqq n} \|(A_i - z)^{-1}\|_{i,0}, (|z| - 1/n)^{-1}\right) < \infty,$$

and thus $z \notin \sigma(Q)$. Since N commutes with Q, $S_0 = N + Q$ is a spectral operator. To complete the proof, we note that since S_i is similar to $J_iN_iJ_i^{-1} + A_i$ for each i, then [**18**, Theorem 2.5] implies that S is quasisimilar to S_0.

REMARK. We note several relationships between the spectrum of a direct sum of spectral operators and that of any spectral operator quasisimilar to it. Let $T = \Sigma \oplus S_i$ denote a direct sum of spectral operators and let S be a spectral operator quasisimilar to T. If $\sigma = \bigcup_i \sigma(S_i)$, then [**11**, Corollary 2.12] implies that $\bar{\sigma} \subset \sigma(S) \subset \sigma(T)$, and [**10**, Lemma 2.1.] implies that each nonempty closed-and-open subset of $\sigma(T)$ has nonempty intersection with σ.

3. Examples concerning quasisimilarity and quasinilpotent operators. In [**11**, Theorem 3.1] it was proved that if T is in $\mathscr{Q}_{qs}$, then T satisfies the following properties: 1) $\sigma(T) = \sigma_e(T)$, 2) if P is a nonzero projection such that $(1 - P)TP = 0$, then $\sigma(T \mid P\mathscr{H})$ is connected and contains 0; if additionally $P \neq 1$, then $\sigma((1-P)T \mid (1-P)\mathscr{H})$ is connected and contains 0, 3) T and T^* are quasitriangular; 4) $(T \mid P\mathscr{H})^*$ is quasitriangular (P as above). It follows from the spectral characterization of quasitriangular operators [**3**] that the preceding conditions are equivalent to the following one: if P is a non-zero projection and $(1-P)TP = 0$, then $0 \in \sigma(T \mid P\mathscr{H})$; if additionally $P \neq 1$, then $0 \in \sigma((1-P)T \mid (1-P)\mathscr{H})$. In the following example we show that $\sigma_e(T \mid P\mathscr{H})$ need not contain 0 and may be disconnected, and that $T \mid P\mathscr{H}$ may be nonquasitriangular.

EXAMPLE 3.1. Let $\{e_n\}_{n=1}^{\infty}$ denote an orthonormal basis for $\mathcal{H}$ and let $k_n = n(n+1)/2$ $(n \geqq 1)$. We define an operator A by the equations $Ae_j = e_{j+1}$ if $k_n \leqq j \leqq k_{n+1} - 2$ $(n \geqq 1)$, and $Ae_j = 0$ if $j = k_{n+1} - 1$ $(n \geqq 1)$. Since A is a direct sum of nilpotent operators, A is quasisimilar to some quasinilpotent operator B. Let $a_n = 1/2^{n/2}$ $(n \geqq 1)$, $x = \sum_{n=1}^{\infty} a_n e_{k_n}$, and $\mathcal{M} = \langle A^n x\rangle_{n=1}^{\infty}$. We will show that $A \mid \mathcal{M}$ is nonquasitriangular. Let $f_n = A^n x$ $(n \geqq 1)$. Then $(f_n, f_m) = 0$ for $n \neq m$, and $\|f_n\|^2 = \sum_{i=n}^{\infty} a_i^2 = \sum_{i=n}^{\infty} 1/2^i = 1/2^{n-1}$ $(n \geqq 1)$. If $g_n = (1/\|f_n\|)f_n$, then $Ag_n = (\|f_{n+1}\|/\|f_n\|)g_{n+1} = (1/2^{1/2})g_{n+1}$ $(n \geqq 1)$. Let U be the unilateral shift on $\mathcal{M}$ defined by $Ug_n = g_{n+1}$ $(n \geqq 1)$. Since $A \mid \mathcal{M} = (1/2^{1/2})U$, $A \mid \mathcal{M}$ is nonquasitriangular and $\sigma_e(A \mid \mathcal{M}) = \{z \mid |z| = 1/2^{1/2}\}$. To show that a part of an operator in $\mathcal{Q}_{qs}$ may have disconnected essential spectrum we may consider $A \oplus (\alpha A)$ for $|\alpha| \neq 1$.

For T in $\mathcal{L}(\mathcal{H})$ we set $\mathcal{M}(T) = \{x \in \mathcal{H}: \|T^n x\|^{1/n} \to 0\}$; also $\mathcal{Q}_{af} = \{T \in \mathcal{L}(\mathcal{H})$: T is a quasiaffine transform of some quasinilpotent operator$\}$, and $\mathcal{Q}_{af}^* = \{T \in \mathcal{L}(\mathcal{H})$: $T^* \in \mathcal{Q}_{af}\}$. In **[2]** C. Apostol proved that if $\mathcal{M}(T^*)^- = \mathcal{H}$, then T is in $\mathcal{Q}_{af}$, and in **[11]** we asked whether $\mathcal{Q}_{qs} = \mathcal{Q}_{af} \cap \mathcal{Q}_{af}^*$, or equivalently, whether $\mathcal{M}(T)^- = \mathcal{M}(T^*)^- = \mathcal{H}$ implies that T is in $\mathcal{Q}_{qs}$. If T is decomposable and $\mathcal{M}(T^*)^- = \mathcal{H}$, then T is in $\mathcal{Q}_{af}$, and [**11**, Corollary 2.12] implies that T is quasinilpotent. If T is hyponormal and $\mathcal{M}(T)^- = \mathcal{H}$, then T^* is in $\mathcal{Q}_{af}$, and [**11**, Theorem 3.6] implies that $T = 0$.

For injective weighted shifts the situation is different. In the following example we show that there is an injective weighted shift T which is not quasinilpotent but for which $\mathcal{M}(T)^- = \mathcal{M}(T^*)^- = \mathcal{H}$. We are unable to decide if T is quasisimilar to a quasinilpotent operator.

EXAMPLE 3.2. For each integer $n \geqq 1$ let $m(n) = n(n+1)/2 + n$. Let T be the injective weighted shift defined by $Te_i = \alpha_i e_{i+1}$ $(i \geqq 1)$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$, $\alpha_i = 1$ if $i \neq m(n)$ for all $n \geqq 1$, and $\alpha_{m(n)} = 1/n^{m(n+1)}$ $(n \geqq 1)$. Since the weight sequence $\{\alpha_i\}$ contains arbitrarily long strings of consecutive 1's, it follows that $\sigma(T)$ is the closed unit disk. Since $T^{*n}e_n = 0$ for $n \geqq 1$, $\mathcal{M}(T^*)^- = \mathcal{H}$. A calculation shows that for $i \geqq m(n)$, $\|T^i e_1\|^{1/i} \leqq 1/n$, and thus e_1 is in $\mathcal{M}(T)$. Using the relation $\|T^i e_{k+1}\|^{1/i} = ((1/\alpha_k)^{1/(i+1)}\|T^{i+1}e_k\|^{1/(i+1)})^{(1+1/i)}$, it follows by induction that each e_k is in $\mathcal{M}(T)$, and thus $\mathcal{M}(T)^- = \mathcal{H}$.

4. Quasisimilarity and hyperinvariant subspaces. We recall two simple facts concerning the similarity of two operators A and B: (i) if J is an invertible operator such that $AJ = JB$, then a subspace $\mathcal{M}$ is invariant for B if and only if $J\mathcal{M}$ is invariant for A; (ii) the restriction of J to a B-invariant subspace $\mathcal{M}$ induces a similarity between $B \mid \mathcal{M}$ and $A \mid J\mathcal{M}$. In this section we examine the extent to which (i) and (ii) have analogues for quasisimilarity. If we persist in considering arbitrary invariant subspaces, the analogue of (ii) cannot be obtained, for there exist quasisimilar operators A and B, and an A-invariant subspace $\mathcal{M}$, such that $A \mid \mathcal{M}$ is not quasisimilar to any part of B. To see this, consider the operators A and B of Example 3.1. Now $A \mid \mathcal{M}$ is nonquasitriangular, while each part of B is quasinilpotent; thus [**11**, Theorem 3.1] implies that $A \mid \mathcal{M}$ is not quasisimilar to any part of B.

The principal occurrence of quasisimilarity in the literature is in relation to Hoover's result [**18**] that if A and B are quasisimilar operators and A has a nontrivial hyperinvariant subspace, then so does B. However, as is discussed in [**8**], the structure of an operator is not likely to be revealed by the presence of a single nontrivial hyperinvariant subspace for B, but more likely by the presence of a collection of hyperinvariant subspaces $\{\mathcal{M}_\alpha\}_{\alpha\in I}$ for which the structure of $B \mid \mathcal{M}_\alpha$ is well understood. Thus the ultimate value of the quasisimilarity relation may lie in the extent to which it preserves the lattices of the hyperinvariant subspaces of quasisimilar operators. In [**21**, Prop. 5.1, pg. 76] it was proved that if A is quasisimilar to a unitary operator B, then there exists an injective mapping of $\mathrm{Lat}_h\,(B)$ into $\mathrm{Lat}_h\,(A)$ which respects the lattice structures. In [**18**] this result was extended to the case when B is a normal operator. Using the proofs of these results as motivation, we next give a generalization of these results to non-normal operators.

For T in $\mathcal{L}(\mathcal{H})$, let $(T)'$ denote the commutant of T, i.e. $(T)' = \{X$ in $\mathcal{L}(\mathcal{H})$: $TX = XT\}$, and let $(T)''$ denote the second commutant of T, i.e. $(T)'' = \{Y$ in $\mathcal{L}(\mathcal{H})$: $YX = XY$ for each X in $(T)'\}$. Let $\varphi(\mathcal{H})$ denote the set of all closed subspaces of $\mathcal{H}$, and let $\gamma(T)$ denote the set $\{\mathcal{M} \in \varphi(\mathcal{H})$: $\mathcal{M} = \overline{Q_{\mathcal{M}}\mathcal{H}}$ for some operator $Q_{\mathcal{M}}$ in $(T)''\}$.

PROPOSITION 4.1. *If T is in $\mathcal{L}(\mathcal{H})$, then $\gamma(T) \subset \mathrm{Lat}_h\,(T)$. If S is quasisimilar to T, then there exists a function $q\colon \gamma(T) \to \varphi(\mathcal{H})$ such that the following properties are satisfied.*

(1) *$q(\mathcal{M})$ is a hyperinvariant subspace of S for each $\mathcal{M}$ in $\gamma(T)$.*

(2) *q is injective.*

(3) *$q(\{0\}) = \{0\}$; $q(\mathcal{H}) = \mathcal{H}$.*

(4) *if $\mathcal{M}, \mathcal{N} \in \gamma(T)$ and $\mathcal{M} \subset \mathcal{N}$, then $q(\mathcal{M}) \subset q(\mathcal{N})$.*

(5) *if $\{\mathcal{M}_\alpha\}_{\alpha\in I} \subset \gamma(T)$ and $\bigcap_\alpha \mathcal{M}_\alpha = \{0\}$, then $\bigcap_\alpha q(\mathcal{M}_\alpha) = \{0\}$.*

(6) *if $\{\mathcal{M}_\alpha\}_{\alpha\in I} \subset \gamma(T)$ and $\vee_\alpha \mathcal{M}_\alpha \in \gamma(T)$, then $q(\vee_\alpha \mathcal{M}_\alpha) = \vee_\alpha q(\mathcal{M}_\alpha)$.*

(7) *$S \mid q(\mathcal{M})$ is a quasiaffine transform of $T \mid \mathcal{M}$ for each $\mathcal{M} \in \gamma(T)$.*

(8) *if R is a nonzero part of $S \mid q(\mathcal{M})$, then each nonempty closed-and-open subset of $\sigma(R)$ has nonempty intersection with $\sigma(T \mid \mathcal{M})$.*

(9) *if V is a nonzero part of $T \mid \mathcal{M}$, then each nonempty closed-and-open subset of $\sigma(V)$ has nonempty intersection with $\sigma(S \mid q(\mathcal{M}))$.*

We acknowledge at the outset that the proofs of (1) and (3)–(6) are essentially the same as the proofs of the corresponding results in [**6**, Theorem 4.5, page 56] and [**21**, Prop. 5.1, page 76]; they are included here for the sake of completeness. We note also that the case $\gamma(T) = \{\mathcal{H}\}$ may arise. In [**13**] Gellar and Herrero give examples of weighted shifts T such that for each $A \neq 0$ commuting with T, A is injective and has dense range; clearly $\gamma(t) = \{\mathcal{H}\}$. We will repeatedly use the simple fact that if $\mathcal{K} \subset \mathcal{H}$ and U is in $\mathcal{L}(\mathcal{H})$, then $\overline{U\overline{\mathcal{K}}} = \overline{U\mathcal{K}}$.

Proof of Proposition 4.1. For $\mathcal{M}$ in $\gamma(T)$ we have $\mathcal{M} = \overline{Q_{\mathcal{M}}\mathcal{H}}$ for some $Q_{\mathcal{M}}$ in $(T)''$. If R commutes with T, then

$$R\mathcal{M} = R\overline{Q_{\mathcal{M}}\mathcal{H}} \subset \overline{R\overline{Q_{\mathcal{M}}\mathcal{H}}} = \overline{RQ_{\mathcal{M}}\mathcal{H}} = \overline{Q_{\mathcal{M}}R\mathcal{H}} \subset \overline{Q_{\mathcal{M}}\mathcal{H}} = \mathcal{M},$$

and thus $\mathcal{M}$ is a hyperinvariant subspace for T.

Suppose that X and Y are quasi-invertible operators such that $SX = XT$ and $TY = YS$. For $\mathcal{M}$ in $\gamma(T)$ we set $q(\mathcal{M}) = \vee_{R \in (S)'} RX\mathcal{M}$; it is clear that q satisfies (1), (3), and (4), and the proofs of (5) and (6) are the same as in the case when T is unitary (see [**21**]). To show that q is injective it suffices to verify that $\overline{Yq(\mathcal{M})} = \mathcal{M}$ for each $\mathcal{M}$ in $\gamma(T)$. First, note that if R is in $(S)'$, then YRX is in $(T)'$, since $YRXT = YRSX = YSRX = TYRX$. Let $Q_{\mathcal{M}}$ in $(T)''$ be such that $\overline{Q_{\mathcal{M}}\mathcal{H}} = \mathcal{M}$. Now

$$\begin{aligned}\overline{Yq(\mathcal{M})} &= \overline{Y \overline{\bigvee_{R \in (S)'} RX\mathcal{M}}} \subset \bigvee_R \overline{YRX\mathcal{M}} = \bigvee_R \overline{YRX\overline{Q_{\mathcal{M}}\mathcal{H}}} = \bigvee_R \overline{YRXQ_{\mathcal{M}}\mathcal{H}} \\ &\subset \bigvee_R \overline{YR\overline{XQ_{\mathcal{M}}\mathcal{H}}} = \bigvee_R \overline{YRXQ_{\mathcal{M}}\mathcal{H}} = \bigvee_R \overline{Q_{\mathcal{M}}YRX\mathcal{H}} \subset \overline{Q_{\mathcal{M}}\mathcal{H}} = \mathcal{M}.\end{aligned}$$

Thus $\overline{Yq(\mathcal{M})} \subset \mathcal{M}$ and so $\overline{YX\mathcal{M}} = \overline{Y\overline{X\mathcal{M}}} \subset \overline{Yq(\mathcal{M})} \subset \mathcal{M}$. Since we also have

$$\overline{YX\mathcal{M}} = \overline{YX\overline{Q_{\mathcal{M}}\mathcal{H}}} = \overline{YXQ_{\mathcal{M}}\mathcal{H}} = \overline{Q_{\mathcal{M}}YX\mathcal{H}} = \overline{Q_{\mathcal{M}}\overline{YX\mathcal{H}}} = \overline{Q_{\mathcal{M}}\mathcal{H}} = \mathcal{M},$$

it follows that $\overline{Yq(\mathcal{M})} = \mathcal{M}$.

The identity $\overline{Yq(\mathcal{M})} = \mathcal{M}$ shows that $Y \mid q(\mathcal{M})\colon q(\mathcal{M}) \to \mathcal{M}$ is a quasiaffinity, and we have $(T \mid \mathcal{M})(Y \mid q(\mathcal{M})) = (Y \mid q(\mathcal{M}))(S \mid q(\mathcal{M}))$; also, $X \mid \mathcal{M}$ is an injective mapping of $\mathcal{M}$ into $q(\mathcal{M})$ and $(S \mid q(\mathcal{M}))(X \mid \mathcal{M}) = (X \mid \mathcal{M})(T \mid \mathcal{M})$. Properties (7)–(9) now follow from the preceding identities and [**11**, Theorem 2.5].

REMARK. The preceding result includes Hoover's as a special case, since when T is normal $\gamma(T) = \mathrm{Lat}_h(T)$. Indeed, suppose $\mathcal{M}$ is in

$\mathrm{Lat}_h(T)$ and let P denote the projection onto $\mathcal{M}$. If S commutes with T, then Fuglede's Theorem implies that S^* commutes with T, and thus S commutes with P. Since $P\mathcal{H} = \mathcal{M}$ and P is in $(T)''$, $\mathcal{M}$ is in $\gamma(T)$.

Let T be an arbitrary operator in $\mathcal{L}(\mathcal{H})$. If f is a function that is analytic in a neighborhood of $\sigma(T)$, then $f(T)$, as defined by the Riesz functional calculus, is in $(T)''$ (see [**19**, pages 26–32]), and thus $\overline{f(T)\mathcal{H}}$ is in $\gamma(T)$. It follows that $\gamma(T)$ contains the Riesz lattice of T defined by $\{E_\sigma\mathcal{H}: \sigma$ is a closed-and-open subset of $\sigma(T)\}$, where $E_\sigma = \chi_\sigma(T)$ and χ_σ is the characteristic function of σ. In view of these remarks, Proposition 4.1 implies the following result.

COROLLARY 4.2. *If S and T are in $\mathcal{L}(\mathcal{H})$ and S is quasisimilar to T, then* $\mathrm{Lat}_h(S)$ *contains a sublattice that is lattice isomorphic to the Riesz lattice of T. If $q: \gamma(T) \to \mathrm{Lat}_h(S)$ is a function that is given by Proposition* 4.1, *and σ is a closed-and-open subset of $\sigma(T)$, then $S \mid q(E_\sigma\mathcal{H})$ is a quasiaffine transform of $T \mid E_\sigma\mathcal{H}$.*

COROLLARY 4.3. *If S is quasisimilar to T and $\mathcal{M}$ is a finite dimensional subspace in $\gamma(T)$, then $S \mid q(\mathcal{M})$ is similar to $T \mid \mathcal{M}$.*

Proof. We retain the notation of the proof of Proposition 4.1. Since $\overline{Yq(\mathcal{M})} = \mathcal{M}$ and Y is injective, $q(\mathcal{M})$ is finite dimensional. Now $Yq(\mathcal{M})$ is closed, so $Yq(\mathcal{M}) = \mathcal{M}$, and in particular $\dim(q(\mathcal{M})) = \dim(\mathcal{M})$. Since $X(\mathcal{M})$ is a subspace of $q(\mathcal{M})$ and $\dim(X(\mathcal{M})) = \dim(\mathcal{M}) = \dim(q(\mathcal{M}))$, we have $X(\mathcal{M}) = q(\mathcal{M})$, and thus $X \mid \mathcal{M}: \mathcal{M} \to q(\mathcal{M})$ induces a similarity between $S \mid q(\mathcal{M})$ and $T \mid \mathcal{M}$.

Let E denote the spectral measure of a spectral operator T in $\mathcal{L}(\mathcal{H})$. The lattice of spectral subspaces of T is determined by $\mathrm{Lat}_{sp}(T) = \{E(\sigma)\mathcal{H}: \sigma$ a Borel set$\}$. Since $E(\sigma)$ is an idempotent in $(T)''$ (see [**18**]), then $\mathrm{Lat}_{sp}(T) \subset \gamma(T)$, and Proposition 4.1 yields the following result.

COROLLARY 4.4. *If T is a spectral operator and S is quasisimilar to T, then* $\mathrm{Lat}_h(S)$ *contains a sublattice that is lattice isomorphic to the lattice of spectral subspaces of T. If $q: \gamma(T) \to \mathrm{Lat}_h(S)$ is a function given by Proposition* 4.1, *and σ is a Borel set, then $S \mid q(E(\sigma)\mathcal{H})$ is a quasiaffine transform of $T \mid E(\sigma)\mathcal{H}$.*

REMARK. For an analogue of Corollary 4.4 for the case when T is only assumed to be decomposable, but the sets are closed rather than Borel, see [**6**, Theorem 4.5, page 56].

5. The quasisimilarity orbit of a class of compact operators. If K is a compact operator in $\mathcal{L}(\mathcal{H})$ and λ is a nonzero member of $\sigma(K)$, let $\mathcal{R}(K, \lambda) = \{x \in \mathcal{H}: (T-\lambda)^n x = 0$ for some integer $n \geqq 1\}$. Let $\mathcal{C}$ denote the set of all compact operators K in $\mathcal{L}(\mathcal{H})$ which satisfy the following properties:

(1) $\vee_{i=1}^{\infty} \mathcal{R}_i = \mathcal{H}$ (where $\{\lambda_i\}_{i=1}^{\infty}$ is the sequence of distinct nonzero members of $\sigma(K)$ and $\mathcal{R}_i = \mathcal{R}(K, \lambda_i)$);

(2) $\bigcap_{i=1}^{\infty} (\vee_{k=i}^{\infty} \mathcal{R}_k) = \{0\}$.

Each injective compact normal operator is in $\mathcal{C}$; moreover, $\mathcal{C}$ is closed under similarity. We next characterize the quasisimilarity orbit of $\mathcal{C}$. Following **[1]**, we say that a sequence $\{\mathcal{M}_i\}$ of closed subspaces of $\mathcal{H}$ is a *basic sequence* if $\mathcal{M}_i$ and $\vee_{k \neq i} \mathcal{M}_k$ are complementary for each i.

THEOREM 5.1. *An operator T in $\mathcal{L}(\mathcal{H})$ is quasisimilar to an operator in $\mathcal{C}$ if and only if T satisfies the following properties*:

(i) *There exists a basic sequence $\{\mathcal{M}_i\}_{i=1}^{\infty}$ of finite dimensional hyperinvariant subspaces of T*;

(ii) $\sigma(T \mid \mathcal{M}_i) = \{\lambda_i\}$, $\lambda_i \neq 0$, *and* $\lambda_i \to 0$;

(iii) $\bigcap_{i=1}^{\infty} (\vee_{k \geqq i} \mathcal{M}_k) = \{0\}$.

We note that noncompact operators may satisfy (i)–(iii); if $\lambda_i \neq 0$, $\lambda_i \to 0$, and if A_i is the 2×2 matrix $\begin{pmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{pmatrix}$, then $A = \Sigma_{i=1}^{\infty} \oplus A_i$ is non-compact and satisfies (i)–(iii).

LEMMA 5.2. *If K is in $\mathcal{C}$, then $\{\mathcal{R}_i\}_{i=1}^{\infty}$ is a basic sequence in $\gamma(K)$ such that $\vee_{i \neq k} \mathcal{R}_i$ and $\vee_{i \geqq k} \mathcal{R}_i$ are in $\gamma(K)$ $(k \geqq 1)$.*

Proof. Let f_i denote the characteristic function of $\{\lambda_i\}$. Then f_i is analytic in a neighborhood of $\sigma(K)$ and $E_i = f_i(K)$ is a finite rank idempotent whose range is $\mathcal{R}_i$ (see [**9**, p. 579], [**20**, p. 424]); in particular, $\mathcal{R}_i$ is in $\gamma(K)$. For $i \neq k$, $(1-f_k)f_i = f_i$, so $\mathcal{R}_i = E_i\mathcal{H} = (1-E_k)E_i\mathcal{H} \subset (1-E_k)\mathcal{H}$ and thus $\vee_{i \neq k} \mathcal{R}_i \subset (1-E_k)\mathcal{H}$. Since $\mathcal{R}_k$ is finite dimensional,

$$\mathcal{H} = \bigvee_{i=1}^{\infty} \mathcal{R}_i = \bigvee_{i \neq k} \mathcal{R}_i + \mathcal{R}_k \subset (1-E_k)\mathcal{H} + \mathcal{R}_k.$$

Since $\mathcal{R}_k$ and $(1-E_k)\mathcal{H}$ are complementary, it follows that $\vee_{i \neq k} \mathcal{R}_i = (1-E_k)\mathcal{H}$ and is thus in $\gamma(K)$; this also shows that $\{\mathcal{R}_i\}_{i=1}^{\infty}$ is a basic sequence. To show that $\vee_{i \geqq k+1} \mathcal{R}_i$ is in $\gamma(K)$, it suffices to verify that $\vee_{i \geqq k+1} \mathcal{R}_i = (1 - E_1 - \cdots - E_k)\mathcal{H}$. For $k = 1$ this identity follows from above, so we assume that $(1 - E_1 - \cdots - E_{k-1})\mathcal{H} = \vee_{i \geqq k} \mathcal{R}_i$. Since $\mathcal{R}_k$ is finite dimensional, $\vee_{i \geqq k} \mathcal{R}_i = \mathcal{R}_k + \vee_{i \geqq k+1} \mathcal{R}_i$, and since, from above, $\vee_{i \geqq k+1} \mathcal{R}_i \subset (1-E_k)\mathcal{H}$, it follows that

$$(1-E_k)\left(\mathcal{R}_k + \bigvee_{i \geqq k+1} \mathcal{R}_i\right) = \bigvee_{i \geqq k+1} \mathcal{R}_i.$$

Now $E_k E_j = \delta_{jk} E_k$, and thus

$$\begin{aligned}(1-E_1-\cdots-E_k)\mathcal{H} &= (1-E_k)(1-E_1-\cdots-E_{k-1})\mathcal{H} \\ &= (1-E_k)\left(\bigvee_{i \geqq k} \mathcal{R}_i\right) = \bigvee_{i \geqq k+1} \mathcal{R}_i,\end{aligned}$$

so the proof is complete.

LEMMA 5.3. *Let $\{\mathcal{M}_n\}_{n=1}^\infty$ be a basic sequence of subspaces of $\mathcal{H}$, and for each n, let P_n denote the* (bounded) *idempotent such that $P_n\mathcal{H} = \mathcal{M}_n$ and* $\ker(P_n) = \bigvee_{i \neq n} \mathcal{M}_i$. *Then $\bigcap_{i=1}^\infty (\bigvee_{k=i}^\infty \mathcal{M}_k) = \{0\}$ if and only if there is no non-zero vector x for which $P_n x = 0$ for each n.*

Proof. Since $\bigvee_{n=1}^\infty \mathcal{M}_n = \mathcal{H}$, for each x in $\mathcal{H}$ we have $x = \lim_{i\to\infty}(\sum_{j=1}^\infty m_{ij})$, where $m_{ij} \in \mathcal{M}_j$, and where for each i, $m_{ij} = 0$ for all but finitely many j. If $P_1 x = 0$, then

$$0 = P_1 x = \lim_{i\to\infty} \sum_{j=1}^\infty P_1 m_{ij} = \lim_{i\to\infty} m_{i1},$$

and so $x = \lim_{i\to\infty} \sum_{j=2}^\infty m_{ij}$. Thus x is in $\bigvee_{k \geqq 2} \mathcal{M}_k$, and by repeating this argument inductively, it follows that if each $P_n x = 0$, then x is in $\bigcap_{i=1}^\infty (\bigvee_{k=i}^\infty \mathcal{M}_k)$. Conversely, if x is in $\bigvee_{k=i}^\infty \mathcal{M}_k$ $(\subset \bigvee_{k \neq i-1} \mathcal{M}_k)$, then $P_{i-1}x = 0$; thus if x is in $\bigcap_{i=1}^\infty (\bigvee_{k=i}^\infty \mathcal{M}_k)$, then $P_i x = 0$ for each i.

Proof of Theorem 5.1. We retain the notation of the proceeding results. Suppose that T is quasisimilar to an operator K in $\mathcal{C}$. Let $q: \gamma(K) \to \mathrm{Lat}_h(T)$ be a function given by Proposition 4.1, and for $i \geqq 1$, let $\mathcal{M}_i = q(\mathcal{R}_i)$. Since $\mathcal{R}_i$ is finite dimensional, Corollary 4.3 implies that $T \mid \mathcal{M}_i$ is similar to $K \mid \mathcal{R}_i$, and thus $\mathcal{M}_i$ is finite dimensional, $\sigma(T \mid \mathcal{M}_i) = \sigma(K \mid \mathcal{R}_i) = \{\lambda_i\}$, $\lambda_i \neq 0$, and $\lambda_i \to 0$. Lemma 5.2 implies that $\bigvee_{k=i}^\infty \mathcal{R}_k$ is in $\gamma(K)$; since $\bigcap_{i=1}^\infty (\bigvee_{k=i}^\infty \mathcal{R}_k) = \{0\}$, Proposition 4.1 shows that

$$\bigcap_{i=1}^\infty \left(\bigvee_{k=i}^\infty \mathcal{M}_k\right) = \bigcap_{i=1}^\infty q\left(\bigvee_{k=i}^\infty \mathcal{R}_k\right) = \{0\}.$$

Similarly, since $\mathcal{R}_k$ and $\bigvee_{i \neq k} \mathcal{R}_i$ are complementary elements of $\gamma(K)$, we have

$$\left(\bigvee_{i \neq k} \mathcal{M}_i\right) \cap \mathcal{M}_k = q\left(\bigvee_{i \neq k} \mathcal{R}_i\right) \cap q(\mathcal{R}_k) = \{0\};$$

by the finite dimensionality of $\mathcal{M}_k$ we also have

$$\mathcal{M}_k + \bigvee_{i \neq k} \mathcal{M}_i = \mathcal{M}_k \vee \bigvee_{i \neq k} \mathcal{M}_i = \bigvee_{i=1}^{\infty} q(\mathcal{R}_i)$$
$$= q\left(\bigvee_{i=1}^{\infty} \mathcal{R}_i\right) = q(\mathcal{H}) = \mathcal{H}.$$

Thus $\{\mathcal{M}_i\}_{i=1}^{\infty}$ satisfies (i)–(iii).

For the converse, since $\mathcal{M}_i$ is finite dimensional, $T \mid \mathcal{M}_i$ is of the form $\lambda_i + N_i$, where N_i is nilpotent. Thus there is a Hilbert space $\mathcal{H}_i$ and an invertible operator $S_i: \mathcal{M}_i \to \mathcal{H}_i$ such that $\|S_i(N_i \mid \mathcal{M}_i)S_i^{-1}\| < 1/i$. It follows that $K = \sum_{i=1}^{\infty} \oplus S_i(T \mid \mathcal{M}_i)S_i^{-1}$ is in $\mathscr{C}$ (with respect to $\mathscr{L}(\mathcal{K})$, where $\mathcal{K} = \sum_{i=1}^{\infty} \oplus \mathcal{H}_i$). We now procede as in **[1]** by choosing numbers $a_i, b_i > 0$ such that $\sum_{i=1}^{\infty} a_i \|S_i^{-1}\| < \infty$, $\sum_{i=1}^{\infty} b_i \|S_i\| \|P_i\| < \infty$, and by defining operators $A: \mathcal{K} \to \mathcal{H}$, $B: \mathcal{H} \to \mathcal{K}$ by $A(\bigoplus_{i=1}^{\infty} h_i) = \sum_{i=1}^{\infty} a_i S_i^{-1} h_i$ (h_i in $\mathcal{H}_i$) and $Bx = \bigoplus_{i=1}^{\infty} b_i S_i P_i x$ (x in $\mathcal{H}$). Property (iii) and Lemma 5.3 imply that B is injective, and it follows readily that A and B are quasiaffinities such that $TA = AK$ and $KB = BT$. Thus T and K are quasisimilar, and the proof is complete.

In **[1]** C. Apostol proved the following characterization of the quasisimilarity orbit of the set of all normal operators in $\mathscr{L}(\mathcal{H})$.

THEOREM 5.4. (Apostol **[1]**). *An operator T in $\mathscr{L}(\mathcal{H})$ is quasisimilar to a normal operator if and only if there exists a basic sequence $\{\mathcal{M}_n\}_{n=1}^{m}$ of T-invariant subspaces such that $T \mid \mathcal{M}_n$ is similar to a normal operator and* $\bigcap_{i=1}^{m}(\bigvee_{n=i}^{m} \mathcal{M}_n) = \{0\}$ if $m = \infty$.

We next give a partial analogue of this result for spectral operators.

THEOREM 5.5. *An operator T in $\mathscr{L}(\mathcal{H})$ is quasisimilar to a spectral operator if there exists a finite or infinite basic sequence $\{\mathcal{M}_n\}_{n=1}^{m}$ of T-invariant subspaces such that* (i) *$T \mid \mathcal{M}_n$ is spectral and* (ii) *$\bigcap_{i=1}^{m}(\bigvee_{n=i}^{m} \mathcal{M}_n) = \{0\}$ if $m = \infty$.*

Proof. By a straightforward modification of Theorem 5.1, it follows that if T satisfies (i) and (ii), then T is quasisimilar to a direct sum of spectral operators. An application of Theorem 2.1 completes the proof.

Question 5.6. Is the converse of Theorem 5.5 true?

The proofs of the results in this section indicate the usefulness of basic sequences of invariant subspaces in constructing a quasisimilarity between a given operator and some reducible operator. In **[18]** Hoover showed that quasisimilarities can be produced by direct sums of similarities and he also showed that not every quasisimilarity arises in this way. Using a theorem of Sz.-Nagy and Foiaş **[21]**, he showed that an irreducible operator may be quasisimilar to a unitary operator. (Gilfeather **[16]** proved what may be regarded as a strong converse to Hoover's example: every normal operator N without point spectrum, on a separable Hilbert space, is the uniform limit of irreducible operators each similar to N.) Thus, in the constructions of all of the quasisimilarities of this section, and in those of **[1]** and **[18]**, at least one of the quasisimilar operators is reducible. It is possible, however, for two irreducible operators to be quasisimilar (but not similar). In Example 4.4 of **[11]** it is shown that there are injective bilateral weighted shifts that are quasisimilar but not similar, and since the weight sequences of these shifts are both nonperiodic, it follows that both shifts are irreducible (see [**17**, page 83]).

Added in proof. (i). E. Azoff has given an alternate proof of Theorem 2.1 based on the following observation of his: if Q is a quasinilpotent operator and $\epsilon > 0$, there exists an invertible operator J in the C^*-algebra generated by Q such that $\|JQJ^{-1}\| < \epsilon$ (for the proof, with $\epsilon = 1$, let $J = (\Sigma_{n=0}^{\infty} Q^{*n}Q^n)^{1/2}$).

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