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FACTORIZATION OF RADONIFYING TRANSFORMATIONS

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It is shown that a linear transformation which carries a cylinder measure on a separable Hilbert space to a Radon measure on a separable Banach space can be factored into a positive-definite Hilbert-Schmidt transformation followed by a measurable linear transformation. Applications to measurable norms are given.

1. Introduction. It is well known that Hilbert–Schmidt transformations carry certain well behaved cylinder measures into Radon measures on Hilbert spaces (see [**8**, **9**]). The problem of characterizing transformations which carry cylinder measures on Hilbert spaces to Radon measures on Banach spaces seems to be more difficult (see [**8**]). In this paper it is shown that such a transformation can be factored into a positive-definite Hilbert–Schmidt transformation followed by a measurable linear transformation. In the last section this type of factorization is applied to abstract Wiener spaces.

2. Radon measures on embeddable spaces. Throughout this paper all topological vector spaces (TVS) will be assumed to be real and locally convex. A Radon measure is a regular Borel measure, and we shall assume that all Radon measures in this paper are positive with total measure 1, i.e. they are probability measures. A topological space X is a *Lusin space* if there is a complete separable metric space Y and a continuous bijective mapping $Y \to X$. Any Borel measure on a Lusin space is a Radon measure ([**9**], p. 122). The Borel subsets for comparable Lusin topologies are identical ([**9**], p. 101).

DEFINITION 2.1. A TVS E is *embeddable* if E is a Lusin space and if there is a continuous linear injection $T: E \to H$ where H is a separable Hilbert space.

Any such mapping T will be called an embedding of E. We can assume that $T(E)$ is dense in H. Kuelbs [**4**] has shown that any separable Banach space is embeddable. In fact we have

LEMMA 2.2. *A TVS is embeddable if and only if it is a Lusin space and there exists a countable bounded set $\{y_i\} \subset E'$, the dual of E, which separates points on E.*

Proof. If $T:E \to H$ is an embedding, then $T^*:H \to E'$, the adjoint of T, is bounded and for any orthonormal basis $\{x_i\}$ of H, $\{T^*x_i\} \subset E'$ separates points on E.

Conversely if $\{y_i\} \subset E'$ is a countable bounded subset which separates points and H is any separable Hilbert space with orthonormal basis $\{x_i\}$, the transformation $T:E \to H$ defined by

$$Tx = \Sigma t_i \langle x, y_i \rangle x_i, \qquad x \in E,$$

where $t_i > 0, \Sigma t_i^2 < \infty$, is an embedding. ($\langle , \rangle$ is the canonical bilinear form on $E \times E'$.)

LEMMA 2.3. *Let E be embeddable with embedding $T:E \to H$. Then for any Borel subset $B \subset E$, $T(B)$ is Borel in H.*

Proof. This is true of injective mappings on Lusin spaces (**[9]** p. 107, Lemma **14**).

DEFINITION 2.4. Let H be a separable Hilbert space with Radon measure μ. Let A be a linear transformation defined on a dense linear subspace $\mathcal{D}_A \subset H$ with image in a *TVS E*. A is a *μ-measurable* linear transformation if $\mu(\mathcal{D}_A) = 1$ and for any Borel set $B \subset E$, $A^{-1}(B)$ is Borel.

We shall use the notation $A:H \to E$ even when $\mathcal{D}_A \neq H$. Any bounded linear transformation is μ-measurable. A μ-measurable linear transformation $A:H \to E$ induces a Borel measure $A(\mu)$ on E defined by $A(\mu)(B) = \mu(A^{-1}(B))$ for any Borel set $B \subset E$.

LEMMA 2.5. *Let E be an embeddable space with embedding $T:E \to H$, and let μ be a Radon measure on E. Then $A = T^{-1}:H \to E$ is $T(\mu)$-measurable.*

Proof. $T(\mu)$ is a Borel measure and therefore a Radon measure on H. $T(\mu)(\mathcal{D}_A) = T(\mu)(T(E)) = 1$. For any Borel set $B \subset E$, $A^{-1}(B) = T(B)$ is Borel in H by Lemma 2.3, so A is $T(\mu)$-measurable.

A cylinder measure γ on a Hilbert space H is *scalarly concentrated on the balls of H* if for all $\epsilon > 0$ there is $N > 0$ such that for any $y \in H$, $\|y\| = 1$,

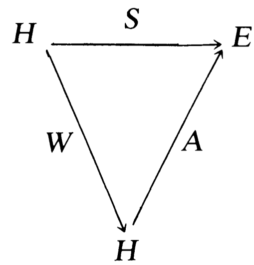
$$\gamma\{x : |\langle x, y \rangle| \leqq N\} > 1 - \epsilon$$

where $\langle , \rangle$ denotes the inner product on H. We shall assume that all cylinder measures discussed in this paper have this property.

LEMMA 2.6. *Let H_1, H_2 be Hilbert spaces, $S: H_1 \to H_2$ be a Hilbert–Schmidt transformation, and γ be a cylinder measure on H_1. Then $S(\gamma)$ can be extended to a Radon measure $\overline{S(\gamma)}$ on H_2.*

Proof. See **[9]** p. 301, or **[8]**.

THEOREM 2.7. *Let E be an embeddable TVS, H be a separable Hilbert space, $S: H \to E$ be a bounded linear transformation, and γ be a cylinder measure on H. Then $S(\gamma)$ can be extended to a Radon measure on E if and only if S can be factored*



where W is a positive-definite Hilbert–Schmidt transformation and A is a closed $\overline{W(\gamma)}$-measurable linear transformation.

Proof. If S can be so factored, then $\overline{A(\overline{W(\gamma)})}$ will be a Radon measure on E. Since $\overline{A(W(\gamma))}$ and $S(\gamma)$ agree on cylinder sets, $S(\gamma)$ can be extended to $\overline{A(W(\gamma))}$.

Conversely, let $T: E \to H_1$ be an embedding. Since H and H_1 are both separable, they are unitarily equivalent, so we can assume $T: E \to H$. If necessary T can be followed by a bijective Hilbert–Schmidt transformation and still remain an embedding, so we can assume that $W = TS$ is Hilbert–Schmidt, and in fact that W is positive-definite.

$\overline{W(\gamma)} = \overline{T(S(\gamma))}$ and $A = T^{-1}: H \to E$ is $\overline{T(S(\gamma))}$-measurable by Lemma 2.5. T is continuous so $A = T^{-1}$ is closed.

REMARK. Theorem 2.7 cannot be improved to state that $S = AW$ where W is Hilbert–Schmidt and A is bounded (see **[6]** p. 107 and **[8]** Proposition (5.20.4)).

The following corollary resembles Theorem 1 of Versik **[10]**, but without the hypothesis that (E, μ) be a "Lebesgue space".

COROLLARY 2.8. *Let E be an embeddable space with Radon mea-*

sure μ. Then there exists a separable Hilbert space H with Radon measure ν and a bijective ν-measurable linear transformation $A: H \to E$ such that $\mu = A(\nu)$.

3. Application to sequentially measurable norms.

DEFINITION 3.1. Let H be a separable Hilbert space and γ be a cylinder measure on H. A continuous norm $\| \ \|$ on H is *sequentially γ-measurable* if for any increasing sequence of finite dimensional orthogonal projections $\{P_n\}$ on H such that $P_n \to I_H$ and $\epsilon > 0$, there is $N > 0$ such that for $m, n > N$, $\gamma\{x : \| P_n x - P_m x \| > \epsilon\} < \epsilon$.

I_H denotes the identity operator on H. To avoid confusion we shall denote the original (Hilbert) norm on H by $\| \ \|_H$.

If γ is the standard Gaussian cylinder measure on H, then a measurable norm (defined in **[2, 3]**) is also sequentially γ-measurable (**[2]** Corollary 5.2).

THEOREM 3.2. (Gross). *Let γ be a cylinder measure on a separable Hilbert space H and let $\| \ \|$ be a sequentially γ-measurable norm on H. Let B denote the Banach space generated by H with norm $\| \ \|$ and let $S: H \to B$ be the natural injection. Then $S(\gamma)$ can be extended to a Radon measure on B.*

Proof. Since H is separable, B is separable and therefore embeddable by Lemma 2.2. Let $T: B \to H$ be an embedding such that TS is a positive-definite Hilbert–Schmidt transformation. $TS(\gamma)$ can be extended to a Radon measure ν on H. We must show that the closed linear transformation $A = T^{-1}: H \to B$ is ν-measurable which, in view of Lemma 2.3, amounts to showing that $\nu(\mathcal{D}_A) = 1$.

Let $\{P_n\}$ be an increasing sequence of finite dimensional orthogonal projections on H such that $P_n \to I_H$ and which commute with TS. Then for $\epsilon > 0$

$$\begin{aligned}
&\gamma\{x : \| P_n x - P_m x \| > \epsilon\} \\
&= \gamma\{x : \| SP_n x - SP_m x \|_B > \epsilon\} \\
&= \gamma\{x : \| ATSP_n x - ATSP_m x \|_B > \epsilon\} \\
&= \gamma\{x : \| AP_n TSx - AP_m TSx \|_B > \epsilon\} \\
&= \nu\{x : \| AP_n x - AP_m x \|_B > \epsilon\}.
\end{aligned}$$

Therefore, if m and n are large enough,

$$\nu\{x : \| AP_nx - AP_mx \|_B > \epsilon\} < \epsilon$$

so AP_nx converges in ν-measure. Then a subsequence, also denoted by AP_nx, converges ν-almost everywhere. Since A is closed and $P_nx \to x$ we must have $AP_nx \to Ax$ ν-almost everywhere. Therefore $\nu(\mathcal{D}_A) = 1$.

THEOREM 3.3. *Let γ be a cylinder measure on a separable Hilbert space H and let $\| \ \|$ be a sequentially γ-measurable norm on H. Let B denote the Banach space generated by H with norm $\| \ \|$, let $S : H \to B$ be the natural injection, and let $\mu = \overline{S(\gamma)}$. Then there exists an orthonormal basis $\{e_k\}$ of H with $e_k = S^*f_k$, $f_k \in B'$, such that*

$$\sum_{k=1}^{n} \langle x, f_k \rangle \, Se_k \to x$$

in μ-measure on B.

Proof. As in the proof of Theorem 3.2 we can define an embedding $T : B \to H$ such that $W = TS$ is a positive-definite Hilbert–Schmidt transformation and $A = T^{-1} : H \to B$ is a closed $\overline{W(\gamma)}$-measurable linear transformation. Let $\{e_k\}$, $k \in \mathbf{N}$, be an orthonormal basis of H consisting of eigenvectors of W such that $We_k = \lambda_k e_k$, $\lambda_k > 0$. Let P_n be the orthogonal projection onto the subspace generated by $\{e_1, \cdots, e_n\}$.

Let $\nu = \overline{W(\gamma)}$ and let $\epsilon > 0$. Then for m and n large enough

$$\begin{aligned} \epsilon &> \gamma\{x : \| P_nx - P_mx \| > \epsilon\} \\ &= \nu\{x : \| AP_nx - AP_mx \|_B > \epsilon\}. \end{aligned}$$

Therefore AP_nx converges in ν-measure. Since A is closed and $P_nx \to x$, it follows that $AP_nx \to Ax$ in ν-measure. But $\nu\{x \in H : \| AP_nx - Ax \|_B > \epsilon\} = \mu\{x \in B : \| AP_nTx - x \|_B > \epsilon\}$ so $AP_nTx \to x$ in μ-measure on B.

Now,

$$\begin{aligned} AP_nTx &= A \sum_{k=1}^{n} \langle Tx, e_k \rangle e_k \\ &= \sum_{k=1}^{n} \langle x, T^*e_k \rangle Ae_k \\ &= \sum_{k=1}^{n} \langle x, \lambda_k f_k \rangle Ae_k \\ &= \sum_{k=1}^{n} \langle x, f_k \rangle Se_k. \end{aligned}$$

If γ is the standard Gaussian cylinder measure on H we obtain the following well known (**[5, 7]**)

COROLLARY 3.4. *With the same hypotheses as Theorem* 3.3 *and* γ *the standard Gaussian cylinder measure on H*

$$\sum_{k=1}^{n} \langle x, f_k \rangle Se_k \to x \qquad \mu\text{-almost everywhere.}$$

Proof. The random variables $\langle \cdot, f_k \rangle$ on (B, μ) are independent (see **[5]**) so by the well known theorem of P. Levy (see **[1]** Theorem 5.3.4, p. 120 and its proof), convergence in measure implies almost everywhere convergence.

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