PACIFIC JOURNAL OF MATHEMATICS
Vol. 71, No. 2, 1977

ON $w\Delta$-SPACES, $w\sigma$-SPACES AND $\Sigma^{\#}$-SPACES

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One of the reasons that paracompactness plays a central rôle in general topology is that it is a property shared by compact spaces and metric spaces. Recently there has been considerable interest in topological properties shared by countably compact spaces and metric spaces. R. W. Heath has introduced a method of describing a generalized metric property of a topological space (X, τ) by means of a function $g: N\times X\to\tau$ and R. E. Hodel has modified this method to obtain important new classes of spaces. Subsequently, J. Nagata obtained a similar characterization of $\Sigma^{\#}$-spaces, and it now appears that the method of Heath and Hodel provides an opportunity to clarify the relationships among those properties that are shared by countably compact spaces and metric spaces. This note seeks to establish some relationships among these properties.

1. **Introduction.** Section 3 concerns $w\Delta$-spaces. We show that every σ-orthocompact $w\Delta$-space is a $\Sigma^{\#}$-space and that every σ-refinable quasi-complete space is a $w\gamma$-space. It follows that every regular σ-refinable space with a G_δ-diagonal that is a $w\gamma$-space (or a quasi-complete space) is a γ-space and that every σ-orthocompact quasi-complete β-space is a $\Sigma^{\#}$-space. In § 2 and 4, respectively, we introduce $w\sigma$-spaces and Θ-spaces. We provide support for the conjecture that $w\sigma$-spaces are exactly the $\Sigma^{\#}$-spaces and characterize the Θ-spaces as the c-semistratifiable θ-spaces.

Throughout this paper we use the following notational conventions. N denotes the set of all natural numbers and if $\mathscr{C}$ is a cover of a space (X, τ) and $x\in X$, then $A_x^{\mathscr{C}} = \cap\{C\in\mathscr{C}\mid x\in C\}$. As is customary, $\langle x_n\rangle$ denotes the sequence whose nth term is x_n.

2. $w\sigma$-**spaces.** An ingenious approach to the study of generalized metric spaces, introduced by R. W. Heath in [7] and pursued by R. E. Hodel [9], [10], is to describe a generalized metric property of a topological space (X, τ) by means of a function $g: N\times X\to\tau$. An extension of this approach, which the authors first used as a mnemonic device, now appears to be useful in further unifying and organizing the study of generalized metric spaces. In particular the extension suggests a natural conjecture that bears upon a problem to be discussed subsequently.

Let (X, τ) be a topological space, let $g: N\times X\to\tau$ be a function such that for each $x\in X$ and $n\in N$, $x\in g(n+1, x)\subset g(n, x)$ and consider the following further conditions on g:

(a) If for each $n \in N$, $\{p, x_n\} \subset g(n, y_n)$, then $\langle x_n \rangle$ has a cluster point.

(b) If for each $n \in N$, $p \in g(n, y_n)$ and $y_n \in g(n, x_n)$, then $\langle x_n \rangle$ has a cluster point.

(c) If for each $n \in N$, $p \in g(n, x_n)$, then $\langle x_n \rangle$ has a cluster point. Let s be any of the conditions (a), (b) or (c) and s^{-1} be the statement obtained by formally interchanging all memberships (e.g., a^{-1} is the condition: If for each $n \in N$, $y_n \in g(n, p) \cap g(n, x_n)$, then $\langle x_n \rangle$ has a cluster point). If $g: N \times X \rightarrow \tau$ satisfies condition s (respectively s^{-1}) for $s = a$, b, or c, we say that g is a *wS-function* (respectively *wS^{-1}-function*) and that (X, τ) is a *wS-space* (respectively *wS^{-1}-space*). Corresponding to each of the above conditions s is the stronger condition, denoted S, in which "then $\langle x_n \rangle$ has a cluster point" is replaced by "then p is a cluster point of $\langle x_n \rangle$." If g satisfies S, we say that g is an *S-function* and that (X, τ) is an *S-space*. *S^{-1}-functions*, and *S^{-1}-spaces* are defined analogously. The following are known,

$A =$ developable space	$B = \sigma$-space	$C =$ semistratifiable space
$A^{-1} =$ Nagata space	$B^{-1} = \gamma$-space	$C^{-1} =$ first countable space
$wA = w\Delta$-space		$wC = \beta$-space
$wA^{-1} = wN$-space	$wB^{-1} = w\gamma$-space	$wC^{-1} = q$-space.

We dub the wB-spaces, for obvious reasons, $w\sigma$-spaces.

DEFINITION [16]. A space (X, τ) is a *$\Sigma^{\#}$-space* if there is a sequence $\langle \mathscr{F}_n \rangle$ of closure preserving closed covers of X such that if $x \in X$ and $x_n \in A_{x_n}^{\mathscr{F}}n$ for each $n \in N$, then $\langle x_n \rangle$ has a cluster point.

PROPOSITION 2.1. *Every $\Sigma^{\#}$-space is a $w\sigma$-space.*

Proof. An immediate consequence of a result of J. Nagata [18].

PROPOSITION 2.2. *Every wN-space is a $w\sigma$-space.*

PROOF. Let g be a wN-function. Suppose that for each $n \in N$, $p \in g(n, y_n)$ and $y_n \in g(n, x_n)$. There is a $q \in X$ such that q is a cluster point of $\langle y_n \rangle$. Thus for each $n \in N$, there is a $j_n > n$ such that $y_{j_n} \in g(n, q)$. Now $y_{j_n} \in g(j_n, x_{j_n}) \subset g(n, x_{j_n})$ so that for each $n \in N$, $g(n, q) \cap g(n, x_{j_n}) \neq \varnothing$. Since g is a wN-function $\langle x_{j_n} \rangle$ has a cluster point. It follows that $\langle x_n \rangle$ has a cluster point.

DEFINITION [20]. A space (X, τ) is a *$\sigma^{\#}$-space* if there is a sequence $\langle \mathscr{F}_n \rangle$ of closure preserving closed collections such that if $x \neq y$, then there is an $F \in \bigcup_{n=1}^{\infty} \mathscr{F}_n$ such that $x \in F$ and $y \notin F$.

DEFINITION [15]. *A* space (X, τ) is *c-semistratifiable* if for each $x \in X$ there is a sequence $\langle g(n, x) \rangle$ of open neighborhoods of x such that for each compact set $K \subset X$, if $g(n, K) = \cup \{g(n, x) \mid x \in K\}$, then $\cap \{g(n, K) \mid n \geqq 1\} = K$. The function $g: N \times X \rightarrow \tau$ is called a *c-semi-stratification* of X.

Every $\sigma^{\#}$-space is c-semistratifiable, but the existence of a c-semistratifiable space that is not a $\sigma^{\#}$-space has not been established.

A comparison of the characterization of $\Sigma^{\#}$-spaces given by J. Nagata [18] to the characterization of σ-spaces given by R. W. Heath and R. E. Hodel [8, Theorem 1.4] suggests the conjecture that every $w\sigma$-space is a $\Sigma^{\#}$-space. The following proposition is further evidence in support of this conjecture, because it is known that every regular $\sigma^{\#}$-space that is a $\Sigma^{\#}$-space is a σ-space.

PROPOSITION 2.3. *Let (X, τ) be a regular $\sigma^{\#}$-space that is a $w\sigma$-space. Then (X, τ) is a σ-space.*

Proof. Since (X, τ) is a $\sigma^{\#}$-space, there is a function r: $N \times X \rightarrow \tau$ such that if $y \in r(n, x)$, then $r(n, y) \subset r(n, x)$ and such that $\bigcap_{n=1}^{\infty} r(n, x) = \{x\}$. Since (X, τ) is a $w\sigma$-space, there is a function $s: N \times X \rightarrow \tau$ such that if $p \in s(n, y_n)$ and $y_n \in s(n, x_n)$, then $\langle x_n \rangle$ has a cluster point. For each $n \in N$, let $g(n, x) = r(n, x) \cap s(n, x)$. Suppose that $p \in g(n, y_n)$ and $y_n \in g(n, x_n)$. Then there is a $q \in X$ such that q is a cluster point of $\langle x_n \rangle$. It suffices to prove that $p = q$. If $p \neq q$, then there exists $k \in N$ such that $p \notin r(k, q)$. Choose $n \geqq k$ such that $x_n \in r(k, q)$. Then $p \in r(n, y_n) \subset r(n, x_n) \subset r(k, x_n) \subset r(k, q)$ so that $p \in r(k, q)$. This is a contradiction. It follows that (X, τ) is a σ-space [8].

3. $w\Delta$-spaces. In this section we investigate two covering properties and their connection with $w\Delta$-spaces.

DEFINITION [5]. A topological space (X, τ) is *σ-orthocompact* provided that every open cover of X has an open refinement $\mathscr{R} = \bigcup_{i=1}^{\infty} \mathscr{R}_i$ such that for each $x \in X$ and each $i \in N$, $A_x^{\mathscr{R}_i} \in \tau$.

PROPOSITION 3.1. *Let (X, τ) be a σ-orthocompact $w\Delta$-space. Then (X, τ) is a $\Sigma^{\#}$-space.*

Proof. Let h be a $w\Delta$-function and for each $n \in N$, let $\mathscr{H}_n =$

$\{h(n, x): x \in X\}$. By [10, Remark 3.3] (X, τ) is countably metacompact so that by [5, Proposition 3.1] for each $n \in N$, there is an open refinement $\mathscr{R}_n$ of $\mathscr{H}_n$ such that for each $x \in X$, $A_x^{\mathscr{R}_n} \in \tau$. Define $g: N \times X \to \tau$ by $g(n, x) = A_x^{\mathscr{R}_n}$. We note that if $y \in g(n, x)$, then $g(n, y) \subset g(n, x)$. Moreover, if $p \in g(n, x_n)$ for each $n \in N$, then there is a $y_n \in X$ such that $\{p, x_n\} \subset g(n, x_n) \subset h(n, y_n)$ and since h is a $w\Delta$-function $\langle x_n \rangle$ has a cluster point. It follows that (X, τ) is a $\Sigma^{\#}$-space [18].

DEFINITION [14]. A topological space (X, τ) is a *σ-refinable* provided that for each open cover $\mathscr{C}$ of X there is a sequence $\langle V_n \rangle$ of reflexive relations on X such that for each $n \in N$ and $x \in X$, $V_n(x) \in \tau$ and such that for each $x \in X$ there exists an $n \in N$ and a $C \in \mathscr{C}$ such that $V_n^2(x) \subset C$. The sequence $\langle V_n \rangle$ is called *σ-refinement*.

Every γ-space is σ-refinable and every σ-refinable Moore space is a γ-space so that the rôle of γ-spaces in the class of σ-refinable spaces is analogous to the rôle of metric spaces in the class of paracompact spaces. In this respect σ-refinability is an appropriate generalization of paracompactness. Indeed, in regular T_1 spaces σ-refinability may be viewed as the nonsymmetric analogue of paracompactness in that if in its definition each V_n is taken to be a symmetric relation, then a characterization of paracompactness is obtained.

PROPOSITION 3.2. *A regular T_1 space is paracompact if, and only if, the following condition holds: For each open cover $\mathscr{C}$ of X there is a sequence $\langle V_n \rangle$ of reflexive symmetric relations on X such that for each $n \in N$ and $x \in X$, $V_n(x) \in \tau$ and such that for each $x \in X$ there exists an $n \in N$ and a $C \in \mathscr{C}$ such that $V_n^2(x) \subset C$.*

Proof. Since, as is well known, every paracompact regular T_1 space admits a uniformity with the Lebesgue property, it is clear that a paracompact regular T_1 space satisfies the condition. Now let (X, τ) be a regular T_1 space that satisfies the condition and let $\mathscr{C}$ be an open cover of X. Let $\langle V_n \rangle$ be a sequence of reflexive, symmetric relations on X such that for each $n \in N$ and $x \in X$, $V_{n+1}(x) \subset V_n(x) \in \tau$ and such that for each $x \in X$ there exists an $n_x \in N$ and a $C \in \mathscr{C}$ such that $V_{n_x}^2(x) \subset C$. Let $\mathscr{R} = \{V_{n_x}(x) \mid x \in X\}$. Let $\langle U_n \rangle$ be a sequence of reflexive, symmetric relations on X such that for each $n \in N$ and $x \in X$, $U_n \subset V_n$, $U_{n+1}(x) \subset U_n(x) \in \tau$ and such that for each $x \in X$ there exists an $m_x \in N$, a $y \in X$ and an $n_y \in N$ such that $U_{m_x}^2(x) \subset V_{n_y}(y)$. For each $m \in N$, let $C_m = \{U_m(x) \mid x \in X\}$. Let $x \in X$; then there exists an $m_x \in N$, a $y \in X$ and an $n_y \in N$ such that

$U^2_{m_x}(x) \subset V_{n_y}(y) \in \mathscr{D}$. Let $m = \max\{m_x, n_y\}$. Then there is a $C \in \mathscr{C}$ such that $\operatorname{st}(U_m(x), \mathscr{G}_m) = U^3_m(x) \subset U_m(U^2_{m_x}(x)) \subset U_m(V_{n_y}(y)) \subset V_m(V_{n_y}(y)) \subset V^2_{n_y}(y) \subset C$. Therefore by [1, Theorem 1], (X, τ) is paracompact.

DEFINITION [4 and 6]. A space (X, τ) is *quasi-complete* provided that there exists a mapping $g: N \times X \to \tau$ such that if $\{x, x_n\} \subset \bigcap_{i=1}^n g(i, y_i)$, then $\langle x_n \rangle$ has a cluster point. The function g is called a *quasi-complete* function.

It is well known that quasi-complete spaces form a generalization of both $w\Delta$-spaces and p-spaces.

PROPOSITION 3.3. *Every σ-refinable quasi-complete space is a $w\gamma$-space.*

Proof. Let (X, τ) be a σ-refinable quasi-complete space and let g be a quasi-complete function such that for each $n \in N$ and $x \in X$, $g(n+1, x) \subset g(n, x)$. Let $\mathscr{A}_n = \{g(n, x) \mid x \in X\}$ and let $\langle V_{m,n} \rangle$ be a σ-refinement of $\mathscr{A}_n$ such that if $i \geqq m$ and $j \geqq n$, then $V_{i,j} \subset V_{m,n}$. Define $f: N \to N \times N$, whose first five terms are $f(1) = (1, 1)$, $f(2) = (1, 2)$, $f(3) = (2, 1)$, $f(4) = (1, 3)$, $f(5) = (2, 2)$, by the recursive formula $f(n+1) = (s+1, t-1)$ if $t \neq 1$ and $f(n) = (s, t)$, and $f(n+1) = (1, s+1)$ if $t = 1$ and $f(n) = (s, t)$. Define $f_1 = \pi_1 \circ f$, $f_2 = \pi_2 \circ f$ and define h by $h(n, x) = V_{f_1(n), f_2(n)}(x)$. Suppose that $x_n \in h(n, y_n)$ and $y_n \in h(n, p)$ for each $n \in N$. Note that $x_n \in V^2_{f_1(n), f_2(n)}(p)$. There is $m_1 \in N$ and $z_1 \in X$ such that $V^2_{m_1, 1}(p) \subset g(1, z)$. Set $k_1 = 1$ and $k_2 = m_1 + 1$. There is $m_2 \geqq m_1$ and $z_2 \in X$ such that $V^2_{m_2, k_2}(p) \subset g(k_2, z_2)$. In general set $k_n = m_{n-1} + k_{n-1}$. Then there is an $m_n \geqq m_{n-1}$ and a $z_n \in X$ such that $V^2_{m_n, k_n}(p) \subset g(k_n, z_n)$. For each $n \in N$ set $j_n = f^{-1}(m_n, k_n)$. Then $\langle x_{j_n} \rangle$ is a subsequence of $\langle x_n \rangle$. Now $\{p, x_{j_n}\} \subset V^2_{m_n, k_n}(p) = \bigcap_{i=1}^n V^2_{m_i, k_i}(p) \subset \bigcap_{i=1}^n g(k_i, z_i) \subset \bigcap_{i=1}^n g(i, z_i)$. Since g is a quasi-complete function, $\langle x_{j_n} \rangle$ and therefore $\langle x_n \rangle$ has a cluster point.

COROLLARY. *Every σ-refinable $w\Delta$-space is a $w\gamma$-space.*

COROLLARY. *Every σ-orthocompact quasi-complete β-space is a $\Sigma^{\#}$-space.*

Proof. Let X be a σ-orthocompact quasi-complete β-space. It follows from Proposition 3.3 that X is a $w\gamma$-space. Since every β, $w\gamma$-space is a $w\Delta$-space [10], the result follows from Proposition 3.1.

DEFINITION [14]. A space (X, τ) satisfies *property A'* provided there is a sequence $\langle V_n \rangle$ of relations on X with the following properties.

(i) For each $x \in X$, $n \in N$, $x \in V_{n+1}(x) \subset V_n(x) \in \tau$

(ii) For each $x \in X$, $\cap \{\overline{V_n^2(x)} \mid n \in N\} = \{x\}$.

We note that any space whose topology contains a Hausdorff γ-space subtopology satisfies property A'.

PROPOSITION 3.4 [14, Theorem 2.4] *Every Hausdorff $w\gamma$-space that satisfies property A', is a γ-space.*

PROPOSITION 3.5. *Let (X, τ) be a regular σ-refinable space that has a G_δ-diagonal. Then (X, τ) satisfies property A'.*

Proof. Since (X, τ) has a G_δ-diagonal, there is a sequence $\{\mathscr{G}_n\}_{n=1}^\infty$ of open covers of X such that for each $x \in X$, $\{x\} = \bigcap_{i=1}^\infty \mathrm{st}(x, \mathscr{G}_n)$ [3, Lemma 5.4]. For each $n \in N$, let $\mathscr{H}_n$ be an open cover of X such that $\{\bar{H} \mid H \in \mathscr{H}_n\}$ refines $\mathscr{G}_{j_n}$. For each $n \in N$, let $\langle V_{m,n} \rangle$ be a σ-refinement of $\mathscr{H}_n$. Let $x \in X$. If $y \neq x$, then there is $n \in N$ such that $y \notin \mathrm{st}(x, \mathscr{G}_n)$. But there is $m \in N$ and $H \in \mathscr{H}_n$ such that $V_{m,n}^2(x) \subset H$. Hence $\overline{V_{m,n}^2(x)} \subset \bar{H} \subset \mathrm{st}(x, \mathscr{G}_n)$. It follows that $\bigcap_{m,n} \overline{V_{m,n}^2(x)} = \{x\}$ for all $x \in X$.

COROLLARY. *Every regular σ-refinable $w\gamma$-space* (or quasi-complete space) *with a G_δ-diagonal is a γ-space.*

The previous propositions may be modified to show that every σ-orthocompact p-space with a G_δ-diagonal admits a nonarchimedean quasi-metric.

The lemmas listed below were announced in [13] where they were used to establish Proposition 3.6. Although this proposition has been established by T. Kotake in a totally different manner, we state the lemmas in the hope that the method of proof that they imply may find wider applicability.

LEMMA. *If (X, τ) is σ-refinable and has a G_δ^*-diagonal, then (X, τ) satisfies property A'.*

LEMMA. *If (X, τ) is a first countable wN-space that satisfies property A', then (X, τ) is a Nagata space.*

Proof. Let (X, τ) be a first countable wN-space that satisfies property A', let $h: N \times X \to \tau$ be a first countable function, let $k: N \times X \to \tau$ be a wN-function and let $\langle V_n \rangle$ be a sequence of relations satisfying the conditions of property A'. Define $g: N \times X \to \tau$

by $g(n, x) = h(n, x) \cap k(n, x) \cap V_n(x)$. We show that g is a Nagata function. Suppose that for each $n \in N$, $g(n, x_n) \cap g(n, p) \neq \emptyset$. Then $\langle x_n \rangle$ has a cluster point q. Suppose that $q \neq p$. Then there is an $m \in N$ such that $p \notin \overline{V_m^2(q)}$ and there is an $n \in N$ such that $g(n, p) \cap V_m^2(q) = \emptyset$. Set $i = \max\{m, n\}$. Then $V_i^2(q) \cap g(i, p) = \emptyset$. There is a $j \geqq i$ such that $x_j \in V_i(q)$. It follows that $g(j, x_j) \subset V_j(x_j) \subset V_j(V_i(q)) \subset V_i^2(q)$. Hence $g(j, x_j) \cap g(j, p) \subset V_i^2(q) \cap g(i, p) = \emptyset$ — a contradiction. Therefore $p = q$ and g is a Nagata function.

PROPOSITION 3.6 [12]. *Every regular semi-stratifiable wN-space is a Nagata space.*

4. Θ-spaces. Let (X, τ) be a topological space and let $g: N \times X \rightarrow \tau$ be a function such that for each $x \in X$ and each $n \in N$, $x \in g(n+1, x) \subset g(n, x)$. For the sake of comparison we list the following further conditions on g.

(1) If for each $n \in N$, $x_n \in g(n, y_n)$ and $y_n \in g(n, p)$, then $\langle x_n \rangle$ has a cluster point.

(2) If for each $n \in N$, $x_n \in g(n, y_n)$ and $\langle y_n \rangle$ has a cluster point, then $\langle x_n \rangle$ has a cluster point.

(3) If for each $n \in N$, $\{x_n, p\} \subset g(n, y_n)$ and $y_n \in g(n, p)$, then $\langle x_n \rangle$ has a cluster point.

(4) If for each $n \in N$, $\{x_n, p\} \subset g(n, y_n)$ and $\langle y_n \rangle$ has a cluster point, then $\langle x_n \rangle$ has a cluster point.

(5) If for each $n \in N$, $\{x_n, p\} \subset g(n, y_n)$ and $y_n \in g(n, p)$, then p is a cluster point of $\langle x_n \rangle$.

(6) If for each $n \in N$, $\{x_n, p\} \subset g(n, y_n)$ and $\langle y_n \rangle$ has a cluster point, then p is a cluster point of $\langle x_n \rangle$.

Functions satisfying (1) (equivalently (2)) characterize $w\gamma$-spaces, those satisfying (3) characterize $w\theta$-spaces and those satisfying (5) characterize θ-spaces [10]. In this section we consider spaces that admit a function satisfying conditions (4) (resp. (6)); we call such spaces $w\Theta$-spaces (resp. Θ-spaces). It is obvious that every T_1, γ-space is a Θ-space and that every Θ-space is a θ-space. Examples 4.12 and 4.13 of [10] show that neither of the implications stated above is reversible. It is easily verified that a space is developable if, and only if, it is a β, Θ-space.

In [10] R. E. Hodel noted that every $w\Delta$-space is a $w\theta$-space and asked whether every β, $w\theta$-space is a $w\Delta$-space. It is evident that every $w\Theta$-space is a $w\theta$-space. Proposition 4.1 shows that the converse of this result would imply an affirmative answer to Hodel's question.

PROPOSITION 4.1. *A space (X, τ) is a $w\Delta$-space if and only if* *it is a β-space and a $w\Theta$-space.*

Proof. Suppose that (X, τ) is a β-space and a $w\Theta$-space. Let g be a β-function and let h be a $w\Theta$-function. Define r by $r(n, x) = g(n, x) \cap h(n, x)$. Suppose that for each $n \in N$, $\{p, x_n\} \subset r(n, y_n)$. Then $\langle y_n \rangle$ has a cluster point since r is a β-function, and since r is also a $w\Theta$-function it follows that $\langle x_n \rangle$ has a cluster point.

PROPOSITION 4.2. *A Hausdorff space (X, τ) is a Θ-space if, and only if, it is a c-semistratifiable θ-space.*

Proof. Suppose that (X, τ) is a Θ-space and that $g: N \times X \to \tau$ is a function satisfying condition (6). Let K be a compact set and suppose that $q \in \bigcap_{n=1}^{\infty} g(n, K)$. Then for each $n \in N$, there is an $x_n \in K$ such that $q \in g(n, x_n)$. Since K is compact, $\langle x_n \rangle$ has a cluster point. Since g satisfies (6), q is a cluster point of $\langle x_n \rangle$. Therefore $q \in K$.

Now suppose that (X, τ) is a c-semistratifiable θ-space and let g be a c-semistratification that satisfies condition (5). Suppose that for each $n \in N$, $\{x_n, p\} \subset g(n, y_n)$ and $\langle y_n \rangle$ has a cluster point q. Since X is first countable, there is a convergent subsequence $\langle y_{j_n} \rangle$ of $\langle y_n \rangle$ such that for each $n \in N$, $y_{j_n} \in g(n, q)$. Then $\{p, x_{j_n}\} \subset g(j_n, y_{j_n}) \subset g(n, y_{j_n})$. If $p = q$, it follows from condition (5) that p is a cluster point of $\langle x_n \rangle$. Suppose that $p \neq q$. Then there is a $k \in N$ such that if $n \geqq k$ then $y_{j_n} \neq p$. Let $K = \{y_{j_n}\}_{n \geqq k} \cup \{q\}$. Then $p \in \bigcap_{n=1}^{\infty} g(n, K) = K$ — a contradiction.

DEFINITION [2]. A sequence $\langle \mathscr{G}_i \rangle$ of collections of open subsets of a topological space is a *quasi-development* for X provided that for each $p \in X$ and each open set R containing p there is a natural number n such that $p \in \cup \mathscr{G}_n$ and such that $\mathrm{st}\,(p, \mathscr{G}_{j_n}) \subset R$. A T_1 space with a quasi-development is called a *quasi-developable space.*

PROPOSITION 4.3. *Every quasi-developable space is a θ-space.*

Proof. Let $\langle \mathscr{G}_n \rangle$ be a quasi-development for (X, τ). Define $h: N \times X \to \tau$ as follows:

$$h(n, x) = \begin{cases} X & x \notin \cup \mathscr{G}_n \\ \text{some element of } \mathscr{G}_n \text{ containing } x & x \in \cup \mathscr{G}_n \end{cases} .$$

Let $g(n, x) = \bigcap_{i=1}^{n} h(i, x)$. We show that g is a θ-function for X. Let $\{p, x_n\} \subset g(n, y_n)$ and let $y_n \in g(n, p)$ for each $n \in N$. To establish that p is a cluster point of $\langle x_n \rangle$, let W be an open neighborhood of p and let $n_0 \in N$. Choose $n_1 \in N$ such that $p \in \mathrm{st}(p, \mathscr{G}_{n_i}) \subset W$. If $n_1 \geqq n_0$, then $x_{n_1} \in W$ and if $n_1 \leqq n_0$, then $x_{n_0} \in W$.

While we have no need of the result here, the proof of Proposition 4.3 shows that quasi-developability may be characterized in terms of a function $g: N \times X \rightarrow \tau$ (where some $g(n, x)$'s may be empty) satisfying a condition similar to (a).

It is natural to ask whether a quasi-developable space is c-semistratifiable. An affirmative answer to this question would show that every quasi-developable β-space is developable. The results of this paper motivate the following additional questions.

QUESTION 1. Is every quasi-complete β-space a $w\Delta$-space?

QUESTION 2. Is every (σ-refinable) $w\Delta$-space a $\Sigma^{\#}$-space?

QUESTION 3. Is every $w\sigma$-space a $\Sigma^{\#}$-space?

QUESTION 4. Is every p-space with a G_δ-diagonal c-semistratifiable?

QUESTION 5. Is there a normal $w\Delta$-space that is not a $w\gamma$-space?

The second author and R. W. Heath have recently shown that Martin's axiom and the negation of the continuum hypothesis imply the existence of a normal Moore space that is not a $w\gamma$-space.

We are indebted to the referee, whose suggestions substantially improved §§ 3 and 4 of this paper.

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Received May 31, 1976 and in revised form August 2, 1976.

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