PACIFIC JOURNAL OF MATHEMATICS
Vol. 71, No. 2, 1977

ON THE RIEMANN-ROCH EQUATION FOR SINGULAR COMPLEX SURFACES

LAWRENCE BRENTON

An explicit constructive algorithm is developed for calculating the Hirzebruch-Riemann-Roch index $\chi(L) = \sum_{i=0}^{2} (-1)^i \dim H^i(X, \mathcal{O}(L))$ of a holomorphic line bundle L on a normal compact two-dimensional complex analytic space $(X, \mathcal{O})$ with singularities, in terms of the standard global topological invariants of X and a "correction term" involving only the local analytic and topological structure of the singular points themselves. The technique is by resolutious of singularities.

In recent years considerable attention has been given to the problem of extending classical results on nonsingular projective algebraic varieties to the more general case of possibly singular complete abstract algebraic varieties, or to that of compact complex analytic spaces. In particular, Paul Baum, William Fulton, and Robert MacPherson have achieved Riemann-Roch theorems for singular varieties by constructing objects in appropriate homology theories and K-theories which play the role of (the duals of) the Chern character of a holomorphic vector bundle E and the Todd class of a complex manifold X in Hirzebruch's formula $\chi(X, E) = [ch(E) \cup Td(X)][(X)]$ ([2], [3], [7]; see also Fulton [8], to appear).

In a rather different spirit one can study the local properties of isolated singular points and inquire how these properties are reflected in the global geometry of compact spaces which contain them. In some sense a "nice" singularity ought to be one whose presence passes unnoticed from a global point of view, while for a compact space with "bad" singularities the classical theorems (like Riemann-Roch) ought to require considerable adjusting. In [4] this tack was taken with respect to Hirzebruch's formula in dimension 2 in an attempt to understand the contribution of normal isolated singular points to global properties of compact surfaces. At that time only the hypersurface case was treated and the proofs depended on rather tedious calculations involving explicit techniques for resolving singularities. The purpose of this paper is to extend the result to arbitrary normal surfaces and to present a more concise and satisfactory proof. Thus we show:

THEOREM 1 (Proposition 2 below). *Let $(X, \mathcal{O})$ be a normal compact two-dimensional complex analytic space. Then there is a rational cohomology class $c_1 \in H^2(X, \boldsymbol{Q})$ such that for every holomorphic* *line bundle L on X we have the Riemann-Roch equation*

$$\chi(\mathcal{O}(L)) = \frac{1}{2}(c(L)^2 + c(L)\cdot c_1)([X]) + \chi(\mathcal{O}) ,$$

where $c(L)$ is the Chern class of L and $[X]$ is the natural generator of $H_4(X, \boldsymbol{Z})$.

THEOREM 2 (Proposition 4 below). *For $(X, \mathcal{O})$ and $c_1 \in H^2(X, \boldsymbol{Q})$ as above, we may compute the integer $\chi(\mathcal{O})$ as follows. Denote by $P = \{x_1, \cdots, x_r\}$ the set of singular points of X and let $\pi: \tilde{X} \to X$ be any normal resolution of singularities. Let $C = \bigcup_{i=1}^s C_i$ be the decomposition of the exceptional curve $C = \pi^{-1}(P)$ into its irreducible branches. Denote by g_i the genus of C_i, by $g(C) = \dim H^1(C, \boldsymbol{R}) - \sum_{i=1}^s \dim H^1(C_i, \boldsymbol{R})$ the number of independent cycles in the dual graph of C, and by $e(C_i)$, C_i^2 respectively, the Euler number and self-intersection. Let $\gamma_i \in H^2(C, \boldsymbol{Q}) \cong \boldsymbol{Q}^s$ be the generator corresponding to the component C_i* (i.e., dual to a point of $C_i - (\bigcup_{j\neq i} C_j)$) *and denote by $\|\ \|_\pi$ the norm on this rational vector space represented in the basis $\{\gamma_1, \cdots, \gamma_s\}$ by the positive definite matrix $(-C_i\cdot C_j)^{-1}$, and by $e(C) + C^2$ the vector $\sum_{i=1}^s (e(C_i) + C_i^2)\gamma_i$. Finally, denote by $R^1\pi_*\mathcal{J}_C$ the first right derived sheaf on X via π of the ideal subsheaf $\mathcal{J}_C \subset \mathcal{O}_{\tilde{X}}$ of the divisor $C \subset \tilde{X}$, and define the topological Euler class c_2 by $c_2 = c_2(X) = e(X)\xi$ for $e(X)$ the Euler number of X and ξ the natural generator of $H^4(X, Z)$. Then:*

$$12\chi(\mathcal{O}) = (c_1^2 + c_2)([X]) + s + 10\sum_{i=1}^s g_i + 11g(C) - \|e(C) + C^2\|_\pi^2 + \sum_{i=1}^r \dim (R^1\pi_*\mathcal{J}_C)_{x_i} .$$

We note that except for the term $(c_1^2 + c_2)([X])$ the terms on the righthand side of the equation depend only on the germs $(x, \mathcal{O}_x)$ of the singular points and that by the theorem their sum is independent of the resolution π. Because of the terms $\dim (R^1\pi_*\mathcal{J}_C)_{x_i}$, however, the index $\chi(\mathcal{O})$ is not determined solely by the topology of X as is the case when X is nonsingular (see [14], Example, page 7). It should also be emphasized that it is the numerical calculations of Theorem 2 that are of primary interest here. That is, our aim is to prescribe a concrete finite algorithm useful for determining, say, the number of sections admitted by particular line bundles on a particular singular surface, rather than to develop a general theory of characteristic classes of vector bundles on analytic spaces. For this the reader may consult, e.g., MacPherson [16], or the references [2], [3] and [7] cited above.

Theorems 1 and 2 then combine to give the main "Composite Riemann-Roch Theorem," Theorem 5 below.

The paper concludes with some remarks and examples, among which we mention here the following result of Laufer which connects these ideas to the notion of the Milnor number μ of an isolated hypersurface singularity.

THEOREM 3 (Laufer [15]). *Let X be a two-dimensional compact complex analytic space each of whose singular points is an isolated hypersurface singularity. Then for K the "canonical" line bundle on X* (the standard bundle of holomorphic 2-forms on the regular points, extended to all of X) *we have for every holomorphic line bundle L on X the equation*

$$\chi(L) = \frac{1}{2} L \cdot (L - K) + \frac{1}{12}(K^2 + e(X) + \mu)$$

for μ the sum of the Milnor numbers of the singular points. Furthermore, μ can be calculated by the formula of Theorem 2 above.

And lastly we give an application to the topic of singular surfaces which are homotopy projective planes:

THEOREM 4 (Proposition 6 below). *Let X be a normal compact two-dimensional complex analytic space with vanishing geometric genus $p_g = \dim H^2(X, \mathcal{O}_X)$ and with integral cohomology ring isomorphic to that of the complex projective plane $\boldsymbol{P}^2$ and generated by the Chern class of the line bundle of a holomorphic divisor. Then X is a rational projective algebraic surface homotopy equivalent to $\boldsymbol{P}^2$, each singular point of X is a rational double point, and, indeed, X is biholomorphic either to*

(a) *$\boldsymbol{P}^2$ itself* (in case X is nonsingular) *or to*

(b) *a singular rational surface obtained from $\boldsymbol{P}^2$ by the successive application of precisely* 8 *monoidal transformations followed by the blowing down of precisely* 8 *nonsingular rational curves, each with self-intersection* 2.

Examples of these last-mentioned spaces are given in [6].

I. A Riemann-Roch theorem for singular surfaces. Let $(X, \mathcal{O}_X)$ be a normal compact complex analytic space (always reduced in this paper). One way to study the properties of X is to produce a resolution $\pi: \tilde{X} \to X$ of the singularities of X, apply the powerful theory of compact complex manifolds to the nonsingular model $\tilde{X}$, and then try to push this information down to X via the map π. This approach is especially fruitful in (complex) dimension 2 where we have at our disposal not only the rich abundance of detail concerning isolated singularities of surfaces and their resolutions (as exposed in [13], e.g.) but also the all but exhaustive classification theory for two-dimensional compact manifolds due principally to Kodaira [12]. In particular the following result relates the topology of a singular surface with that of its nonsingular model. A complete proof is given in [5] (Lemma 1 of that paper).

LEMMA 1. *Let X be a compact two-dimensional complex analytic space with only isolated singular points $x_1, \cdots, x_r$. Let $\pi: \tilde{X} \to X$ be any resolution of the singularities of X. Put $C = \pi^{-1}(\{x, \cdots, x_r\})$ the exceptional curve in $\tilde{X}$. Then there is an exact sequence*

$$(^*Z) \cdots \longrightarrow H^i(X, Z) \xrightarrow{\pi_i} H^i(\tilde{X}, Z) \xrightarrow{\varepsilon_i} H^i(C, Z) \xrightarrow{\delta_i} H^{i+1}(X, Z) \longrightarrow \cdots ,$$

$i \geqq 1$, with π_i naturally induced by π and ε_i by the inclusion $C \subset \tilde{X}$. Furthermore,

(a) *π_1 is injective, π_3 surjective, and π_4 an isomorphism;*

(b) *if $(^*Z)$ is tensored with $\boldsymbol{Q}$, in the resulting sequence*

$$(^*\boldsymbol{Q}) \cdots \longrightarrow H^i(X, \boldsymbol{Q}) \xrightarrow{\pi_i'} H^i(X, \boldsymbol{Q}) \xrightarrow{\varepsilon_i'} H^i(C, \boldsymbol{Q}) \xrightarrow{\delta_i'} H^{i+1}(X, \boldsymbol{Q}) \longrightarrow \cdots$$

ε_2' is surjective and π_3' an isomorphism.

This result alone gives us the first version of the theorem.

PROPOSITION 2 (Theorem 1 of the introduction). *Let $(X, \mathcal{O}_X)$ be a normal compact two-dimensional complex analytic space. Then there is a rational cohomology class $c_1 \in H^2(X, \boldsymbol{Q})$ such that for any holomorphic line bundle L on X we have the Hirzebruch-Riemann-Roch formula*

$$\chi(\mathcal{O}_X(L)) = \frac{1}{2}(c(L)^2 + c(L) \cdot c_1)([X]) + \chi(\mathcal{O}_X)$$

for $c(L)$ the Chern class of L and $[X]$ the canonical generator of $H_4(X, \boldsymbol{Q})$. (In particular, the expression $1/2(c(L)^2 + c(L) \cdot c_1)$ lies in the image of the natural mapping $H^*(X, Z) \to H^*(X, \boldsymbol{Q})$ for every L, even though the rational class c_1 may not.)

Proof. Denote by $P = \{x_1, \cdots, x_r\}$ the set of singular points of

X, let $\pi: \tilde{X} \to X$ be a resolution of singularities, and let $C = \bigcup_{i=1}^{s} C_i$ be the decomposition of the exceptional curve $C = \pi^{-1}(P)$ into its irreducible branches. Now the exact sequence

$$H^2(X, \boldsymbol{Q}) \xrightarrow{\pi^*} H^2(X, \boldsymbol{Q}) \xrightarrow{\varepsilon^*} H^2(C, \boldsymbol{Q}) \longrightarrow 0$$

of Lemma 1 admits the natural splitting $\varepsilon^{*^{-1}}$ given (in the obvious bases) by the inverse to the nonsingular intersection matrix $(C_i \cdot C_j)$. Thus we have the internal direct sum

$$H^2(\tilde{X}, \boldsymbol{Q}) = \operatorname{im}(\pi^*) \oplus \mathscr{C}$$

for $\mathscr{C}$ the rational subspace spanned by the Poincaré duals C_i^* of the curves C_i. In particular for $\tilde{c}_1 \in H^2(\tilde{X}, \boldsymbol{Q})$ the first Chern class of the compact complex manifold $\tilde{X}$ we have

$$(1) \qquad \tilde{c}_1 = \pi^*(c_1) + \sum_{i=1}^{s} t_i C_i^*$$

for some cohomology class $c_1 \in H^2(X, \boldsymbol{Q})$ and some rational numbers t_i. Note then that for any $\alpha \in H^2(X, \boldsymbol{Q})$,

$$(2) \qquad \tilde{c}_1 \cdot \pi^*(\alpha) = \pi^*(c_1) \cdot \pi^*(\alpha) .$$

Next, the Leray spectral sequence for the sheaf $\mathscr{O}_{\tilde{X}}(\pi^* L)$ and the map π gives

$$(3) \qquad \chi(\mathscr{O}_X(L)) - \chi(\mathscr{O}_{\tilde{X}}(\pi^* L)) = \dim H^0(X, R^1\pi_*\mathscr{O}_{\tilde{X}}(\pi^* L)) .$$

(Here $\pi_*\mathscr{O}_{\tilde{X}}(\pi^* L) = \mathscr{O}_X(L)$ by normality.) But since π is a biholomorphism off C the sheaf $R^1\pi_*\mathscr{O}_{\tilde{X}}(\pi^* L)$ (the first derived sheaf of $\mathscr{O}_{\tilde{X}}(\pi^* L)$ via π) is supported on P, and since $\mathscr{O}_X(L)$ is free in a neighborhood of P we have in fact

$$R^1\pi_*\mathscr{O}_{\tilde{X}}(\pi^* L) \approx R^1\pi_*\mathscr{O}_{\tilde{X}} .$$

Thus (3) becomes

$$(4) \qquad \begin{aligned} \chi(\mathscr{O}_X(L)) - \chi(\mathscr{O}_{\tilde{X}}(\pi^* L)) &= \dim H^0(X, R^1\pi_*\mathscr{O}_{\tilde{X}}) \\ &= \chi(\mathscr{O}_X) - \chi(\mathscr{O}_{\tilde{X}}) , \end{aligned}$$

or

$$(5) \qquad \chi(\mathscr{O}_{\tilde{X}}(\pi^* L)) - \chi(\mathscr{O}_{\tilde{X}}) = \chi(\mathscr{O}_X(L)) - \chi(\mathscr{O}_X) .$$

Finally we note that for $[X]$, $[\tilde{X}]$ the natural generators of $H_4(X, \boldsymbol{Q})$, $H_4(\tilde{X}, \boldsymbol{Q})$, respectively, the fact that π is orientation preserving and a homeomorphism off a proper subvariety means that

$$(6) \qquad \pi_*([\tilde{X}]) = [X] .$$

Putting (2), (5), and (6) together with Riemann-Roch on $\tilde{X}$ for the line bundle π^*L and naturality of Chern classes and cup products now completes the proof. Namely, if c, $\tilde{c}$ denote the Chern class maps respectively on X and $\tilde{X}$, then for $c_1 \in H^2(X, \boldsymbol{Q})$ as in (1), if $L \to X$ is any holomorphic line bundle we have

$$\begin{aligned}\frac{1}{2}(c(L)^2 + c(L) \cdot c_1)([X]) &= \frac{1}{2}(\tilde{c}(\pi^*L)^2 + \tilde{c}(\pi^*L) \cdot \pi^*(c_1))([\tilde{X}]) \\ &= \frac{1}{2}(\tilde{c}(\pi^*L)^2 + \tilde{c}(\pi^*L) \cdot \tilde{c}_1)([\tilde{X}]) \\ &= \chi(\mathcal{O}_{\tilde{X}}(\pi^*L)) - \chi(\mathcal{O}_{\tilde{X}}) \\ &= \chi(\mathcal{O}_X(L)) - \chi(\mathcal{O}_X) .\end{aligned}$$

REMARK. The class $c_1 \in H^2(X, \boldsymbol{Q})$ thus selected is unique only up to an element in the kernal of the mapping $\pi_2' \colon H^2(X, \boldsymbol{Q}) \to H^2(\tilde{X}, \boldsymbol{Q})$. Of course the class $c_1 \cdot \alpha \in H^2(X, \boldsymbol{Q})$ is nevertheless well-defined $\forall \alpha \in H^2(X, \boldsymbol{Q})$, for $\ker \pi_2'$ is precisely the null eigen space of the cup product pairing (not necessarily nonsingular for singular surfaces) on X. c_1 is obviously independent of the particular resolution π.

II. Calculation of the term $\chi(\mathcal{O}_X)$. For singular surfaces X the analytic index $\chi(\mathcal{O}_X)$ is evidently not determined by a nice set of global topological invariants. In this section we calculate this index by an algorithm which exhibits explicitly its dependence on the nature of the singular points.

For any irreducible compact analytic space Y of dimension n denote by $e(Y) = \sum_{i=1}^{2n} (-1)^i \dim H^i(Y, \boldsymbol{R})$ the topological Euler number. By both abuse and confusion of notation, then, we may define the "nth Chern class" $c_n(Y)$ of Y by simply putting

$$c_n(Y) = e(Y)\xi(Y)$$

for $\xi(Y)$ either the natural generator of $H^{2n}(Y, Z)$ or its natural image in $H^{2n}(Y, \boldsymbol{Q})$, or in $H^{2n}(Y, \boldsymbol{R})$, etc., as convenience dictates.

Now if Σ is the singular set of Y, Y as above, a resolution $\pi \colon \tilde{Y} \to Y$ of the singularities of Y is called *normal* if the hypersurface $Z = \pi^{-1}(\Sigma)$ consists of a collection of manifolds Z_i meeting (if at all) transversally, with $Z_i \cap Z_j$ connected $\forall i, j$ and with $\dim (Z_i \cap Z_j \cap Z_k) \leqq n - 3$, i, j, k distinct. In particular, if Y is a surface and Z a curve, then Z_i meets Z_j in at most one point and $Z_i \cap Z_j \cap Z_k = \varnothing$. Normal resolutions always exist by Hironaka [10].

PROPOSITION 3. *Let X be a normal compact two-dimensional* *complex analytic space with singular set* $P = \{x_1, \cdots x_r\}$ *and let* π: $\tilde{X} \to X$ *be any normal resolution of the singularities of* X *with exceptional curve* $C = \bigcup_{i=1}^{s} C_i$, C_i *irreducible. Denote by* $\gamma_i \in H^2(C, \boldsymbol{Q}) \cong \boldsymbol{Q}^s$ *the generator corresponding to the curve* C_i, *by* $e(C) + C^2$ *the vector* $\sum_{i=1}^{s} (e(C_i) + C_i^2)\gamma_i$, *and by* $\|\ \|_\pi$ *the norm on this rational vector space given in the basis* $\{\gamma_1, \cdots, \gamma_s\}$ *by the inverse* (E_{ij}) *to the positive definite matrix* $-(C_i \cdot C_j)$. *Let* $c_1 \in H^2(X, \boldsymbol{Q})$ *be as in Proposition 2 and let* $c_2 = c_2(X) \in H^4(X, \boldsymbol{Q})$ *be as immediately above the topological Euler class. Then*

$$\chi(\mathcal{O}_X) = \frac{1}{12}((c_1^2 + c_2)([X]) + e(C) - r - \|e(C) + C^2\|_\pi^2) + \sum_{x \in P} \dim (R^1\pi_*\mathcal{O}_{\tilde{X}})_x .$$

Proof. The proof is a straightforward calculation. By equation (4) of the proof of Proposition 2

$$\begin{aligned}\chi(\mathcal{O}_X) &= \chi(\mathcal{O}_{\tilde{X}}) + \dim H^0(X, R^1\pi_*(\mathcal{O}_{\tilde{X}})) \\ &= \chi(\mathcal{O}_{\tilde{X}}) + \sum_{x \in P} \dim (R^1\pi_*\mathcal{O}_{\tilde{X}})_x .\end{aligned}$$

By Riemann-Roch on $\tilde{X}$,

$$\chi(\mathcal{O}_{\tilde{X}}) = \frac{1}{12}(\tilde{c}_1^2 + \tilde{c}_2)([\tilde{X}]) .$$

From the exact sequence

$$\cdots \longrightarrow H^i(X, \boldsymbol{Q}) \longrightarrow H^i(\tilde{X}, \boldsymbol{Q}) \longrightarrow H^i(C, \boldsymbol{Q}) \longrightarrow \cdots ,$$

$i \geqq 1$, of Lemma 1 we obtain

$$e(\tilde{X}) - \dim H^0(\tilde{X}, \boldsymbol{R}) = (e(X) - \dim H^0(X, \boldsymbol{R})) + (e(C) - \dim H^0(C, \boldsymbol{R})) .$$

But X and $\tilde{X}$ are connected, while C has as many topological components as X has singular points, namely r. Thus

$$(\tilde{c}_2)([\tilde{X}]) = (c_2)([X]) + e(C) - r .$$

And finally, putting $\tilde{c}_1 = \pi^*(c_1) + \sum_{i=1}^{s} t_i C_i^*$ as per the definition of c_1, we have

$$\begin{aligned}(\tilde{c}_1^2)([X]) &= (\pi^*(c_1) + \Sigma t_i C_i^*)^2([\tilde{X}]) \\ &= (\pi^*(c_1))^2([\tilde{X}]) + \sum_{i,k=1}^{s} t_i(C_i \cdot C_j)t_j \\ &= (c_1)^2([X]) + \sum_{i,j=1}^{s} (\sum_{l=1}^{s} t_l c_l \cdot c_j)(-E_{ji})(\sum_{k=1}^{s} c_i \cdot c_k)t_k \\ &= (c_1)^2([X]) - \sum_{i,j=1}^{s} (\tilde{c}_1|_{C_j})([C_j])(E_{ji})(\tilde{c}_1|_{C_i})([C_i])\end{aligned}$$

$$= (c_1)^2([X]) - \sum_{i,j=1}^{s} (e(C_j) + C_j^2)(E_{ji})(e(C_i) + C_i^2)$$
$$= (c_1)^2([X]) - ||e(C) + C^2||_\pi^2 .$$

Putting this list of facts together proves the theorem.

We want to restate this result slightly, the better to exhibit its "algorithmic" character. The advantage below in substituting the sheaf $R^1\pi_*\mathscr{J}_C$ for $R^1\pi_*\mathscr{O}_{\tilde{X}}$ is simply that it more often vanishes. (A simple sufficiency test for $(R^1\pi_*\mathscr{J}_C)_x = 0$ is given, following Laufer [13], in [4], page 49.)

PROPOSITION 4 (Theorem 2 of the introduction). *For X, $\pi: \tilde{X} \to X$ as in Proposition 3, the index $\chi(\mathscr{O}_X)$ satisfies*

$$12\chi(\mathscr{O}_X) = (c_1^2 + c_2)([X]) + s + 10\sum_{i=1}^{s} g_i + 11g(C) - ||e(C) + C^2||_\pi^2 + \sum_{x \in P} \dim (R^1\pi_*\mathscr{J}_C)_x ,$$

where s is the number of components of the exceptional curve $C = \pi^{-1}(P)$, g_i is the genus of C_i, $g(C)$ the "number of cycles in the dual graph of C" $= \dim H^1(C, \boldsymbol{R}) - \sum_{i=1}^{s} \dim H^1(C_i, \boldsymbol{R})$; where the vector $e(C) + C^2$ and the norm $|| \ ||_\pi$ are as in Proposition 3, and where $\mathscr{J}_C$ is the locally principle ideal subsheaf of $\mathscr{O}_{\tilde{X}}$ consisting of those germs which vanish on C.

Proof. We continue calculating where we left off in Proposition 3:

$$(*) \quad e(C) - r = \dim H^0(C, \boldsymbol{R}) - \dim H^1(C, \boldsymbol{R}) + \dim H^2(C, \boldsymbol{R}) - r$$
$$= r - ((\sum_{i=1}^{s} 2g_i) + g(C)) + s - r$$
$$= s - 2(\sum_{i=1}^{s} g_i) - g(C) .$$

To relate $R^1\pi_*\mathscr{O}_{\tilde{X}}$ to $R^1\pi_*\mathscr{J}_C$ we consider a connected component C' of C and a (small) strongly pseudoconvex neighborhood $\tilde{U}$ of $\tilde{C}$ such that $\pi(\tilde{U})$ is Stein. Without loss of generality we may suppose that $C' = \bigcup_{i=1}^{s'}$ with $C_i \cap (\bigcup_{j<i} C_j) \neq \varnothing$ for $i > 1$. For each $i = 1, 2, \cdots, s'$ put

$$n_i = \sum_{j<i} C_j \cdot C_i = \text{number of curves } C_j \text{ that meet } C_i \text{ for } j < i.$$

Following Laufer [13] denote by $\mathscr{J}_i$ the ideal sheaf of C_i and successively for $i = 1, 2, \cdots, s'$ consider the exact sequences

$$0 \longrightarrow \bigotimes_{j \leq i} \mathscr{J}_j \longrightarrow \bigotimes_{j \leq i-1} \mathscr{J}_j \longrightarrow \mathscr{O}_{C_i} \otimes \mathscr{O}_{\tilde{X}}(\bigotimes_{j<i} [C_j]^{-1}) \longrightarrow 0 ,$$

for $[C_j]^{-1}$ the dual of the line bundle of the divisor C_j on $\tilde{X}$. On $\tilde{U}$ we have the induced sequences

$$\cdots \longrightarrow H^0(\tilde{U}, \bigotimes_{j\leqq i-1} \mathscr{I}_j) \longrightarrow H^0(C_i, \mathscr{O}_{C_i}(\bigotimes_{j<i} [C_j]^{-1}|_{C_i})$$
$$\longrightarrow H^1(\tilde{U}, \bigotimes_{j\leqq i} \mathscr{I}_j) \xrightarrow{\phi_i} H^1(\tilde{U}, \bigotimes_{j\leqq i-1} \mathscr{I}_j) \xrightarrow{\psi_i} H^1(C_i, \mathscr{O}_{C_i}(\bigotimes_{j<i} [C_j]^{-1}|_{C_i}))$$
$$\longrightarrow H^2(\tilde{U}, \bigotimes_{j\leqq i} \mathscr{I}_j) \longrightarrow \cdots .$$

Since U is pseudoconvex, this last group vanishes ([1], Proposition 27, page 256, or [9], Satz 1, page 355, supplies the proof), so ψ_i is always surjective. Also, ϕ_i is injective. for if $i = 1$ we have simply the sequence $\mathscr{I}_{C_1} \to \mathscr{O}_{\tilde{X}} \to \mathscr{O}_{C_1}$, and C_1, being compact, admits only constant holomorphic functions, while if $i > 1$, then the bundle $\bigotimes_{j<i} [C_j]^{-1}|_{C_i}$ is strictly negative and admits only the zero section. Thus since all these groups are finite dimensional we may simply count dimensions:

$$\dim H^1(\tilde{U}, \bigotimes_{j\leqq i-1} \mathscr{I}_j) = \dim H^1(\tilde{U}, \bigotimes_{j\leqq 1} \mathscr{I}_j) + \dim H^1(C_i, \mathscr{O}_{C_i}(\bigotimes_{j<1} [C_j]^{-1}|_{C_i}) ,$$

and by Riemann-Roch on C_i this last group has dimension

$$\dim H^1(C_i, \mathscr{O}_{C_i}(\bigotimes_{j<i} [C_j]^{-1}|_{C_i}) = \begin{cases} n_i - (1 - g_i) & \text{for } \quad i > 1 \\ g_1 & \text{for } \quad i = 1 . \end{cases}$$

Adding these equations (and noting that the sheaf $\bigotimes_{j\leqq i-1} \mathscr{I}_j = \mathscr{O}_{\tilde{X}}$ for $i = 1$ and $\bigotimes_{j\leqq i} \mathscr{I}_j = \mathscr{I}_{C'}$ for $i = s'$) then gives the conclusion

$$\dim H^1(\tilde{U}, \mathscr{O}_X) = \dim H^1(\tilde{U}, \mathscr{I}_{C'}) + \sum_{i=1}^{s'} g_i + \sum_{i=2}^{s'} (n_i - 1) .$$

But it is easy to check that this last number is just the number $g(C)$ of cycles in the dual graph of C'. Thus shrinking U to the singular point $x = \pi(C')$ and summming over the connected components of C yields the relation

$$(**) \qquad \sum_{x\in P} \dim R^1\pi_*\mathscr{O}_{\tilde{X}} = \sum_{x\in P} \dim R^1\pi_*\mathscr{I}_C + \sum_{i=1}^{s} g_i + g(C) .$$

(cf. [5], Lemma 2, where we take a closer look at this relation for negatively embedded curves C on surfaces).

Putting equations (*) and (**) together with the conclusion to Proposition 3 now completes the proof.

COMPOSITE RIEMANN-ROCH THEOREM 5. *Let $(X, \mathscr{O}_X)$ be a normal two-dimensional compact complex analytic space. Then there is a rational cohomology class $c_1 \in H^2(X, \boldsymbol{Q})$, and an integer $R(X)$ depending only on the germs of the singular points, such that for* *any holomorphic line bundle L on X we have the "Hirzebruch-Riemann-Roch" equation*

$$\chi(L) = \left(\frac{1}{2}(c(L)^2 + c(L)\cdot c_1) + \frac{1}{12}(c_1^2 + c_2)\right)([X]) + \frac{1}{12}R(X)$$

for $\chi(L)$ *the analytic index of* L, $c_2 \in H^4(X, \boldsymbol{Q})$ *the Euler class, and* $[X] \in H_4(X, \boldsymbol{Q})$ *the natural positive generator. Furthermore,* $R(X)$ *be computed from any normal resolution* $\pi\colon \widetilde{X} \to X$ *of the singularities of* X *according to the formula*

$$R(X) = s + 10\sum_{i=1}^{s} g_i + 11g(C) - ||e(C) + C^2||_\pi^2 + \sum_{x\in P}\dim(R^1\pi_*\mathscr{I}_C)_x\ ,$$

with notation as in the previous propositions.

III. Remarks and applications. (1) At first glance it might appear that this "Riemann-Roch correction term" $R(X)$ is faily complicated. In any particular example, however, its calculation is completely straightforward. Given a singular point $x \in X$ there are at least two standard methods for explicitly constructing a resolution π of singularities. (One is by successive monoidal transformations and the other is Hirzebruch's method [11] of piecing together branched coverings of C^2.) Given the resolution, the numbers s, g_i, and $g(C)$ can be read off immediately, as can the intersection matrix $(C_i \cdot C_j)$ whose inverse gives the norm $||\ ||_\pi$. The last term $\dim(R^1\pi_*\mathscr{I}_C)_x$ may be trickier but it can always be computed by the technique of the last part of the proof of Proposition 4, as is shown in [13], Chapter 6. Furthermore, for some special kinds of singularities we may get a simplification of the formula for R due to vanishing of some of the terms. If X has only rational double points, for instance, everything in sight vanishes and we have simply

$$R(X) = s\ ,$$

the number of curves in the minimal resolution.

(2) In some cases the cohomology class c_1 is actually the Chern class of a holomorphic line bundle K. For instance if the singularities of X are *Gorenstein* (i.e., if the canonical bundle K_0 on the set of regular points X_0 of X is trivial in a neighborhood of each singular point), then, as is clear from the definition of c_1, c_1 is just the Chern class of the bundle $K = K_0$ extended (uniquely up to an analytic isomorphism) to X. In this case we may call the bundle K so defined the "canonical line bundle of X" and obtain the formula

$$\begin{aligned}\chi(\mathcal{O}(L)) &= \frac{1}{2}L\cdot(L-K)+\chi(\mathcal{O})\\ &= \frac{1}{2}L\cdot(L-K)+\frac{1}{12}(K^2+e(X)+R(X)) .\end{aligned}$$

To specialize further to the case where X is locally a hypersurface, Laufer shows in [15] that the term $R(X)$ is equal to the sum μ of the Milnor numbers $\mu(x_i)$ of the singular points. This gives the pretty result

$$(\dagger)\qquad \chi(L)=\frac{1}{2}L\cdot(L-K)+\frac{1}{12}(K^2+e(X)+\mu)$$

mentioned in the introduction. Since in general Milnor's μ is to be regarded as the natural analogue in higher dimensions of the notion of "multiplicity" in the classical theory of plane curves, the result (†) and its variants can be interpreted as a generalization to dimension two of a part of the theory surrounding the various formulas involving the "number of nodes and cusps" of an algebraic curve. Our "first Chern class" c_1, for instance, corresponds to the "virtual genus" $\hat{g}=K_S\cdot D+D^2$ of a singular curve D contained in a nonsingular surface S in the formula

$$g=\hat{g}+\frac{1}{2}\Sigma\mu_i(\mu_i-1)$$

for g the genus of D (i.e., of the nonsingular model) and the μ_i the multiplicities of the singular points.

The formula (†) also shows that in this case the correction term $R(X)$ is always positive—a fact that is not immediately obvious from the definition of $R(X)$ as given in Theorem 5.

(3) The conclusion that the expression $(1/2)(c(L)^2+c(L)\cdot c_1)([X])$ must always be an integer is sometimes a useful one. We close with an illustration of this fact.

PROPOSITION 6 (Theorem 4 of the introduction). *Let X be a normal compact two-dimensional complex analytic space with vanishing geometric genus $p_g=\dim H^2(X,\mathcal{O}_X)$ and with $H^*(X,Z)\cong H^*(\boldsymbol{P}^2,Z)$ and generated by the Chern class of the line bundle of a holomorphic divisor Γ. Then X is a rational projective algebraic surface homotopy equivalent to $\boldsymbol{P}^2$ and each singular point of X is a rational double point. Indeed, either*

(a) *X is biholomorphic to $\boldsymbol{P}^2$ or*

(b) *X is biholomorphic to a singular surface obtained from $\boldsymbol{P}^2$ by the successive application of precisely 8 monoidal transformations followed by the collapsing to one or more points of precisely*

8 nonsingular rational curves, each with self-intersection -2. *In this case* Γ *may be taken to be a nonsingular elliptic curve contained in the regular points of* X.

Proof. The desired conclusion is proved in [6], Theorem 6, under the assumption that the singularities are already known to be rational double points (in this case $p_g = 0$ is redundant). Thus it suffices to show that the hypotheses of the present proposition force this condition upon the singularities.

Let $\pi: \tilde{X} \to X$ be a resolution of singularities. From the exact sequence

$$(*) \qquad \cdots \longrightarrow H^i(X, Z) \longrightarrow H^i(\tilde{X}, Z) \longrightarrow H^i(C, Z) \longrightarrow \cdots$$

of Lemma 1, vanishing of $H^3(X, Z)$ implies the same for $H^3(\tilde{X}, Z)$, whence Poincaré duality on $\tilde{X}$ gives $b_1(\tilde{X}) = 0$, b_1 the first Betti number. Thus also $q(\tilde{X}) = (1/2)b_1(\tilde{X}) = 0$ for $q(X) = \dim H^1(X, \mathcal{O}_X)$ the irregularity ([12], I, Theorem 3). But in any case $q(X) \leqq q(\tilde{X})$ and $p_g(\tilde{X}) \leqq p_g(X)$ ([5], Corollary 3), so in our case all four of these numbers vanish. In particular, then,

$$\sum_{x \in P} \dim (R^1\pi_*\mathcal{O}_{\tilde{X}})_x = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_{\tilde{X}}) = 0 \ ,$$

so each singular point is rational.

As in the proof of Proposition 1 above put

$$(**) \qquad \tilde{c}_1 = \pi^*(c_1) + \sum_{i=1}^{s} t_i C_i^* \ ,$$

$t_i \in \boldsymbol{Q}$, $c_1 \in H^2(X, \boldsymbol{Q})$. I claim that in fact c_1 is integral. To see this write $c_1 = tc([\Gamma])$, $t \in \boldsymbol{Q}$, and apply proposition 1 to the line bundle $[\Gamma]$ of the divisor Γ:

$$\begin{aligned}\chi(\mathcal{O}_X([\Gamma])) &= \frac{1}{2}(c([\Gamma])^2 + c([\Gamma])\cdot c_1)([X]) + \chi(\mathcal{O}_X) \\ &= \frac{1}{2}(1 + t) + \chi(\mathcal{O}_X) \ .\end{aligned}$$

(Here we have used the fact that $H^*(X, Z) \cong H^*(\boldsymbol{P}^2, Z)$ (as rings) and that $H^2(X, Z) \cong Z$ is generated by $c([\Gamma])$.) Since $1/2(1 + t)$ must thus be an integer, so must t.

Now the map $\varepsilon_2: H^2(\tilde{X}, Z) \to H^2(C, Z)$ is surjective ($H^3(X, Z) = 0$ in (*)), so for each i $\exists$ an integral cohomology class $\alpha_i \in H^2(\tilde{X}, Z)$ such that

$$C_j^* \cdot \alpha_i - \begin{cases} 1 & \text{if} \quad j = i \\ 0 & \text{otherwise.} \end{cases}$$

Hence the rational numbers t_i in (**) satisfy

$$t_i = (\sum_{j=1}^{s} t_j C_j^*) \cdot \alpha_i = \tilde{c}_1 \cdot \alpha_i - \pi^*(c_1) \cdot \alpha_i .$$

Since both $\tilde{c}_1$ and $\pi^*(c_1) = tc(\pi^*[\Gamma])$ are integral, $t_i \in Z \forall i$. But then $q(\tilde{X}) = q(X) = 0$ together with (**) gives the bundle isomorphism

$$K_{\tilde{X}} = \pi^*[\Gamma]^{-t} \otimes (\bigotimes_{i=1}^{s} [C_i]^{-t_i}) .$$

Restricting this equation to $X - C$ shows that the canonical bundle K_{X_0} on the regular points X_0 of X is isomorphic to the bundle $[\Gamma]^{-t}|_{X_0}$. In particular, K_{X_0} is trivial in a neighborhood of each singular point—i.e., the singularities of X are Gorenstein. (The support of Γ may of course pass through singular points of X, but since by assumption Γ is a (Cartier) divisor this does not compromise local triviality of $[\Gamma]$.)

But Laufer show in [14], Theorem 4.3 and its proof that a rational singularity is Gorenstein if and only if it is a double point, and thus the proof is complete.

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Received December 29, 1976 and in revised form March 16, 1977. The research here reported was done in part while I was a visitor at Tulane University in New Orleans for the autumn semester, 1976. I would like to thank the Mathematics Faculty and the College of Arts and Sciences for a most profitable and enjoyable term.

WAYNE STATE UNIVERSITY
DETROIT, MI 48202