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AN UNEXPECTED SURGERY CONSTRUCTION OF A LENS SPACE

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A useful method of constructing-3-dimensional manifolds is to remove the interior of a tubular neighbourhood $V \subset S^3$ of a knot K in the 3-sphere and sew it back differently, via a homeomorphism $h: \partial V \to \partial V$. This surgery construction, due to M. Dehn, yields the manifold

$$M^3 = (S^3 - \operatorname{int} V) \bigcup_h V ,$$

where $x \in \partial V \subset V$ is identified with $h(x) \in \partial V \subset S^3 - \operatorname{int} V$. For example surgery along a trivial knot yields, for various choices of h, exactly the class of lens spaces $L(p, q)$, including the extreme cases $L(1, 0) \cong S^3$ and $L(0, 1) \cong S^2 \times S^1$.

Louise Moser has shown in [3] that certain surgeries along nontrivial torus knots also yield lens spaces. Strong circumstantial evidence led her to conjecture a converse.

Moser's conjecture. If M^3 is a lens space obtained by surgery along K, then K is a torus knot.

The purpose of this paper is to present a counterexample to this conjecture (it is also a counterexample to the other two conjectures of [3]). This conjecture also has been put forward by J. P. Neuzil [4].

THEOREM. *The lens space $L(23, 7)$ is the result of an appropriate surgery along the knot of Figure* 1, *which is not a torus knot.*

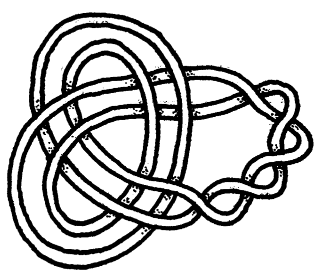

FIGURE 1

To explain the proof, some notational conventions need to be stated. Suppose K is a tame knot in S^3 and V is a solid torus neighbourhood of K. Assume for the moment that K is oriented.

A *preferred longitude* is an oriented curve λ in ∂V which is homotopic in V to the centerline K and has linking number zero with K. A *preferred meridian* is an oriented curve μ in ∂V which is homotopically trivial in V and links K once in, say, a right-hand screw sense. Their homology classes, which we also denote by λ and μ, form a distinguished basis for $H_1(\partial V) \cong Z \oplus Z$. Any oriented *simple* closed curve in ∂V has homology class $l\lambda + m\mu$, where l and m are relatively prime integers. To designate the manifold obtained by removing the interior of V from S^3 and replacing it via a homeomorphism $h: \partial V \rightarrow \partial V$ we need only specify the knot K and the *ratio*

$$r = m/l \quad \text{where} \quad h_*(\mu) = l\lambda + m\mu .$$

The possibility $r = \pm 1/0 = \infty$ is allowed, corresponding to the "trivial" surgery in which V is replaced using the identity map on the boundary. The choice of orientation of K is irrelevant to this definition of r, which well call the *surgery coefficient* assigned to K. More generally one can specify a 3-manifold, well-determined up to homeomorphism, by choosing a tame link $L = L_1 \cup \cdots \cup L_n$ in S^3 and surgery coefficients $r_1, \cdots, r_n$ which describe, as above, how to remove and replace disjoint tubular neighbourhoods of the components of L. According to [2], all closed, connected, orientable 3-manifolds arise in this manner, even if one requires each r_i to be ± 1.

As an example, the lens space $L(p, q)$ is the result of surgery on a single unknotted curve using coefficient p/q.

The proof also employs a trick whereby a given surgery description in S^3 may be transformed into another surgery description which yields the *same* 3-manifold. Locate an *unknotted* component L_i of the surgery link L. The complement of an open tubular neighbourhood of L_i is therefore a solid torus. Give this complementary solid torus a twist so that the meridian μ_i of V_i is carried to a curve of type $\tau\lambda_i + \mu_i$. The integer τ describes the number of complete twists, and is positive or negative according as the twist is in a right-or left-handed sense. This changes, in general, the other components of L, forming (with L_i) a new link L'. Figure 2 illustrates the case $\tau = 1$. This twist also changes the appropriate surgery coefficients (so that L' yields the same 3-manifold) according to the formulas (derived in [6]):

$$r'_i = \cfrac{1}{\tau + \cfrac{1}{r_i}}$$

$$r'_j = r_j + \tau(lk(L_j, L_i))^2 \quad \text{if} \quad j \neq i .$$

These formulas are consistent with the conventions $1/0=\infty$, $1/\infty=0$, etc. Another trick which is used is that, of course, any component of a surgery description link which has a coefficient ∞ may simply be erased without changing the homeomorphism type of the 3-manifold thus described.

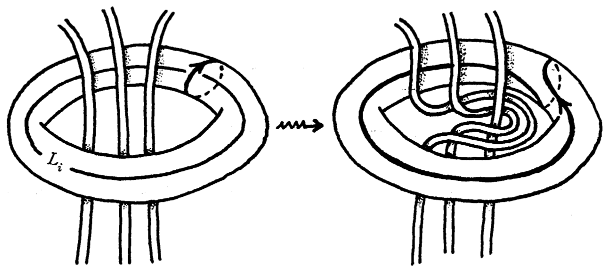

FIGURE 2. Twisting the complement of an unknotted component L_i of L.

Proof of the theorem. The "appropriate" surgery is that corresponding to the surgery coefficient -23. The bulk of the proof is contained in Figure 3. Each of the six surgery descriptions yields the same 3-manifold, according to the discussion above. The knot

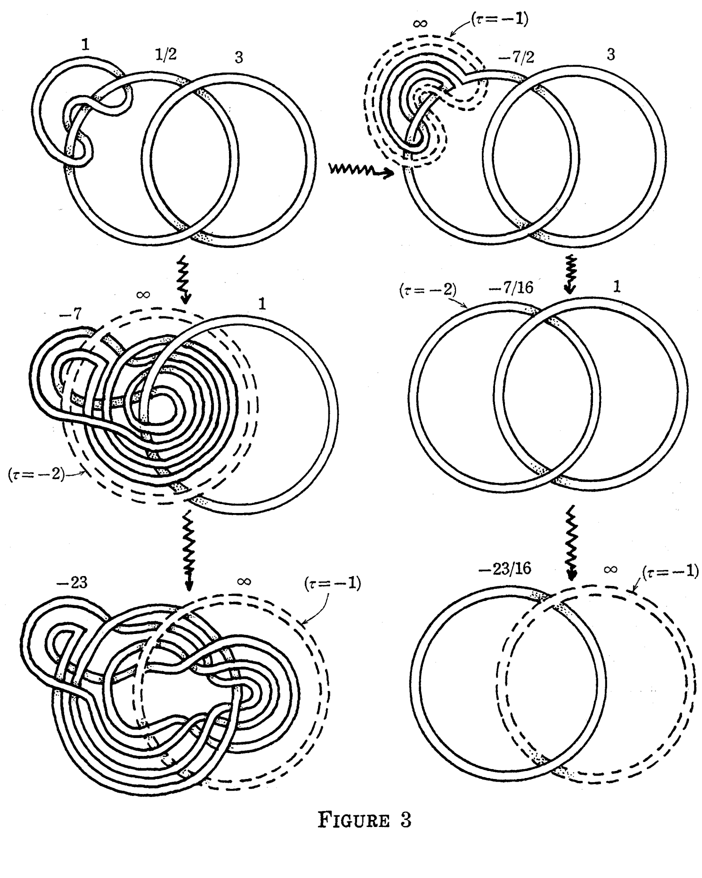

FIGURE 3

at lower left is the same as the knot of Figure 1. The surgery description at lower right shows that the 3-manifold in question is a lens space of type $L(-23, 16)$, which is homeomorphic with $L(23, 7)$. Finally, the knot of Figure 1 is not a torus knot. An easy way to see this is to calculate its Alexander polynomial. Since it is an 11, 2 cable on a trefoil, we calculate by the method of [7]:

$$\begin{aligned}\Delta(t) &= \frac{(t^{22}-1)(t-1)}{(t^{11}-1)(t^2-1)}(t^4-t^2+1) \\ &= t^{14}-t^{13}+t^{10}-t^9+t^8-t^7+t^6-t^5+t^4-t+1\,.\end{aligned}$$

Since this is not of the form $(t^{pq}-1)(t-1)/(t^p-1)(t^q-1)$, it is not a torus knot (cf. [1]).

REMARK. The referee has kindly pointed out an interesting connection between our example and recent work of J. P. Neuzil [5]. He states:

COROLLARY 2. *If K is a knot in S^3 with polynomial $\Delta(t)=a_0+\cdots+a_pt^p$ and $|\alpha|=|a_0+a_2+\cdots|>1$, then $\pi_1(M^3)$ is never a finite cyclic group of even order* (where M denotes any manifold obtained from S^3 by surgery along K).

In our example, $\alpha=6$ and $\pi_1(M^3)$ is a finite cyclic group of order 23.

Added in proof. Jon Simon has also recently discovered our example using different methods, which show that lens spaces arise from surgery on certain iterated cable knots. Can one construct a lens space from a knot which is not a cable of a cable of $\cdots$ etc.?

REFERENCES

1. W. Burau, *Kennzeichnung der Schlauchknoten*, Hamburger Abh., **36** (1932), 125-133.
2. W. B. R. Lickorish, *A representation of orientable combinatorial 3-manifolds*, Ann. of Math., (2) **76** (1962), 531-540.
3. L. Moser, *Elementary surgery along a torus knot*, Pacific, J. Math., **38** (1971), 737-745.
4. J. P. Neuzil, *Elementary surgery manifolds and elementary ideals I*, to appear.
5. ———, *Elementary surgery manifolds and elementary ideals II*, to appear.
6. D. Rolfsen, *Knots and Links*, Publish or Perish Lecture Notes Series, Box 7108 Berkeley 94707, 1976.
7. H. Seifert, *On the homology invariants of knots*, Quart. J. Math. Oxford, (2) **1** (1950), 23-32.

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