PACIFIC JOURNAL OF MATHEMATICS
Vol. 72, No. 2, 1977

AFFINE OPEN ORBITS, REDUCTIVE ISOTROPY GROUPS, AND DOMINANT GRADIENT MORPHISMS; A THEOREM OF MIKIO SATO

FRANK SERVEDIO

An algebraic proof is given for a theorem of M. Sato. The theorem gives criteria for the open orbit in a prehomogeneous vector space under a reductive group to be an affine variety. The following conditions are equivalent:

1. $O(G)$ the open orbit is an affine variety.

2. G_X the isotropy subgroup of X in $O(G)$ is reductive.

3. There exists a semi-invariant form P of degree $r \geqq 2$ such that grad P: $V \rightarrow V^*$ is a dominant morphism of affine varieties.

In 1965, Mikio Sato stated a theorem giving characterizations of open affine orbits in real or complex vector spaces under the actions of reductive linear Lie groups. The statement has not appeared published in a European language, but appeared as a remark in Japanese in [8]. "Let (G, V) be a prehomogeneous pair; assume that G is a reductive real or complex algebraic group. The following conditions are equivalent:

(i) H_X, the isotropy subgroup of X in the open dense orbit, is reductive.

(ii) S, the union of singular G-orbits in V, is a union of hypersurfaces $Z(P_1) \cup Z(P_2) \cup \cdots \cup Z(P_m)$.

(iii) There exists a semi-invariant form P for G such that the mapping grad P/P: $V - Z(P) \longrightarrow V^*$ is dominant."

By a prehomogeneous pair (G, V) we mean an algebraic subgroup $G \subseteqq GL(V)$ acting on V, a finite dimensional vector space over $\boldsymbol{R}$ or $\boldsymbol{C}$ such that there is an *open dense orbit* $O(G)$ *in* V; see [9]. A proof of the theorem was not known. The result is striking in that the conditions are superficially quite different; also they are entirely algebraic whereas the theorem appears in the Sugaku article [8] where the techniques are analytic. The theorem is restated and provided with an algebraic proof. The author wishes to gratefully acknowledge the observations and assistance of Takuro Shintani.

Let k be an algebraically closed field of characteristic 0. k shall denote the multiplicative group $k - \{0\}$. V shall always denote a finite dimensional k-vector space and V^* shall be its k-dual. $G \subset GL(V)$ shall denote a closed algebraic subgroup defined over k. The topologies used are always the Zariski topologies on the spaces. A prehomogeneous pair (G, V) is defined as above with this modification. Let $k[V]$ denote the graded affine k-algebra of polynomial functions on V. If $P \in k[V]$, reserve the notations "$Z(P)$" for the Zariski closed subset of V consisting of zeroes of the function P and "U_P" for the Zariski open subset $U_P = V - Z(P)$. If $P \neq 0$, U_P is known to be an affine algebraic variety defined over k, Zariski dense in V; see [7]. Let "$O(G)$" denote the Zariski open orbit of G in V for a prehomogeneous pair (G, V). G acts as a group of automorphisms of $k[V]$ by $\lambda_g P(X) = P(g^{-1}X)$ for all $g \in G$, $P \in k[V]$ and $X \in V$. P *is semi-invariant for* G if there exists a $\chi \in k[G]$ which is a unit in $k[G]$ such that for all $g \in G$, $\lambda_g P = \chi(g)^{-1}P$. $\chi\colon G \to k^{\cdot}$ is a rational character. Define the morphism $\operatorname{grad} P\colon V \to V^*$ of the canonical affine variety structures on V and V^* by setting $(\operatorname{grad} P)(X)$ to be the element of V^* given by $(\operatorname{grad} P)(X)(Z) = (D_Z P)(X)$, for all $Z \in V$, where $D_Z\colon k[V] \to k[V]$ is the k-derivation of degree -1 on the k-algebra $k[V]$. $k[V]$ is canonically isomorphic to the symmetric algebra $S_k(V^*)$ and in either description D_Z is defined by requiring $D_Z(Y) = Y(Z)$ for all $Y \in V^*$. If a basis $\mathscr{B} = \{X_1, \cdots, X_n\}$ is chosen in V and a dual basis $\mathscr{B}^* = \{Y_1, \cdots, Y_n\}$ in V^* such that $Y_j(X_i) = \delta_{ij}$, then $k[V]$ is naturally isomorphic to the polynomial algebra $k[Y_1, \cdots, Y_n]$ and $(\operatorname{grad} P)(X) = \sum_{i=1}^n \partial P/\partial Y_i(X) Y_i$, or in coordinates $(\operatorname{grad} P)(X) = (\partial P/\partial Y_1(X), \cdots, \partial P/\partial Y_n(X))$.

SATO'S THEOREM. *Let (G, V) be a prehomogeneous pair such that G is a reductive algebraic group containing $k^{\cdot}I_V$. The following are equivalent:*

(1) *$O(G)$ is an affine variety defined over k.*

(1′) *$O(G)$ is equal to U_P, for P a nonzero semi-invariant form of degree $r \geqq 2$ for G.*

(2) *For $X \in O(G)$, $G_X = \{g \in G \mid gX = X\}$, the subgroup fixing X in G, is a reductive closed subgroup of G.*

(3) *There exists a nonzero form P of degree $r \geqq 2$ in $k[V]$ semi-invariant for G such that* $\operatorname{grad} P\colon V \to V^*$ *is a dominant morphism.*

(3′) *There exists a nonzero form P in $k[V]$ of degree $r \geqq 2$ semi-invariant for G such that* $\operatorname{grad} P/P\colon V \to V^*$, $X \mapsto 1/P(X)\,(\operatorname{grad} P)(X)$ *is a dominant rational mapping.*

REMARKS AND EXAMPLES. (a) The condition that $\operatorname{grad} P\colon V \to V^*$ is a dominant morphism is equivalent to the condition that the forms $\partial P/\partial Y_i$; $i = 1, \cdots, n$ be algebraically independent over k.

(b) LEMMA. *For a form $P \in k[V]$,* $\operatorname{grad} P\colon V \to V^*$ *is a* *dominant morphism of affine algebraic varieties if and only if* $\operatorname{grad} P/P\colon V \to V^*$ *is a dominant rational mapping.*

Proof. The proof is straightforward in view of the fact that the dominance of the rational mapping is equivalent to the algebraic independence of the rational functions $\partial P/\partial Y_i/P$; $i = 1, \cdots, n$.

This lemma enables us to conclude immediately that (3) and (3′) are equivalent.

(c) The theorem as stated in the *Sugaku* article [8], contains a "non-fatal" error. Statement "(ii)" lacks the requirement that m, the number of hypersurfaces, be greater than 1 or if $m = 1$, that the degree of the form P_1 be greater than 1.

(d) EXAMPLES. (i) If $G = GL(V)$ and $\dim V \geqq 2$, then all statements (1), $\cdots$, (3′) are false; if $\dim V = 1$, then all statements are true with $G = k^{\cdot}$, $G_X = 1$, and $P = Y_1^2$.

(ii) Let $R = Y_1^2 + Y_1^2 + \cdots + Y_n^2$ be a quadratic form on k^n, $G = k^{\cdot} I_V \cdot O(n)$ where $O(n)$ is the orthogonal group of R. Then all statements of the theorem are true. (1) and (1′) are applications of Witt's theorem; $G_X \cong O(n-1)$ a reductive group and $\operatorname{grad} R$ gives a linear isomorphism since R is a nondegenerate quadratic form.

(iii) For $V = k^{4\times 3}$ and $G = k^{\cdot} I_V \cdot \mathrm{Sp}(4) \times O(3)$ there is a semi-invariant form P for G of degree 4. With $X = (X_1, X_2, X_3)$ and $X_i \in k^4$, $P(X) = [X_1, X_2]^2 + [X_2, X_3]^2 + [X_3, X_1]^2$ where $[\ ,\ ]$ is the skew bilinear form on k^4 defining the symplectic group, Sp (4). In this case we have

(1) $O(G)$ is not affine.

(1′) $O(G) \subsetneqq U_P$; in fact $X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in U_P$ but GX, the G-orbit of X has codimension 2 in U_P. $O(G) \subset U_P - GX$.

(2) For $X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in O(G)$, G_X is a unipotent algebraic group of dimension 2.

(3) $\operatorname{grad} P$ is not a dominant morphism. The closure of the image of $\operatorname{grad} P$ has codimension 2 in V^*. See [8], page 141.

Proof of Sato's theorem. (1) *if and only if* (1′). Only "(1) implies (1′)" needs justification. Since $O(G)$ is open in V and is an affine variety, by the result in [5], $V - O(G)$ is an algebraic set of pure codimension 1. Since $k[V]$ is a unique factorization domain, $V - O(G) = Z(P)$ by [7]. Thus $O(G) = U_P$ for some $P \neq 0$ in $k[V]$. Clearly P must be G-semi-invariant, and P must be a form since $k \cdot I_V \subset G$. The form P must have degree $r \geqq 2$; for if $r = 1$, we may assume $P = Y_1$ and then $U_P = \{X \in V \mid Y_1(X) \neq 0\}$. $Z(Y_1) = \{X \in V \mid Y_1(X) = 0\}$ is a G-invariant subspace of codimension 1. Since G is reductive there exists a complementary G-invariant subspace, a line $Z(Y_2, \cdots, Y_n)$ on appropriate choice of basis $\mathscr{B}$. But $Z(Y_2, \cdots, Y_n) \cap U_{Y_1}$ is nonempty unless $\dim V = 1$, where the theorem has been verified. However, $Z(Y_2, \cdots, Y_n) \cap U_{Y_1}$ being nonempty contradicts $O(G) = U_P$.

(2) *implies* (1). Since G_X is a closed subgroup of G acting on G by right translation and since G_X is reductive, Mumford's theorem enables us to conclude that the quotient variety G/G_X is an affine variety; see [4]. However, the action of G_X on the image of the orbit mapping Gor: $XG \to O(G)$ is isomorphic

$$g \longmapsto gX$$

to the action of G_X on G by right translation and thus is a quotient morphism in the sense of [1]. Hence, $G/G_X \cong O(G)$. Therefore, $O(G)$ is affine.

(1) implies (2). As above, $G/G_X \cong O(G)$. With G reductive and k of characteristic O, and $O(G)$ affine, Theorem 3.5 in [2] allows us to conclude that G_X is reductive.

The equivalence of (3) and conditions (1) and (2) is seen more easily if the following lemmas are established. First, fix some notation. Let $A \in \mathrm{Hom}_k(V, W)$, "$A^*$" shall always denote the transpose of A. Thus $A^* \in \mathrm{Hom}_k(W^*, V^*)$ is defined by the requirement that $(A^*Y)(X) = Y(AX)$ for all $X \in V$ and all $Y \in W^*$.

LEMMA 1. *There is a k-linear isomorphism $T: V \to V^*$ such that $T^* = T$ and an automorphism $i: G \to G$ of order 2 over k such that for all $g \in G$, and for all $X \in V$, $T(gX) = (i(g)^*)^{-1}T(X)$.*

Proof. There is a $k = C$ version of this in [9], Lemma 1.1 on page 135. One can justify the result for k by proving Lemma 2 below and then using it to obtain the result for G, whose Lie algebra is L by imitating the techniques used in [10].

LEMMA 2. *Let L be a reductive algebraic Lie subalgebra of $LGL(V)$, the Lie algebra of $GL(V)$. There is a k-linear isomorphism $T: V \to V^*$ such that $T = T^*$ and a Lie algebra automorphism i' of L of order 2 such that for all $A \in L$, for all $X \in V$,*

$T(AX) = -i'(A)^* T(X)$.

Sketch of proof of Lemma 2. $L \cong \tau \times L'$ where τ is an algebraic torus and L' is the derived subalgebra of L, k-split semi-simple; see [3]. For i', send elements of τ to their negatives and specify i' on L' by sending each root to its negative and extend on a system of canonical generators of L' as described in [6]. T is specified by sending each element of a basis of weight vectors of L' in V to its correspondent in a dual basis of V^*. This suffices to verify Lemma 2.

LEMMA 3. *If P is a semi-invariant form in $k[V]$ for G, then for all $g \in G$, for all $X, U \in V$,* $\operatorname{grad} P(gX)(gU) = \chi(g) \operatorname{grad} P(X)(U)$. *Equivalently, for all $g \in G$, for all $X \in V$,* $\chi(g) g^{*-1} \operatorname{grad}(P) = \operatorname{grad} P(gX)$.

Proof. Let t be transcendental over k. $\operatorname{grad} P(X)(U)$ is the coefficient of t in the $k[t]$-polynomial $P(X + tU)$; see [11]. The identity $\chi(g)P(X + tU) = P(g(X + tU)) = P(gX + tgU)$ establishes the lemma.

Let $G^* = \{g^* \mid g \in G\}$. From *Lemma* 1, it follows that (G^*, V^*) is a prehomogeneous pair. Let $O(G^*)$ be the open orbit in V^*. Since k is algebraically closed and $T^* = T$, there exists a choice of basis $\mathscr{B} = \{X_1, \cdots, X_n\}$ such that $T\mathscr{B} = \{TX_1, \cdots, TX_n\}$ is the dual basis to $\mathscr{B}$, namely $(TX_i)(X_j) = \delta_{ij}$ for $i, j = 1, \cdots, n$. Such a basis $\mathscr{B}$ will becalled an *orthogonal basis*. Any change of basis by an orthogonal transformation results again in an orthogonal basis. As above, let $\mathscr{B}^* = \{Y_1, \cdots, Y_n\}$ denote the dual basis of $\mathscr{B}$.

LEMMA 4. *For (G, V) prehomogeneous with G reductive and P a semi-invariant for G, there exists an orthogonal basis $\mathscr{B}$ for V with $X_1 \in O(G)$ and $c \neq 0$ such that* $\operatorname{grad} P(X_1) = cY_1$ *if and only if* $\operatorname{grad} P\colon V \to V^*$ *is a dominant morphism.*

Proof. For a basis $\mathscr{B}$ let the $n \times 1$ matrix of coordinates or basis coefficients for $X \in V$ be denoted by $X_{\mathscr{B}}$, the $n \times n$ matrix of $A \in \operatorname{End}_k(V)$ be denoted by $A_{\mathscr{B}}$ and the $1 \times n$ matrix of dual basis coefficients of $Y \in V^*$ be denoted by $Y_{\mathscr{B}}$. Note that $Y(AX) = Y_{\mathscr{B}} A_{\mathscr{B}} X_{\mathscr{B}}$. For an orthogonal basis $\mathscr{B}$, $Y_{\mathscr{B}}^{\text{transpose}} = (T^{-1}Y)_{\mathscr{B}}$. The conditions of Lemmas 1 and 3 give $i(g)X = T^{-1}g^{-1*}TX$ and

$$\operatorname{grad} P(gX)_{\mathscr{B}}^{\text{transpose}} = (\chi(g)T^{-1}g^{-1*} \operatorname{grad} P(X))_{\mathscr{B}} .$$

Hence if $\mathscr{B}$ is an orthogonal basis and $X_1 \in O(G)$ and $\operatorname{grad} P(X_1) = cY_1 = cTX_1$ with $c \neq 0$, then

$$\operatorname{grad} P(gX_1)^{\text{transpose}}_{\mathscr{B}} = c(i(g)X_1)_{\mathscr{B}} = ci(g)_{\mathscr{B}} X_{1\mathscr{B}} \; .$$

Since i is an automorphism of G and $X_1 \in O(G)$, the first column of coordinate functions of G in basis $\mathscr{B}$ are algebraically independent. Hence the coordinate functions of $\operatorname{grad} P$ are algebraically independent.

Conversely, if $\operatorname{grad} P$ is dominant, then the rational mapping $\operatorname{grad} P/P$: $V \longrightarrow V^*$ has the property that $\operatorname{grad} P/P(O(G))$ contains a Zariski open subset U of V^* such that $k^{\cdot} U \subset U$. Hence by the proposition below $\operatorname{grad} P/P(O(G))$ contains a vector Y_1 which may be completed to an orthogonal basis. Let X_1 be such that

$$\frac{1}{r} \frac{\operatorname{grad} P}{P}(X_1) = Y_1 \; .$$

Since $O(G) \subset U_P$,

$$Y_1(X_1) = \frac{1}{r} \frac{\operatorname{grad} P}{P}(X_1)(X_1) = 1 \; .$$

Now complete $\{X_1\}$ to an orthogonal basis for V.

PROPOSITION. *Let U be a Zariski open subset of V such that $k^{\cdot} U \subset U$, and let R be a nondegenerate quadratic form on V. Then U contains an orthogonal basis with respect to R.*

Proof. $U \cap U_R$ is open and nonempty. Therefore there is an $X_1 \in U \cap U_R$ such that $R(X_1) = 1$. Let $Y_1 = R(X_1, \ )$ be the linear (polynomial) function on V given by the symmetric bilinear form associated to R. $Z(Y_1)$ is the closed subset of V with underlying point set equal to the vector space $Y_1^{\perp}$. $R_1(=R$ restricted to $Y_1)$ is a nondegenerate quadratic form. Consider $U \cap U_R \cap Z(Y_1)$. If the latter is nonempty choose X_2 as above in the choice of X_1 for this vector space $Y_1^{\perp}$. If $U \cap U_R \cap Z(Y_1)$ is empty, then $Z(Y_1) \subset Z(R) \cup S$, where $S = V - U$. $Z(Y_1)$ is an irreducible closed set. Hence $Z(Y_1) \subset S = Z(R_1, \cdots, R_m)$, where R_i, $i = 1, \cdots, m$ are forms in $k[V]$. Equivalently the following inclusion of ideals holds;

$$(Y_1) \supset (R_1, \cdots, R_m) \; .$$

It is clear now that an X_1' could be chosen, as was X_1 for which $(Y_1') \not\supset (R_1, \cdots, R_m)$ so that $U \cap U_R \cap Z(Y_1')$ is not empty. Proceed inductively until an orthogonal basis is chosen in V.

In characteristic 0, it is well known that if a closed algebraic subgroup of $GL(V)$ has a reductive Lie algebra, then that subgroup is reductive; see [4], Proposition 3.31 and [3].

(3) *implies* (2). For this part of the proof we use the Lie algebras of $GL(V)$, G and G_X which we denote by $LGL(V)$, L and L_X respectively. These are algebraic Lie algebras over k. We show G_X reductive by showing L_X reductive. $LX = \{AX \mid A \in L\}$ is canonically isomorphic to the tangent space of the orbit GX at X. Hence $L_X = \{A \in L \mid AX = 0\}$. For $X \in O(G)$, L_X has codimension n in L since the dimension of the orbit $O(G) = GX$ is n. We use the following criterion of reductivity for algebraic Lie algebras.

LEMMA 5. *Let $L \subset LGL(V)$ be an algebraic Lie subalgebra. L is reductive if and only if the trace form restricted to $L \times L$ is nondegenerate.*

Proof. See [3].

The trace form is nondegenerate when restricted to L. We need show that the trace form restricted to L_X is nondegenerate for $X \in O(G)$. We show that L_X can be defined under the trace form as the subspace orthogonal and complementary to a subspace of L of codimension n. As above, choose an orthogonal basis where $X_1 \in O(G)$ and $\operatorname{grad} P(X_1) = cY_1$, $c \neq 0$. Since $\operatorname{grad} P(gX) = \chi(g)g^{*-1} \operatorname{grad} P(X)$, we see that $L_X = L_{\operatorname{grad} P(X)}$ where $L_{\operatorname{grad} P(X)} = \{A \in L \mid A^* \operatorname{grad} P(X) = 0$ in $V\}$. $\operatorname{grad} P$ is dominant implies that $\operatorname{grad} P(X_1)$ lies in the open orbit $O(G^*)$ in V^* and hence $L_{\operatorname{grad} P(X_1)}$ is also of codimension n in L. Hence $L_{X_1} = L_{\operatorname{grad} P(X_1)}$. With the basis chosen as above, $AX_1 = 0$ if and only if $A_{i1} = 0$ for $i = 1, \cdots, n$ if and only if Trace $AE_{1j} = 0$ for $j = 1, \cdots, n$ where E_{1j} is the $n \times n$ matrix with first row $(0, \cdots, 0, 1, 0, \cdots, 0)$ with 1 in the jth place and other rows zero if and only if Trace $AE'_{1j} = 0$ where $E'_{1j} \equiv E_{1j}$ modulo the annihilator of L under the trace form and $E'_{1j} \in L$. Let M_{X_1} be the subspace spanned by E'_{1j} in L. The criterion $L_{X_1} = L_{Y_1}$ implies immediately that $L_{X_1} \cap M_{X_1} = 0$. Hence L_{X_1} is reductive.

(1′) *implies* (3). We assume that $O(G) = U_P$. Recall that the dual pair (G^*, V^*) is a prehomogeneous vector space with a corresponding form $Q \in k[V^*]$ of degree r; Lemma 1 gives this. $\operatorname{grad} P$ sends G orbits to G^* orbits; i.e., $\operatorname{grad} P(GX) = G^* \operatorname{grad} P(X)$ for all $X \in V$. Lemma 3 implies this easily. Let R be the quadratic form associated to the k-vector space mapping $T\colon V \to V^*$ of Lemma 1, so that $R(X) = T(X)(X)$. We may choose an $X_1 \in O(G) \cap U_R$ and assume that $P(X_1) = 1$ and that X_1 is a member of an orthogonal basis $\mathscr{B}$. Then $P = Y_1^r + Y_1^{r-1}P_1 + \cdots + Y_1P_{r-1} + P_r$ with $P_i \in k[Y_2, \cdots, Y_n]$ of degree i. We compute easily that $\operatorname{grad} P(X_1) = rY_1 + P_1$. Since $\mathscr{B}$ is an orthogonal basis, $Q = X_1^r + X_1^{r-1}Q_1 + \cdots + X_1Q_{r-1} + Q_r$ where $Q_i \in k[X_2, \cdots, X_n]$ is of degree i and is the correspondent of P_i. Thus Q_i is P_i with Y replaced by X. We establish that $Q(\text{grad } P(X_1)) \neq 0$. For any $g \in G$,

$$Q(\text{grad } P(gX_1)) = Q(\chi(g)g^{-1} * \text{grad } P(X_1)) = \chi(g)^r Q(g^{-1} * (rY_1 + P_1)) .$$

It suffices to compute $Q(g * (rY_1 + P_1))$.

$$\begin{aligned} Q(g*(rY_1 + P_1)) &= X_1^r(g * (rY_1 + P_1)) \\ &\quad + X_1^{r-1}(g*(rY_1 + P_1))Q_1(g*(rY_1 + P_1)) + \cdots + Q_r(g*(rY_1 + P_1)) \\ &= (gX_1)^r(rY_1 + P_1) + (gX_1)^{r-1}(rY_1 + P_1)gQ_1(rY_1 + P_1) \\ &\quad + \cdots + gQ_r(rY_1 + P_1) \\ &= \left(\sum_{i=1}^n g_{i1}X_i\right)^r(rY_1 + P_1) + \left(\sum_{i=1}^n g_{i1}X_i\right)^{r-1}(rY_1 + P_1)gQ_1(rY_1 + P_1) \\ &\quad + \cdots + gQ_r(rY_1 + P_1) \\ &= \left(rg_{11} + \sum_{i=2}^n g_{i1}X_i(P_1)\right)^r + \left(rg_{11} + \sum_{i=2}^n g_{i1}X_i(P_1)\right)^{r-1} gQ_1(rY_1 + P_1) \\ &\quad + \cdots + gQ_r(rY_1 + P_1) . \end{aligned}$$

The latter is a nonzero polynomial expression of the type

$$r^r g_{11}^r + g_{11}^{r-1}S_{r-1}(g) + \cdots + g_{11}S_1(g) + S_0(g)$$

with $S_i(g)$ polynomial expressions in the coordinate functions g_{1m} with $(1, m) \neq (1, 1)$. This polynomial cannot be the zero polynomial, since otherwise g_{11} is algebraically dependent on the g_{1m} with $(1, m) \neq (1, 1)$ and this contradicts that the point $X_1 \in O(G)$. This completes the proof of the theorem.

A description of all prehomogeneous pairs (G, V) over k with G acting irreducibly on V is being sought. The examples such as (iii) with $\mathrm{Sp}\,(2n)$, $n \geqq 2$, are the only ones known where there exists a semi-invariant P and the condition $O(G) \subsetneqq U_P$ maintains. We have shown that grad $P(O(G))$ is contained in a proper G-invariant closed subvariety of U_Q in V^*. In general grad P restricted to $Z(P)$ fails to have the property of being a dominant mapping to $Z(Q)$ even when the conditions of the theorem hold; an example is $G \cong k^\cdot \cdot SL(n) \times SL(n)$ acting on $k^{n\times n}$ with $(c, g_1, g_2)X = cg_1Xg_2^{-1}$ and $P =$ determinant.

References

1. Armand Borel, *Linear Algebraic Groups*, 1969, W. A. Benjamin, (6.1) and (6.8).
2. Armand Borel and Harish Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math., **75** no. 3, (1962), Theorem 3.5.
3. Bourbaki, *Elements of Mathematics*, Lie Groups and Lie Algebras, part I, Addison-Wesley Publishing Company, 1975. Ch. I, § 6.4.
4. John Fogarty, *Invariant Theory*, W. A. Benjamin, 1969, p. 159.

5. Jacob Eli Goodman, *Affine open subsets of algebraic varieties*, Ann. of Math., **89** (1969), 162.
6. Nathan Jacobson, *Lie Algebras*, Interscience Publishers, 1962, Ch. IV, §3 and Theorem 3, 127.
7. David Mumford, *Introduction to Algebraic Geometry*, Harvard University Lecture Notes, Propn. 4, p. 87.
8. Mikio Sato, *The theory of prehomogeneous vector spaces*, Sugaku no Ayumi 15-1 (1970), 85-157, notes by T. Shintani.
9. Takuro Shintani, *On Dirichlet series whose coefficients are class numbers of integral binary cubic forms*, J. Math. Soc. Japan, **24**, no. 1, 132-188.
10. R. Steinberg, *Lectures on Chevalley Groups*, Yale University Lecture Notes, 1968, Theorem 29.
11. Hermann Weyl, *The Classical Groups*, Princeton University Press, 1946, p. 5.

Received November 4, 1976 and in revised form March 23, 1977.

McMaster University
Hamilton, Ontario, Canada