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ANOTHER NOTE ON EBERLEIN COMPACTS

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An Eberlein compact is a compact space that can be embedded in a Banach space with its weak topology. It is shown that: If X is compact and if $X=M_1 \cup M_2$ with M_1 and M_2 metrizable, then $\bar{M}_1 \cap \bar{M}_2$ is metrizable and X is an Eberlein compact. This answers a question of Arhangel'skiĭ.

1. **Introduction.** An *Eberlein compact*, or *EC*, is a compact space[1] which can be embedded in a Banach space with its weak topology. For background and various properties of these spaces, the reader is referred to [1] or the authors' preceding note [3].

Since every metrizable space can be embedded in a Banach space with its norm topology, every metrizable compact space is clearly an *EC*. The purpose of this note is to prove the following stronger result, thereby answering a question of A. V. Arhangel'skiĭ.

THEOREM 1.1. *If X is compact, and if $X = M_1 \cup M_2$ with M_1 and M_2 metrizable, then $\bar{M}_1 \cap \bar{M}_2$ is metrizable and X is an EC.*

In contrast to Theorem 1.1, a compact space which is the union of *three* metrizable subsets need *not* be an *EC*, or even a Fréchet space[2] (see [2, Example 6.2][3]). However, it was shown in [5] that a compact space which is the union of countably many metrizable subsets must at least be sequential (a property somewhat weaker than being a Fréchet space).

2. **Proof of Theorem 1.1.** We first show that $M = \bar{M}_1 \cap \bar{M}_2$ is metrizable. For $i = 1, 2$, let $\mathscr{U}_i$ be a σ-discrete—hence σ-disjoint—base for M_i. For each $U \in \mathscr{U}_i$, choose an open set $\phi_i(U)$ in X such that $\phi_i(U) \cap M_i = U$. Let $\mathscr{U} = \{\phi_i(U) \cap M: U \in \mathscr{U}_i, i = 1, 2\}$. Then $\mathscr{U}$ is easily seen to be a σ-disjoint $1—m$ hence point-countable $1—m$ base for M. Since M is compact, it must therefore be metrizable by a result of A. S. Miščenko [4].

Since M is compact and metrizable, it has a countable base (B_n). For each pair (m, n) such that $\bar{B}_m \cap \bar{B}_n = \emptyset$, pick an open F_σ-set

[1] All spaces in this paper are Hausdorff.

[2] X is a *Fréchet* space if, whenever $x \in \bar{A}$ in X, then $x_n \to x$ for some $x_n \in A$. Every *EC* is a Fréchet space by a theorem of Eberlein and Šmulian (see [1, Theorem 4.1]).

[3] In this example, the three metrizable subsets are actually discrete, and one of them is an open set whose complement is (necessarily, by Theorem 1.1) an *EC*.

$V_{m,n}$ in X such that $B_m \subset V_{m,n}$ and $V_{m,n} \cap B_n = \emptyset$. Let $\mathscr{V}$ be collection of all such $V_{m,n}$.

Let $Y = X - M$. Then Y is the union of the two disjoint, open metrizable subsets $X - \bar{M}_1$ and $X - \bar{M}_2$, so Y is metrizable. Since Y is open in X, it therefore has a σ-disjoint base $\mathscr{W}$ such that $\bar{W} \subset Y$ for all $W \in \mathscr{W}$. Clearly each $W \in \mathscr{W}$ is an open F_σ in X.

Finally, let $\mathscr{S} = \mathscr{V} \cup \mathscr{W}$. Then $\mathscr{S}$ is a σ-disjoint, separating (in the sense of [3, Definition 1.3]) cover of X by open F_σ-sets, so X is an *EC* by a characterization of H. P. Rosenthal (see [6, Theorem 3.1] or [3, Theorem 1.4]).

3. Concluding remarks.

(3.1) The proof of Theorem 1.1 actually establishes the following somewhat sharper results.

(a) If X is regular, and if $X = \bigcup_{n=1}^{\infty} X_n$ with each X_n having a σ-disjoint base, then $\bigcap_{n=1}^{\infty} \bar{X}_n$ has a σ-disjoint base.

(b) If X is compact, and if $X = \bigcup_{n=1}^{\infty} X_n$ with each X_n metrizable, then $\bigcap_{n=1}^{\infty} \bar{X}_n$ is metrizable.

(c) If X is compact, and $X = X_1 \cup X_2$ with X_1 and X_2 having σ-disjoint bases, then X has a σ-disjoint, separating collection of open F_σ-subsets.

Observe that not every *EC* staisfies the conclusion of (c), as can be seen from the space of all points in $\{0, 1\}^{\omega_1}$ which have at most two nonzero coordinates.

(3.2). Somewhat in the spirit of Theorem 1.1, one can show that if $X = \bigcup_{i=1}^{n} X_i$, and if each X_i is an *EC*, then X is an *EC*: In fact, X is then the image under the obvious perfect map of the topological sum $\sum_{i=1}^{n} X_i$, and this sum is clearly an *EC*, so X must be an *EC* by [1, Theorem 2.1] (see also [3, Theorem 1.1]).

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