PACIFIC JOURNAL OF MATHEMATICS
Vol. 72, No. 2, 1977

ON THE METRIZABILITY OF k_ω-SPACES

STANLEY P. FRANKLIN AND BARBARA V. SMITH THOMAS

The metrizability of a k_ω-space can be characterized in terms of its k_ω-structure by whether or not it contains one of two canonical subspaces.

A natural generalization of countable *CW*-complexes, the k_ω-spaces, have recently appeared in papers on topological groups. (See, for example, [2], [5], and [6].) A decomposition of a space $X = \bigcup_{n=1}^{\infty} X_n$ where each X_n is a compact Hausdorff space, the X_n are increasing, and X has the weak topology of the X_n is called a *k_ω-decomposition* of X. Any space with a k_ω-decomposition is a *k_ω-space*. For example, $\boldsymbol{R} = \bigcup_{n=1}^{\infty} [-n, n]$ is a k_ω-space.

One metrizability theorem for k_ω-spaces is a corollary of known results. *A k_ω-space is metrizable iff it is second countable.* (Morita has shown that each k_ω-space is (normal and) Lindelöf [4], and hence, if metrizable, is second countable. Conversely, a second countable regular space is metrizable.)

Since k_ω-spaces are composed of compact pieces, it is natural to hope that the metrizability of the pieces will yield that of the space. The example of a nonlocally finite countable *CW*-complex shows this to be a vain hope. We are left with the question of exactly how the process can fail, which is the subject of this note.

We will need two examples. The first, called the *sequential fan*, is the union of countably many convergent sequences with their limit points identified. The pieces (finitely many convergent sequences plus limit point) are metrizable. The sequential fan is a k_ω-space (Morita has characterized them as quotients of locally compact Lindelöf spaces [4]). However it is not first countable and hence not metrizable.

The second example will require more description. Think of S_2, as it is called [1], as consisting of a sequence $\{s_j\}$ converging to a point s_0, together with another sequence of isolated points $\{s_{j,i}\}$ converging to each s_j. Take the topology resulting from thinking of S_2 as a quotient of a disjoint sum of countably many convergent sequences with limits. Clearly, then S_2 is a k_ω-space. One k_ω-decomposition is given by $S_2 = \bigcup_{n=1}^{\infty} X_n$, where

$$X_n = \{s_0\} \cup \{s_j \mid j \in N\} \cup \{s_{j,i} \mid j \leqq n,\ i \in N\}\ .$$

Each X_n is metrizable but S_2 is not even a Fréchet space.

Are there other examples? Not any essentially different ones.

THEOREM. *A k_ω-space with metrizable "pieces" is metrizable iff it contains no copy of S_2 and no sequential fan.*

For the proof we will use a short sequence of lemmas and propositions.

LEMMA 1 (Steenrod [7]). *If $X = \bigcup_{n=1}^{\infty} X_n$ is a k_ω-decomposition, then each compact subset of X is contained in some X_n.*

LEMMA 2. *Any subsequence of a k_ω-decomposition is again a k_ω-decomposition.*

In fact, one easily shows that given a k_ω-decomposition $X = \bigcup_{n=1}^{\infty} X_n$ and another increasing cover $X = \bigcup_{n=1}^{\infty} X'_n$, the X'_n form a k_ω-decomposition iff each X_n is contained in some X'_n. The lemma follows.

LEMMA 3. *A closed subspace Y of a k_ω-space $X = \bigcup_{n=1}^{\infty} X_n$ has a k_ω-decomposition $Y = \bigcup_{n=1}^{\infty} (X_n \cap Y)$.*

The heart of the matter lies in the following.

PROPOSITION 1. *Suppose $X = \bigcup X_n$ is a k_ω-decomposition with each X_n first countable and that X is not first countable. If X is Fréchet it contains a sequential fan. If X is not Fréchet, it contains a copy of S_2*

Proof. If a point $x_0 \in X$ has no countable neighborhood base, then each of its neighborhoods must meet cofinally many X_n. Let $T_n = X \backslash \bigcup_{i=1}^{n} X_i$ be the nth tail of the k_ω-decomposition. If m is the least integer with $x_0 \in X_m$, thet $x_0 \in \operatorname{cl} T_n \backslash T_n$ for each $n \geqq m$. If X is Fréchet, some sequence $\mathscr{S}_1$ in T_m must converge to x_0. Since $\mathscr{S}_1 \cup \{x_0\}$ is compact, Lemma 1 says $\mathscr{S}_1$ is wholly contained in some X_{n_1} with $n_1 > m$. But $x \in \operatorname{cl} T_{n_1} \backslash T_{n_1}$ so that some sequence S_2 in T_{n_1} also converges to x_0. In this way we construct a sequence of distinct sequences $\{\mathscr{S}_k\}$, each converging to X_0, such that $\mathscr{S}_k \subseteqq X_{n_k} \cap T_{n_{k-1}}$ with $m < n_1 < n_2 < \cdots < n_k < \cdots$. Then $F = \{x_0\} \cup \bigcup \mathscr{S}_k$ meets each X_{n_k} in a finite union of convergent sequences and is therefore closed (by Lemma 2) in X. Let $D_n = \{x_0\} \cup \bigcup_{k=1}^{n} \mathscr{S}_k$. By Lemma 3 $F = \bigcup D_n$ is a k_ω-decomposition of F and hence F is a quotient of $\bigoplus D_n$. F then is a sequential fan.

The proof of the second assertion is more delicate. It depends on the fact that every sequential space which is not Fréchet contains a subset whose sequential coreflection is $\mathscr{S}_2$ ([1], Prop. 3.1). X is certainly sequential since it is the quotient of a first countable space, namely the disjoint sum of its "pieces". If X is not Fréchet take a subset A, with sequential coreflection $\mathscr{S}_2$, and write it as $A = \{x_0\} \cup A_1 \cup A_2$ with $A_1 = \{x_n | n \in N\}$ and $A_2 = \{x_{n,j} | (n, j) \in N \times N\}$ all distinct points. Sequential convergence in A is the same as in $\mathscr{S}_2$. (See the description of $\mathscr{S}_2$.) Thus x_0 has no countable neighborhood base since $x_0 \in \mathrm{cl}\, A_2$ and no sequence in A_2 converges to it. Thus x_0 can belong to the interior of no X_n. Hence if n_0 is the least integer with $x_0 \in X_{n_0}$, then $x \in \mathrm{bdy} X_n$ for each $n \geqq n_0$. Similarly, no infinite subset of A_1 is contained in the interior of any X_n. Otherwise we would have $x_0 \in X_n - \mathrm{cl}\,(A_2 \cap X_n)$ with no sequence in $A_2 \cap X_n$ converging to x_0, contradicting the first countability of X_n. However, $\{x_0\} \cup A_1$ is compact and thus, by Lemma 1, is contained in some X_n. Let n_1 be the least such n. Then for each $n \geqq n_1$, $\{x_0\} \cup A_1 \subseteqq \mathrm{bdy} X_n$. Write A_2^i for the sequence $\{x_{ij} | j \in N\}$ in A_2. A_2^i converges to x_i. For $n \geqq n_1$, at most finitely many A_2^i can meet X_n infinitely many times because of the first countability of X_n. However, each A_2^i is contained in some X_n since $\{x_i\} \cup A_2^i$ is compact. Choose $m_1 \geqq n_1$ with $A_2^i \subseteqq X_{m_1}$. Let $B_1 = A_2^i$ and let $i_1 = 1$. Let i_2 be the least i such that $A_2^{i2} \cap X_{m_1}$ is infinite. Choose $m_2 > m_1$ with $A_2^{i_2} \subseteqq X_{m_2}$. Let $B_2 = A_2^{i_2} \backslash X_{m_1}$. In this way we recursively define a subsequence $\{X_{m_k}\}$ of $\{X_n\}$ and a sequence $\{B_k\}$ with each B_k an infinite subset of some $A_2^{i_k}$ and with $B_k \subseteqq X_{m_k} \backslash X_{m_{k-1}}$. Let $B' = \{x_{i_k} | k \in N\}$ and $B = \{x_0\} \cup B' \cup \bigcup B_k$. By (9) $X = \bigcup X_{m_k}$ is a k_ω-decomposition of X and

$$B \cap X_{m_k} = \{x_0\} \cup B' \cup \bigcup_{p=1}^{k} B_p$$

is the union of finitely many compact sets and hence is closed. Thus B is closed in X. Being closed B is sequential and thus its own sequential coreflection, in this case clearly $\mathscr{S}_2$. (B, containing x_0, a subsequence of the x_n, and for each such n the corresponding $x_{n,j}$ is sequentially homeomorphic to $\mathscr{S}_2$.)

To complete the proof of the theorem we need only the following analogue of a well known fact about CW-complexes.

PROPOSITION 2. *If $X = \bigcup_{n=1}^{\infty} X_n$ is a k_ω-space with each X_n metrizable, then X is metrizable iff it is first countable.*

Sketch of proof. First countability gives local compactness (Ordman [6]) which, in turn, implies that each X_n is contained in the interior of some subsequent one (cover each point of X_n with a compact neighborhood, reduce to a finite subcover, union and apply Lemma 1). Using Lemma 2, we may assume that $X_n \subseteqq \mathrm{int}\, X_{n+1}$. Choose a countable base for each $\mathrm{int}\, X_n$. Their union is a countable base for X which is hence metrizable.

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Received October 25, 1976 and in revised form March 18, 1977.

MEMPHIS STATE UNIVERSITY
MEMPHIS, TN 38152