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A DECOMPOSITION OF ADDITIVE SET FUNCTIONS

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In this paper it is demonstrated that if μ is an additive function from a field F into the nonnegative reals, then μ can be separated into two mutually singular parts, μ_1 and μ_2, where μ_1 is representable in the sense that its Lebesgue decomposition projection operator has a refinement integral representation and μ_2 is such that for each $E \in F$ the contraction of μ_2 to E is representable iff $\mu_2(E) = 0$. If μ_2 is maximal, then the decomposition is unique.

1. **Introduction.** Suppose S is a set, F a field of subsets of S, $b(F)$ the set of bounded functions from F into $\boldsymbol{R}$, and $ba(F)$ the set of functions in $b(F)$ which are additive on disjoint elements of F. For $H \subseteq ba(F)$ denote by H^+ the set of nonnegative valued elements of H and let μ be in $ba(F)^+$. For $\lambda \in ba(F)^+$ denote by A_λ the set of elements in $ba(F)$ which are absolutely continuous with respect to λ and by α_λ the Lebesgue decomposition projection operator for λ, i.e., for $\eta \in ba(F)$, $\alpha_\lambda(\eta)$ is that part of η which is absolutely continuous with respect to λ [5]. For $\lambda \in ba(F)^+$ we say that λ is representable if there exists a $g: F \to \boldsymbol{R}$ such that $\alpha_\lambda(\eta) = \int g\eta$ for each $\eta \in ba(F)$ in which case g will be said to represent λ.

2. **Preliminary theorems.** All integrals in this paper are refinement limits of sums over finite subdivision of S by elements of F. If $\beta: F \to \boldsymbol{R}$ and $\int_S \beta(I)$ exists we will denote by $\int \beta$ the function $\left\{\left(v, \int_v \beta(I)\right) \middle| v \in F\right\}$. For further details concerning the integral and 2. K. 1 and 2. K. 2 below see [1].

THEOREM 2. K. 1. *If $\alpha: F \to \boldsymbol{R}$ and $\int_S \alpha(I)$ exists, then*

$$\int_S \left|\alpha(v) - \int_v \alpha(I)\right|$$

exists and is zero. Consequently, if $\beta \in b(F)$ and $v \in F$, then $\int_v \beta(I) \int_I \alpha(J)$ exists iff $\int_v \beta(I)\alpha(I)$ exists in which case they are equal.

Proof. [9].

COROLLARY 2. K. 2. *If $\alpha: F \to \boldsymbol{R}$ and $\beta: F \to \boldsymbol{R}$ and each of*

$\int_S \alpha(I)$ *and* $\int_S \beta(I)$ *exists and* M *is either max or min then* $\int M\{\alpha, \beta\}$ *exists iff* $\int M\left\{\int \alpha, \int \beta\right\}$ *exists in which case they are equal.*

Proof. [1].

Notice that if h represents μ, then for $\lambda \in ba(F)^+$ we have $0 \leqq \alpha_\mu(\lambda) = \int h\lambda \leqq \lambda$ so that $\int h\lambda = \int \max\{0, \min\{h, 1\}\}\lambda$ and therefore h can be replaced by a bounded function. Also any representation for μ which is valid for $ba(F)^+$ is valid for $\eta \in ba(F)$ since $\alpha_\mu(\eta) = \alpha_\mu(\eta^+) - \alpha_\mu(\eta^-)$ [5] where η^+ and η^- are the positive and negative parts of η, respectively. Consequently we will restrict our attention to $ba(F)^+$.

We will also have need of the following theorem due to Appling.

THEOREM 2.A. *If* $\mu \in ba(F)^+$, $\eta \in A_\mu$, $\beta \in b(F)$ *and* $\int \beta\mu$ *exists, then* $\int \beta\eta$ *exists.*

Proof. [3].

If in subsequent statements the existence of a given integral or its equivalence to a given integral is an immediate consequence of the statements of this section, the integral will often only be written and the proof of existence or equivalence will be left to the reader.

3. Two lemmas. By the remarks of the previous section if μ has a representing function, then it has a bounded representing function which, by the following lemma we may assume to be the characteristic function of some subset of F.

LEMMA 3.1. *Suppose* $h \in b(F)$ *and for each* $\lambda \in ba(F)^+$ *we have* $\int h\lambda$ *exists and is equal to* $\int h \int h\lambda$. *Then there exists a* $g\colon F \to \{0, 1\}$ *such that for each* $\lambda \in ba(F)^+$ *we have* $\int g\lambda$ *exists and is equal to* $\int h\lambda$.

Proof. Let $\alpha = h^2$, $\beta = \min\{\alpha, 1\}$ and suppose $\lambda \in ba(F)^+$. It is an easy consequence of 2.K.1 and 2.K.2 that

$$\int \alpha\lambda = \int \alpha^2\lambda = \int \alpha \int \alpha\lambda = \int h\lambda \quad \text{and} \quad \int \beta\lambda = \int \beta \int \beta\lambda = \int \beta^2\lambda\,.$$

Also

$$\int \alpha\lambda \leqq \int \max\{\alpha, 1\}\lambda - \lambda + \lambda \leqq \int \max\{\alpha, 1\}(\max\{\alpha, 1\} - 1)\lambda + \lambda = \lambda$$

hence $\int \beta\lambda = \int \min\{\alpha, 1\}\lambda = \int \alpha\lambda = \int h\lambda$. Now

$$\begin{aligned} 0 &\leqq \int \min\{\beta, 1-\beta\}\lambda = \int (1-\beta)\min\{\beta, 1-\beta\}\lambda + \int \beta \min\{\beta, 1-\beta\}\lambda \\ &= \int \min\{\beta - \beta^2, (1-\beta)^2\}\lambda + \int \min\left\{\int \beta^2\lambda, \int \beta\lambda - \int \beta^2\lambda\right\} \\ &= \int \min\left\{\int \beta\lambda - \int \beta^2\lambda, \int (1-\beta)^2\lambda\right\} + 0 = 0\,. \end{aligned}$$

For each $v \in F$ let $l(v) = \begin{cases}\beta(v) & \text{if } \beta(v) \leqq 1/2 \\ 0 & \text{otherwise.}\end{cases}$ Then $0 \leqq l \leqq \min\{\beta, 1-\beta\}$ so that $\int l\lambda$ exists and is zero. For each $v \in F$ let

$$g(v) = \begin{cases} 1 & \text{if } \beta(v) > \dfrac{1}{2} \\ 0 & \text{if } \beta(v) \leqq \dfrac{1}{2} \end{cases} = \min\{2(\beta(v) - l(v)), 1\}\,.$$

Now by 2.K.2. $\int g\lambda$ exists and we have

$$\begin{aligned} \int g\lambda &= \int \min\{2(\beta - l), 1\}\lambda = \int \min\left\{2\int \beta\lambda - 2\int l\lambda, \lambda\right\} \\ &= \int \min\left\{2\int \beta\lambda, \lambda\right\} = \int \beta\lambda + \int \min\left\{\int \beta\lambda, \lambda - \int \beta\lambda\right\} \\ &= \int \beta\lambda - \int \min\{\beta, 1-\beta\}\lambda = \int \beta\lambda\,. \end{aligned}$$

If D is a subdivision of S, i.e., a finite disjoint subset of F whose union is S, then H is a refinement of D, $H \ll D$, means that H is a subdivision of S and for each $v \in D$ there exists a subset H_v of H whose union is v.

LEMMA 2. *Suppose $\lambda \in ba(F)^+$, (E_i) is a disjoint sequence in F, $B > 0$ and for each $i \in N$ we have $g_i\colon F \to [0, B]$ and $\int g_i\lambda$ exists. Suppose also that if $i \in N$, $I \in F$ and $g_i(I) \neq 0$, then $I \subseteq E_i$. Then for each $v \in F$, $\int_v g(I)\lambda(I)$ exists and is $\sum_{i=1}^{\infty} \int_v g_i(I)\lambda(I)$, where $g(I) = \sum_{i=1}^{\infty} g_i(I)$ for each $I \in F$.*

Proof. Let $v \in F$ and $c > 0$. Let n be such that $\sum_n^{\infty} \lambda(E_i \cap v) < c/4B$. For each $i \leqq n$ let $D_i \ll \{E_i \cap v\}$ be such that if $K \ll D_i$, then $\left|\sum_K g_i(I)\lambda(I) - \int_{v \cap E_i} g_i(I)\lambda(I)\right| < c/2n$. Let

$$D = (\bigcup_{i=1}^{n} D_i) \cup \{v \sim \bigcup_{i=1}^{n} E_i\}$$

and suppose $H \ll D$. Let $H_i = \{I \in H \mid I \subseteqq E_i\}$ for each i and $H' = H \sim \bigcup_{i=1}^{n} H_i$. Note that if $I \in H_i$, then $g_i(I) = g(I)$. Now

$$\begin{aligned}
&\left|\sum_H g(I)\lambda(I) - \sum_1^\infty \int_v g_i(I)\lambda(I)\right| \\
&\quad \leqq \left|\sum_{i=1}^{n}\sum_{H_i} g(I)\lambda(I) - \sum_1^n \int_{v\cap E_i} g_i(I)\lambda(I)\right| \\
&\qquad + \left|\sum_{H'} g(I)\lambda(I)\right| + \left|\sum_{n+1}^{\infty}\int_{v\cap E_i} g_i(I)\lambda(I)\right| \\
&\quad \leqq \sum_1^n \left|\sum_{H_i} g_i(I)\lambda(I) - \int_{v\cap E_i} g_i(I)\lambda(I)\right| \\
&\qquad + \sum \{g(I)\lambda(I) \mid I \in H',\, I \subseteqq E_j \cap v \quad \text{and} \quad j > n\} \\
&\qquad + \sum_{n+1}^{\infty} B\lambda(E_i \cap v) \\
&\quad < \sum_1^n c/2n + \sum_{n+1}^{\infty} B\lambda(E_j \cap v) + B\cdot c/4B \\
&\quad \leqq c/2 + B\cdot c/4B + c/4 = c\,.
\end{aligned}$$

For $v \in F$ denote by x_v the characteristic function of $\{I \in F \mid I \subseteqq v\}$ and by $c_v(\mu)$ the contraction of μ to v, i.e., $c_v(\mu) = \int x_v \mu$.

A linear transformation, T, from $ba(F)$ into $ba(F)$ is in the class $\mathscr{C}$ [2] iff there exists a $K > 0$ such that for each $v \in F$ and ξ in $ba(F)$ we have

$$\int_v |T(\xi)(I)| \leqq K \int_v |\xi(I)|\,.$$

THEOREM 3.A. *If $T \in \mathscr{C}$, $\eta \in ba(F)^+$ and $\delta \in A_\eta$, then $T(\delta) = \int (T(\eta)/\eta)\delta$.*

Proof. [2].

In [4] it was shown that the elements of $\mathscr{C}$ commute. Now, if $v \in F$ and $\lambda \in A_\mu^+$, then c_v, α_μ and α_λ are clearly in $\mathscr{C}$. Therefore for $\xi \in ba(F)$ we have $\alpha_\lambda(\xi) \in A_\mu$, so that

3.c.1. $$\alpha_\lambda(\xi) = \alpha_\mu(\alpha_\lambda(\xi)) = \alpha_\lambda(\alpha_\mu(\xi)) = \int (\alpha_\lambda(\mu)/\mu)\alpha_\mu(\xi)\,,$$

consequently if μ is representable, then λ is also. If we replace λ, in 3.c.1, by $c_v(\mu)$ we have:

3.c.2. $$\alpha_{c_v(\mu)}(\xi) = (c_v \circ \alpha_\mu)(\xi)\,,$$

hence we may say that if g represents $c_v(\mu)$ and $I \in F$ is such that $I \subseteqq v$, then $g \cdot x_I$ represents $c_I(\mu)$.

4. The decomposition. Suppose $R \subseteqq F$ is a ring of subsets of S such that $I \in F$ and $I \subseteqq v \in R$ implies that $I \in R$, then if f is the characteristic function of R and $\lambda \in ba(F)^+$ the expression $\sum_D f(I)\lambda(I)$ is nondecreasing for successive refinements and bounded by $\lambda(S)$ so that $\int f\lambda$ exists.

THEOREM 1. *Suppose $R \subseteqq F$ is a ring of subsets of S for which $I \in F$ and $I \subseteqq v \in R$ imply $I \in R$. Suppose further that $c_v(\mu)$ is representable for each $v \in R$ and $\int f\mu = \mu$ where f is the characteristic function of R. Then μ is representable.*

Proof. Since $\mu = \int f\mu$ we have for each n there exists $D_n \ll \{S\}$ such that if $E \ll D_n$, then $\mu(S) - \sum_E f(I)\mu(I) < 1/n$ and D_n can be chosen so that $D_n \ll D_{n-1}$. Therefore if $v_n = \bigcup \{I \in D_n \mid f(I) = 1\}$, then $v_n \subseteqq v_{n+1}$ and $\mu(S \sim v_n) < 1/n$ for each n. Let $E_1 = v_1$ and $E_i = v_i \sim v_{i-1}$ for $i > 1$. For each i let $\mu_i = c_{E_i}(\mu)$ and $g_i \colon F \to \{0, 1\}$ be such that $g_i \cdot x_{E_i} = g_i$ and $\alpha_{\mu_i}(\lambda) = \int g_i\lambda$ for each $\lambda \in ba(F)^+$. Let $g = \sum_1^\infty g_i$ and suppose $\lambda \in ba(F)^+$. By Lemma 2, $\int g\lambda$ exists and is $\sum_1^\infty \int g_i\lambda$ and for each i we have $\alpha_{\mu_i}(\lambda) = \int g_i\lambda \in A_{\mu_i} \subseteqq A_\mu$ and therefore $\int g\lambda \in A_\mu$. Thus, if $\lambda = \int g\lambda$, then $\lambda \in A_\mu$.

Now suppose $\lambda \in A_\mu^+$. Let $c > 0$ and n be such that $\mu(I) < 1/n$ implies that $\lambda(I) < c$. Then

$$\begin{aligned} 0 &\leqq \lambda(S) - \int_S g(I)\lambda(I) \leqq \lambda(S) - \sum_1^n \int_S g_i(I)\lambda(I) = \lambda(S) - \sum_1^n \alpha_{\mu_i}(\lambda)(S) \\ &= \lambda(S) - \sum_1^n c_{E_i} \circ \alpha_\mu(\lambda)(S) = \lambda(S) - \sum_1^n \alpha_\mu(\lambda)(E_i) \\ &= \lambda(S) - \sum_1^n \lambda(E_i) = \lambda(S) - \lambda(v_n) = \lambda(S \sim v_n) < c \,. \end{aligned}$$

Therefore $\lambda \in A_\mu$ iff $\lambda = \int g\lambda$.

Now, as previously established, $\int g\lambda \in A_\mu$. Since $\int g\lambda \leqq \lambda$ it follows that $\int g\lambda \leqq \alpha_\mu(\lambda) = \int g\alpha_\mu(\lambda) \leqq \int g\lambda$, hence g represents μ.

If η and δ are in $ba(F)^+$ we will say that they are mutually singular if whenever $\lambda \in ba(F)^+$ and $\lambda \leqq \eta$ and $\lambda \leqq \delta$, then $\lambda = 0$. This is the notion of singularity used in [5] and [10] which is equivalent to that of [6]. It is also equivalent to $\int \min\{\eta, \delta\} = 0$.

Since η and δ are only finitely additive we cannot, necessarily, obtain disjoint sets s_1 and s_2 such that $\eta(s_1) = \delta(s_2) = 0$ with $s_1 \cup s_2 = t$.

THEOREM 2. *There exist μ_1 and μ_2 in $ba(F)^+$ such that:*

(1) *μ_1 and μ_2 are mutually singular and $\mu = \mu_1 + \mu_2$.*

(2) *μ_1 is representable.*

(3) *For each $v \in F$ we have $c_v(\mu_2)$ is representable iff $\mu_2(v) = 0$.*

(4) *If μ_3 is in $ba(F)^+$, $\mu_2 \leqq \mu_3 \leqq \mu$ and for each $v \in F$ we have $c_v(\mu_3)$ is representable iff $\mu_3(v) = 0$, then $\mu_2 = \mu_3$.*

Proof. If I, $v \in F$, $I \subseteqq v$ and h represents $c_v(\mu)$, then by 3.c.2. $x_I \cdot h$ represents $c_I(\mu)$. Consequently $R = \{v \in F \mid c_v(\mu) \text{ is representable}\}$ is a ring satisfying the conditions of Theorem 1 since for I and v in R with h, k representing $c_I(\mu)$ and $c_v(\mu)$ respectively we have $h + x_{v\sim I} \cdot k$ represents $c_{I\cup v}(\mu)$. Let f be the characteristic function of R and $\mu_1 = \int f\mu$. Then for each $v \in R$ we have

$$c_v(\mu_1) = \int x_v \mu_1 = \int x_v \int f\mu = \int x_v f\mu = \int x_v \mu = c_v(\mu)$$

so that $c_v(\mu_1)$ is representable. Also

$$\int f\mu_1 = \int f \int f\mu = \int f^2\mu = \int f\mu = \mu_1$$

and thus, by Theorem 1, μ_1 is representable. Let $\mu_2 = \mu - \mu_1 = \int (1 - f)\mu$ and note that μ_1 and μ_2 are mutually singular since $\min\{f, 1 - f\} = 0$. Therefore for $\lambda \in ba(F)^+$ we have $\alpha_{\mu_1}(\lambda)$ and $\alpha_{\mu_2}(\lambda)$ are mutually singular hence

$$\alpha_{\mu_1}(\lambda) + \alpha_{\mu_2}(\lambda) = \int \max\{\alpha_{\mu_1}(\lambda), \alpha_{\mu_2}(\lambda)\} \leqq \alpha_\mu(\lambda) \leqq \alpha_{\mu_1}(\lambda) + \alpha_{\mu_2}(\lambda)\,,$$

i.e., $\alpha_{\mu_1} + \alpha_{\mu_2} = \alpha_\mu$. Now suppose $v \in F$ and $c_v(\mu_2)$ is representable, then $c_v(\mu) = c_v(\mu_1) + c_v(\mu_2)$ is representable so that $v \in R$. Therefore $f(I) = 1$ for each $I \in F$ for which $I \subseteqq v$. Hence

$$\mu_2(v) = \int_v (1 - f(I))\mu(I) = 0\,.$$

Finally suppose $\mu_3 \in ba(F)^+$ and $\mu_2 \leqq \mu_3 \leqq \mu$ and $c_v(\mu_3)$ is representable iff $\mu_3(v) = 0$. For each $v \in R$ we have $c_v(\mu_3)$ is representable by 3.c.1. so that $\mu_3(v) = 0$. Therefore

$$\mu_3 = \int f\mu_3 + \int (1 - f)\mu_3 \leqq 0 + \int (1 - f)\mu = \mu_2\,.$$

This decomposition differs from those of [6], [7] and [10] in that it does not give rise to a normal subspace [5]. To see that this is true suppose that the set R of those elements of $ba(F)$ whose total variations are representable is a normal subspace and note that if $a \in ba(F)^+$ and for each $v \in F$ we have $a(v) \in \{0, a(S)\}$, then $a \in R$. Therefore for any summable sequence, (a_n), of such elements we have $\lambda = \sum_{n=1}^{\infty} a_n \in R$. Consequently α_λ has an integral representation. However by [4] this is true iff the linear functional $\eta \rightarrow \alpha_\lambda(\eta)(S)$ has an integral representation and in [8] it was shown that this is not always true.

REFERENCES

1. W. D. L. Appling, *Summability of real-valued set functions*, Rivista Mathematica, University of Parma, VIII, No. 2 (1967), 77-100.
2. ———, *Concerning a class of linear transformations*, J. London Math. Soc., **44** (1969), 385-396.
3. ———, *Addendum to: 'Some integral characterizations of absolute continuity,'* Proc. Amer. Math. Soc., **24** (1970), 788-793.
4. ———, *An isomorphism and isometry theorem for a class of linear functionals*, Trans. Amer. Math. Soc., ICC (1974), 131-140.
5. S. Bochner and R. S. Phillips, *Additive set functions and vector lattices*, Ann. of Math., (2) **42** (1941), 131-140.
6. Richard B. Darst, *A decomposition of finitely additive set functions*, J. Reine Angew. Math., **210** (1962), 31-37.
7. E. Hewitt and K. Yoshida, *Finitely additive measures*, Trans. Amer. Math. Soc., **72** (1952), 46-66.
8. Michael Keisler, *Integral representation for elements of the dual of ba(c, Σ)*, preprint.
9. A. N. Kolmogoroff, *Untersuchen über den Integralbegriff*, Math. Annals, CIII (1930), 654-696.
10. Detlef Plachky, *Decomposition of Additive Set Functions*, Transactions of the Sixth Prague Conference on Information Theory, Satistical Decision Functions, Random Processes (Technical Univ. Prague, 1971; dedicated to the memory of Antonin Spacek), pp. 715-719. Academic, Prague, 1973.

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