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PLANE PARTITIONS (II): THE EQUIVALENCE OF THE BENDER-KNUTH AND MACMAHON CONJECTURES

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A plane partition of the integer n is a representation of n in the form $n = \sum_{i,j\geqq 1} n_{ij}$ where the integers n_{ij} are non-negative and $n_{ij} \geqq n_{i,j+1}$, $n_{ij} \geqq n_{i+1,j}$. In 1898, MacMahon conjectured that the generating function for the number of symmetric plane partitions (i.e., $n_{ij} = n_{ji}$) with each part at most m (i.e., $n_{11} \leqq m$) and at most s rows (i.e., $n_{ij} = 0$ for $i > s$) has a simple closed form. In 1972, Bender and Knuth conjectured that a simple closed form also exists for the generating function for plane partitions having at most s rows, $n_{11} \leqq m$ and strict decrease along rows (i.e., $n_{ij} > n_{i,j+1}$ whenever $n_{ij} > 0$). The main theorem of this paper establishes that each conjecture follows immediately from the other.

In the first paper of this series, MacMahon's conjecture was proved. Hence a corollary of the main theorem here is the truth of the Bender-Knuth conjecture; the Bender-Knuth conjecture has also been proved in a different manner by Basil Gordon.

1. Introduction. In the 1898, P. A. MacMahon [8] conjectured that the generating function $\mu(m, s; q)$ of $M(m, s; n)$ the number of plane partitions $\sum_{i,j\geqq 1} n_{ij}$ of $n(n_{ij} \geqq n_{i,j+1};\ n_{i,j} \geqq n_{i+1,j})$ such that $n_{ij} = n_{ji}$, $n_{ij} = 0$ if $i > s$, and $n_{11} \leqq m$ satisfies

$$\begin{aligned}\mu(m, s; q) &\equiv \sum_{n\geqq 0} M(m, s; n)q^n \\ &= \prod_{i=1}^{s}\left\{\frac{(1-q^{m+2i-1})}{(1-q^{2i-1})}\prod_{h=i+1}^{s}\frac{(1-q^{2(m+i+h-1)})}{(1-q^{2(i+h-1)})}\right\}.\end{aligned} \tag{1.1}$$

The first paper in this series [2] was devoted to the proof of (1.1).

In 1972, E. Bender and D. Knuth [3] conjectured that the generating function $\beta(m, s; q)$ of $B(m, s; n)$ the number of row-strict plane partitions $\sum_{i,j\geqq 1} n_{ij}$ of $n(n_{ij} > n_{i,j+1}$ if $n_{ij} > 0$, i.e., strict decrease along rows; $n_{ij} \geqq n_{i+1,j})$ such that, $n_{ij} = 0$ if $i > s$ and $n_{11} \leqq m$, satisfies

$$\beta(m, s; q) = \sum_{n\geqq 0} B(m, s; n)q^n = \prod_{i=1}^{m}\prod_{j=i}^{m}\frac{(1-q^{s+i+j-1})}{(1-q^{i+j-1})}. \tag{1.2}$$

Our object in this paper is to prove the following theorem and its corollary:

THEOREM 1.

$$\mu(s, m; q) \prod_{i=1}^{m} \frac{(1 + q^{s+2i-1})}{(1 + q^{2i-1})} = \beta(m, x; q^2) . \tag{1.3}$$

This is our asserted "equivalence" of the two conjectures. Since the MacMahon conjecture is true [2], we see immediately the following:

COROLLARY 1 (Gordon's theorem). *The Bender-Knuth conjecture is true*, i.e., *equation* (1.2) *is valid.*

In [10; p. 265], R. Stanley mentions that B. Gordon possesses (unpublished) a proof of the Bender-Knuth conjecture; however, the implication from Stanley's comments is that Gordon's methods differ substantially from ours. The limiting case $m \to \infty$ was done for $s = 2$ by Gordon [4], and for general s by Gordon and Houten [6], [7]. Professor Gordon informs me that his proof of the Bender-Knuth conjecture will be published shortly.

In §2, we shall fill in some of the details needed to get a determinant representation for $\beta(m, s; q)$. Bender and Knuth [3; p. 50] state this determinant representation without supplying the intermediate steps. In §3, we present several simple recurrences for Gaussian polynomials which suffice for the proof of Theorem 1. In §4, we prove the equivalence theorem (Theorem 1) and Gordon's theorem (Corollary 1).

2. The determinant representation. We consider the Gaussian polynomials

$$\begin{bmatrix} n \\ m \end{bmatrix}_r = \begin{cases} \dfrac{(1 - q^{rn})(1 - q^{r(n-1)}) \cdots (1 - q^{r(n-m+1)})}{(1 - q^{rm})(1 - q^{r(m-1)}(\cdots (1 - q^r)} & 0 < m < n , \\ 1 \text{ if } m = 0 \text{ or } n \\ 0 \text{ if } m < 0 \text{ or } m > n . \end{cases} \tag{2.1}$$

Now

$$q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_1 \tag{2.2}$$

is the generating function for partitions with exactly k distinct parts each $\leqq n$, [9; p. 10], and therefore

$$q^{-k} \cdot q^{k(k+1)} \begin{bmatrix} n \\ k \end{bmatrix}_2 = q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_2 \tag{2.3}$$

is the generating function for partitions with distinct odd parts each $\leqq 2n - 1$.

Now we make B. Gordon's observation [5; p. 158] that Sylvester's mapping of self-conjugate partitions into partitions with distinct odd parts may be directly extended to plane partitions to show that $M(j, m; n)$ is also the numbe of plane partitions of n with strict decrease along rows where each part is odd and at most $2m-1$ and there are at most j rows.

The assertion by Bender and Knuth [3; p. 50] that

$$\mu(2n, m; q) = \det (C_{i-j} + C_{i+j-1})_{n\times n}\ , \tag{2.4}$$

$$\begin{aligned}&\mu(2n+1, m; q)\\ &\quad = (1+q)(1+q^3)\cdots(1+q^{2m-1}) \det (C_{i-j} - C_{i+j})_{n\times n}\ ,\end{aligned} \tag{2.5}$$

where

$$C_k = q^{k^2}\begin{bmatrix} 2m \\ m+k\end{bmatrix}_2$$

is now immediate from the above remarks, Lemma 3 of [3; p. 49] and the corollary of Theorem 4 [3; p. 49] once we remark that

$$\sum_{k\geqq 0} q^{k^2}\begin{bmatrix} m \\ k\end{bmatrix}_2 q^{(k+v)^2}\begin{bmatrix} m \\ k+v\end{bmatrix}_2 = q^{v^2}\begin{bmatrix} 2m \\ m+v\end{bmatrix}_2 , \tag{2.6}$$

an identity equivalent to the q-analog of the Chu-Vandermonde summation [1; p. 469, Th. 4.2].

In exactly the same way, we find

$$\beta(m, 2n; q) = \det (C'_{i-j} + C'_{i+j-1})_{n\times n} \tag{2.7}$$

$$\begin{aligned}&\beta(m, 2n+1; q)\\ &\quad = (1+q)(1+q^2)\cdots(1+q^{m}) \det (C'_{i-j} - C'_{i+j})_{n\times n}\ ,\end{aligned} \tag{2.8}$$

where

$$C'_v = \sum_{k\geqq 0} q^{k(k+1)/2}\begin{bmatrix} m \\ k\end{bmatrix}_1 q^{(k+v)(k+v+1)/2}\begin{bmatrix} m \\ k+v\end{bmatrix}_1 . \tag{2.9}$$

This is not a summation that has a known closed form; however, the determinants in (2.7) and (2.8) may be simplified as follows:

$$\begin{aligned} C'_v + C'_{v-1} &= \sum_{k\geqq 0} q^{k(k+1)/2}\begin{bmatrix} m \\ k\end{bmatrix}_1 q^{(k+v)(k+v-1)/2}\\ &\qquad \times\left(q^{k+v}\begin{bmatrix} m \\ k+v\end{bmatrix}_1 + \begin{bmatrix} m \\ k+v-1\end{bmatrix}_1\right)\\ &= \sum_{k\geqq 0} q^{k(k+1)/2}\begin{bmatrix} m \\ k\end{bmatrix}_1 q^{(k+v)(k+v-1)/2}\begin{bmatrix} m+1 \\ k+v\end{bmatrix}_1 \end{aligned} \tag{2.10}$$

$$= q^{v(v-1)/2}\begin{bmatrix}2m+1\\ m+v\end{bmatrix}_1$$
$$\equiv \gamma_v(q)\ .$$

Furthermore for $v > 0$,

$$\begin{aligned}C'_{1-v} &= \sum_{k\geqq 0} q^{k(k+1)/2}\begin{bmatrix}m\\ k\end{bmatrix}_1 q^{(k+v)(k-v+2)/2}\begin{bmatrix}m\\ k+1-v\end{bmatrix}_1\\ &= \sum_{k\geqq v-1} q^{k(k+1)/2}\begin{bmatrix}m\\ k\end{bmatrix}_1 q^{(k-v+1)(k-v+2)/2}\begin{bmatrix}m\\ k+1-v\end{bmatrix}_1\\ &\quad\times\left(\text{since } \begin{bmatrix}m\\ r\end{bmatrix}=0 \quad\text{for}\quad r<0\right)\\ &= \sum_{k>0} q^{(k+v-1)(k+v)/2}\begin{bmatrix}m\\ k+v-1\end{bmatrix}_1 q^{k(k+1)/2}\begin{bmatrix}m\\ k\end{bmatrix}_1\\ &= C'_{v-1}\ .\end{aligned}$$

Hence the first row of the determinant in (2.7) is

$$C'_{1-j} + C'_j = C'_{j-1} + C'_j = \gamma_j(q)\ , \tag{2.11}$$

and in (2.8) it is

$$\begin{aligned}C'_{1-j} - C'_{j+1} = C'_{j-1} - C'_{j+1} &= C'_{j-1} + C'_j - C'_j - C'_{j+1}\\ &= \gamma_j(q) - \gamma_{j+1}(q)\\ &= \gamma_{1-j}(q) - \gamma_{j+1}(q) \quad (\text{by } (2.10))\ .\end{aligned} \tag{2.12}$$

Finally by adding the ith row of each determinant to the $(i+1)$st row successively as i runs from $n-1$ down to 1, we find that ith row in (2.7) becomes (for $i > 1$)

$$C'_{k-j-1} + C'_{i+j-2} + C'_{1-j} + C'_{i+j-1} = \gamma_{i-j}(q) + \gamma_{i+j-1}(q)\ , \tag{2.13}$$

while the ith row in (2.8) becomes (for $i > 1$)

$$C'_{i-j-1} - C'_{i+j-1} + C'_{i-j} - C'_{i+j} = \gamma_{i-j}(q) - \gamma_{i+j-1}(q)\ . \tag{2.14}$$

Hence by (2.11) and (2.13) we see that

$$\beta(m, 2n; q) = \det\left((1-\delta_{i1})\gamma_{i-j}(q) + \gamma_{i+j-1}(q)\right)_{n\times n}\ , \tag{2.15}$$

where $\delta_{i1} = 1$ if $i = 1$ and equals 0 if $i > 1$, and by (2.12) and (2.14)

$$\begin{aligned}\beta(m, 2n+1; q) = (1+q)(1+q^2)\cdots(1+q^m)\det(\gamma_{i-j}(q)\\ - \gamma_{i+j}(q))_{n\times n}\ .\end{aligned} \tag{2.16}$$

These last two equations are the forms given this generating function by Bender and Knuth [3; p. 50].

3. Recurrence lemmas. Here we require four quite elementary lemmas. Each result involves the C_k (defined just after equation (2.5)) and the $\gamma_k(q^2)$ (defined in (2.10)). To simplify matters we shall utilize the standard notation $(q^2, q^2)_n = (1-q^2)(1-q^4)\cdots(1-q^{2n})$.

LEMMA 1.

$$C_{1-j} + C_j = \frac{(1+q^{2j-1})q^{-j+1}}{(1+q^{2m+1})}\gamma_j(q^2)\,.$$

Proof.

$$\begin{aligned} C_{1-j} + C_j &= q^{(1-j)^2}\begin{bmatrix} 2m \\ m+1-j \end{bmatrix}_2 + q^{j^2}\begin{bmatrix} 2m \\ m+j \end{bmatrix}_2 \\ &= \frac{q^{(j-1)^2}(q^2, q^2)_{2m}}{(q^2; q^2)_{m+j}(q^2; q^2)_{m+1-j}}\{(1-q^{2m+2j}) + q^{2j-1}(1-q^{2m+2-2j})\} \\ &= \frac{q^{(j-1)^2}(q^2; q^2)_{2m}(1+q^{2j-1})(1+q^{2m+1})}{(q^2; q^2)_{m+j}(q^2; q^2)_{m+1-j}} \\ &= \frac{(1+q^{2j-1})q^{-j+1}}{(1+q^{2m+1})}q^{j^2-j}\begin{bmatrix} 2m+1 \\ m+j \end{bmatrix}_2 \\ &= \frac{(1+q^{2j-1})q^{-j+1}}{(1+q^{2m+1})}\gamma_j(q^2)\,. \end{aligned}$$

LEMMA 2.

$$C_{1-j} - C_{1+j} = \frac{(1+q^{2j})q^{-j+1}}{(1+q^{2m+2})}(\gamma_{1-j}(q^2) - \gamma_{1+j}(q^2))\,.$$

Proof.

$$\begin{aligned} &C_{1-j} - C_{1+j} \\ &= q^{(1-j)^2}\begin{bmatrix} 2m \\ m+1-j \end{bmatrix}_2 - q^{(1+j)^2}\begin{bmatrix} 2m \\ m+1+j \end{bmatrix}_2 \\ &= \frac{q^{(j-1)^2}(q^2; q^2)_{2m}((1-q^{2m+2+2j})(1-q^{2m+2j}) - q^{4j}(1-q^{2m+2-2j})(1-q^{2m-2j}))}{(q^2; q^2)_{m+1+j}(q^2; q^2)_{m+1-j}} \\ &= \frac{q^{(j-1)^2}(q^2; q^2)_{2m}(1-q^{4j})(1-q^{4m+2})}{(q^2; q^2)_{m+1+j}(q^2; q^2)_{m+1-j}} \\ &= \frac{q^{-j+1}(1+q^{2j})}{(1+q^{2m+2})}\cdot\frac{q^{j^2-j}(q^2; q^2)_{2m+1}(1-q^{2j})(1+q^{2m+2})}{(q^2; q^2)_{m+1+j}(q^2; q^2)_{m+1-j}} \\ &= \frac{q^{-j+1}(1+q^{2j})}{(1+q^{2m+2})}\cdot\frac{q^{j^2-j}(q^2; q^2)_{2m+1}((1-q^{2m+2j+2}) - q^{2j}(1-q^{2m+2-2j}))}{(q^2; q^2)_{m+1+j}(q^2; q^2)_{m+1-j}} \\ &= \frac{q^{-j+1}(1+q^{2j})}{(1+q^{2m+2})}\left(q^{j^2-j}\begin{bmatrix} 2m+1 \\ m+1-j \end{bmatrix}_2 - q^{j^2+j}\begin{bmatrix} 2m+1 \\ m+j+1 \end{bmatrix}_2\right) \\ &= \frac{q^{-j+1}(1+q^{2j})}{(1+q^{2m+2})}(\gamma_{1-j}(q^2) - \gamma_{1+j}(q^2))\,. \end{aligned}$$

LEMMA 3. *Let*

$$X_i = \frac{q^{2i-2}(1+q^{2m-2i+3})}{(1+q^{2m+2i-1})}. \tag{3.1}$$

Then

$$X_i \cdot C_{i-j-1} + C_{i-j} = \frac{q^{i-j}(1+q^{2j-1})}{(1+q^{2m+2i-1})}\gamma_{i-j}(q^2). \tag{3.2}$$

Proof.

$$\begin{aligned}
&\frac{q^{i-j}(1+q^{2j-1})}{(1+q^{2m+2i-1})}\gamma_{i-j}(q^2) - C_{i-j} \\
&\quad = \frac{q^{i-j}(1+q^{2j-1})q^{(i-j)(i-j-1)}}{(1+q^{2m+2i-1})}\begin{bmatrix} 2m+1 \\ m+i-j\end{bmatrix}_1 - q^{(i-j)^2}\begin{bmatrix} 2m \\ m+i-j\end{bmatrix}_2 \\
&\quad = \frac{q^{(i-j)^2}(q^2;q^2)_{2m}((1+q^{2j-1})(1-q^{4m+2}) - (1+q^{2m+2i-1})(1-q^{2m+2-2i+2j}))}{(1+q^{2m+2i-1})(q^2;q^2)_{m+i-j}(q^2;q^2)_{m+1-i+j}} \\
&\quad = \frac{q^{(i-j)^2+2j-1}(q^2;q^2)_{2m}(1+q^{2m-2i+3})(1-q^{2m+2i-2j})}{(1+q^{2m+2i-1})(q^2;q^2)_{m+i-j}(q^2;q^2)_{m+1-i+j}} \\
&\quad = \frac{q^{2i-2}(1+q^{2m-2i+3})}{(1+q^{2m+2i-1})}\cdot q^{(i-j-1)^2}\begin{bmatrix} 2m \\ m+i-j-1\end{bmatrix} \\
&\quad = X_i \cdot C_{i-j-1}.
\end{aligned}$$

LEMMA 4. *Let*

$$Y_i = \frac{q^{2i-1}(1+q^{2m-2i+2})}{(1+q^{2m+2i})}. \tag{3.3}$$

Then

$$Y_i \cdot C_{i-j-1} + C_{i-j} = \frac{q^{i-j}(1+q^{2j})}{(1+q^{2m+2i})}\gamma_{i-j}(q^2). \tag{3.4}$$

Proof.

$$\begin{aligned}
&\frac{q^{i-j}(1+q^{2i})}{1+q^{2m+2i}}\gamma_{i-j}(q^2) - C_{i-j} \\
&\quad = \frac{q^{(i-j)^2}(q^2;q^2)_{2m}((1+q^{2j})(1-q^{4m+2}) - (1+q^{2m+2i})(1-q^{2m+2-2i+2j}))}{(1+q^{2m+2i})(q^2;q^2)_{m+i-j}(q^2;q^2)_{m-i+1+j}} \\
&\quad = \frac{q^{(i-j)^2+2j}(q^2;q^2)_{2m}(1-q^{2m+2i-2j})(1+q^{2m-2i+2})}{(1+q^{2m+2i})(q^2;q^2)_{m+i-j}(q^2;q^2)_{m-i+1+j}} \\
&\quad = \frac{q^{2i-1}(1+q^{2m-2i+2})}{(1+q^{2m+2i})}\cdot q^{(i-j-1)^2}\begin{bmatrix} 2m \\ m+i-j-1\end{bmatrix} \\
&\quad = Y_i \cdot C_{i-j-1}.
\end{aligned}$$

4. Proof of main results. We restate Theorem 1 for convenience:

THEOREM 1.

$$\mu(s, m; q) \prod_{i=1}^{m} \frac{(1 + q^{s+2i-1})}{(1 + q^{2i-1})} = \beta(m, s; q^2) .$$

Proof. If $s = 1$, then the assertion is trivial since clearly $\mu(1, m; q) = \prod_{i=1}^{m} (1 + q^{2i-1})$, and $\beta(m, 1; q) = \prod_{i=1}^{m} (1 + q^{2i})$.

Next let $s = 2n \geqq 2$. Then

$$\mu(2n, m; q) = \det (C_{i-j} + C_{i+j-1})_{n\times n} \qquad \text{(by (2.4))} .$$

The first row of this determinant is

$$\frac{(1 + q^{2j-1})q^{1-j}}{(1 + q^{2m+1})}\gamma_j(q^2) , \quad 1 \leqq j \leqq n \text{ (by Lemma 1)} ,$$

and if we multiply the $(i-1)$st row by X_i and add it to the ith row (letting i run from n down to (2) we find that by Lemma 3, the resulting ith row is

$$\frac{q^{i-j}(1 + q^{2j-1})}{(1 + q^{2m+2i-1})}\gamma_{i-j}(q^2) + \frac{q^{i+j-1}(1 + q^{-2j+1})}{(1 + q^{2m+2i-1})}\gamma_{i+j-1}(q^2)$$
$$= \frac{q^{i-j}(1 + q^{2j-1})}{(1 + q^{2m+2i-1})}(\gamma_{i-j}(q^2) + \gamma_{i+j-1}(q^2)) .$$

Therefore

$$\begin{aligned}
\mu(2n, m; q) &= \det \left(\frac{q^{i-j}(1 + q^{2j-1})}{(1 + q^{2m+2i-1})}((1 - \delta_{i1})\gamma_{i-j}(q^2) + \gamma_{i+j-1}(q^2))\right)_{n\times n} \\
&= \left\{\prod_{h=1}^{n} \frac{(1 + q^{2h-1})}{(1 + q^{2m+2h-1})}\right\} \det ((1 - \delta_{i1})\gamma_{i-j}(q^2) + \gamma_{i+j-1}(q^2))_{n\times n} \\
&= \beta(m, 2n; q^2) \prod_{j=1}^{n} \frac{(1 + q^{2j-1})}{(1 + q^{2m+2j-1})} , \\
&= \beta(m, 2n; q^2) \frac{\left(\prod_{j=1}^{n} (1 + q^{2j-1})\right)\left(\prod_{j=1}^{m} (1 + q^{2j-1})\right)}{\prod_{j=1}^{m+n} (1 + q^{2j-1})} \\
&= \beta(m, 2n; q^2) \prod_{j=1}^{m} \frac{(1 + q^{2j-1})}{(1 + q^{2n+2j-1})} ,
\end{aligned}$$

which is the desired result for $s = 2n$.

Finally we let $s = 2n + 1 \geqq 3$. Then

$$\mu(2n + 1, m; q) = \left(\prod_{h=1}^{m} (1 + q^{2h-1})\right) \det (C_{i-j} - C_{i+j})_{n\times n} \quad \text{(by (2.5))} .$$

The first row of this determinant is

$$\frac{(1+q^{2j})q^{1-j}}{(1+q^{2m+2})}(\gamma_{1-j}(q^2)-\gamma_{1+j}(q^2))\,, \quad 1 \leqq j \leqq n \text{ (by Lemma 2)}\,,$$

and if we multiply the $(i-1)$st row by Y_i and add it to the ith row (letting i run from n down to (2) we find that by Lemma 4, the resulting ith row is

$$\frac{q^{i-j}(1+q^{2j})}{(1+q^{2m+2i})}\gamma_{i-j}(q^2) - \frac{q^{i+j}(1+q^{-2j})}{(1+q^{2m+2i})}\gamma_{i+j}(q^2)$$
$$= \frac{q^{i-j}(1+q^{2j})}{(1+q^{2m+2i})}(\gamma_{i-j}(q^2)-\gamma_{i+j}(q^2))\,.$$

Therefore

$$\mu(2n+1, m, q) = \left\{\prod_{k=1}^{m}(1+q^{2k-1})\right\}\det\left(\frac{q^{i-j}(1+q^{2j})}{(1+q^{2m+2i})}(\gamma_{i-j}(q^2)-\gamma_{i+j}(q^2))\right)_{n\times n}$$
$$= \left\{\prod_{k=1}^{m}(1+q^{2k-1})\right\}\left\{\prod_{h=1}^{n}\frac{(1+q^{2h})}{(1+q^{2m+2h})}\right\}\det(\gamma_{i-j}(q^2)-\gamma_{i+j}(q^2))_{n\times n}$$
$$= \beta(m, 2n+1, q^2)\left\{\prod_{k=1}^{m}\frac{(1+q^{2k-1})}{(1+q^{2k})}\right\}\frac{\left(\prod_{h=1}^{n}(1+q^{2h})\right)\left(\prod_{j=1}^{m}(1+q^{2h})\right)}{\prod_{h=1}^{m+n}(1+q^{2h})}$$
$$= \beta(m, 2n+1; q^2)\prod_{h=1}^{m}\frac{(1+q^{2h-1})}{(1+q^{2n+2h})}\,,$$

which is the desired result for odd $s = 2n+1$.

COROLLARY 1 (Gordon's theorem). *The Bender-Knuth conjecture is true*, i.e., *equation* (1.2) *is valid.*

Proof. Equation (1.1) was proved in [2]. Hence by Theorem 1,

$$\beta(m, s; q^2) = \mu(s, m; q)\prod_{i=1}^{m}\frac{(1+q^{s+2i-1})}{(1+q^{2i-1})}$$
$$= \prod_{i=1}^{m}\left\{\frac{(1-q^{2(s+2i-1)})}{(1-q^{2(2i-1)})}\prod_{h=i+1}^{m}\frac{(1-q^{2(s+i+h-1)})}{(1-q^{2(i+h-1)})}\right\}$$
$$= \prod_{i=1}^{m}\prod_{h=i}^{m}\frac{(1-q^{2(s+i+h-1)})}{(1-q^{2(i+h-1)})}\,,$$

which is just (1.2) with q^2 replacing q.

We should point out that if one starts with Gordon's forthcoming proof of the Bender-Knuth conjecture then obviously Theorem 1 implies the MacMahon conjecture.

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