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KIRILLOV THEORY FOR COMPACT p-ADIC GROUPS

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The purpose here is to describe a method by which one may obtain a reasonably explicit and "global" picture of the unitary representation theory of compact p-adic groups, and to indicate some applications. (By p-adic, we refer to Q_p, or local fields of characteristic zero.) The basic inspiration for such a description goes back to Kirillov's work on nilpotent lie groups. The main ingredients are the exponential map and the co-adjoint action. The Campbell-Hausdorff formula is used heavily as a tool.

The basic results indicated here are to some extent implicit in [1], but it seems desirable to have them put in as sharp a form as possible with a view to their application to representations of non-compact groups. Also with this in mind, we have included some facts to allow calculation of intertwining numbers of induced representations. Moreover, since the theory as presented here depends heavily on the exponential map and the Campbell-Hausdorff formula, it is inadequate to deal with all compact p-adic groups. Only "sufficiently compact" groups are covered. Roughly speaking, this means that given a p-adic group, we can analyze completely some open subgroup. Therefore, we have included some propositions showing how the theory of representations of group extensions applies to this situation. These results are all dealt with in section one.

In the succeeding sections we describe some applications. Using Sl_n, we illustrate how compact Cartan subgroups of semisimple groups give rise to series of supercuspidal representations. Then we show how the Kirillov picture can be used to help analyze representations of certain arithmetic groups, like $Sl_n(\boldsymbol{Z})$, $n \geqq 3$. Finally we compute completely the representations of p-adic division algebras of prime degree to show in a simple case how the Kirillov picture can be used as a guide to a full analysis of the representations of a p-adic group. In this section, we place no restriction on the characteristic of the ground field, and the results strongly suggest that although the theory presented here does not apply to groups over fields of positive characteristic, nevertheless the representation theory there is very much like the characteristic zero theory. This, of course, reinforces the experience with Gl_2, Sl_2, etc.

I. Let F be a p-adic field, that is, a finite extension of Q_p, the p-adic numbers. Let R be the valuation ring of F, π a prime element. $|\ |$ will be the natural absolute value on F, and $\mathrm{ord}_F(\ )$ the corresponding valuation. Let $\alpha = [\mathrm{ord}_F(p)/(p-1)] + 1$, where $[\ ]$ here denotes greatest integer. Thus α is the smallest integer such that $|\pi^{\alpha(p-1)}| < |p|$. If A is a ring, let $M_n(A)$ denote the $n \times n$ matrices with entries in A. Then $\exp: M_n(\pi^\alpha R) \to 1 + M_n(\pi^\alpha(R))$, and log going backwards, can be defined by the usual formulas, and have the usual properties. In particular, they are homeomorphisms (in fact isometries, when both sets are given the metric induced from the ultrametric operator norm on M_n), and they satisfy the Campbell-Hausdorff formula (hereafter abbreviated to C.H.).

Now let $L \subseteqq M_n(F)$ be a lie subalgebra and let $K \subseteqq L \cap M_n(\pi^\alpha R)$ be an open additive subgroup of L, closed under the bracket operation. Then we may consider K to be a lie algebra (over R) in its own right. Let $C = \exp K$ be the image of K under exp. According to C.H. ([2], [7]), a sufficient condition that C be a group is $[K, K] \subseteqq \pi^\beta K$. Here $[\ ,\ ]$ denotes the lie algebra bracket operation and β is defined as $[3\ \mathrm{ord}_F(p)/2(p-1)] + 1$ unless $p = 3$, when $\beta = \mathrm{ord}_F(3) + 1$. Note that $\alpha \leqq \beta \leqq 2\alpha$. If K satisfies this condition, we will call it (or C) e.e.

Now suppose K is e.e. Then conjugation in C pulls back by exp to give an action (the adjoint action, written $\mathrm{Ad}(C)$) of C on K, which of course extends to a linear action on L. We also get an action $\mathrm{Ad}{*}C$ on $L*$, the dual vector space to L, which preserves the lattice $K*$, consisting of $l* \in L*$ such that $\langle l*, K\rangle \subseteqq R$. Here $\langle\ ,\ \rangle$ is the pairing $L* \times L \to F$. On the other hand, we have an action $\mathrm{Ad}{*}C$ on $\hat{K}$, the Pontryagin dual of K. If X is a character of the additive group of F, we may use X to define an isomorphism $\theta: L* \to \hat{L}$, by the formula $\theta(l*)(l) = X(\langle l*, l\rangle)$. If $\pi^m R$ is the conductor of X, then this also accomplishes an identification of $\pi^m K* \subseteqq L*$ with $K^\perp \subseteqq \hat{L}$, $K^\perp$ being the annihilator of K in $\hat{L}$. Thus finally we have an identification, also called θ, of $\hat{L}/K^\perp \simeq \hat{K}$ and $L*/\pi^m K*$. Using exp, we may think of elements of $\hat{K}$ as being functions on C. More particularly exp sends Haar measure to Haar measure, and so defines an isometry of L_2-spaces. If $J \subseteqq K$ is another e.e. algebra, and $B = \exp J$, then we have the restriction map $r: \hat{K} \to \hat{J}$. If B is normal in C, then $\mathrm{Ad}{*}C$ acts on $\hat{J}$.

We want to match $\mathrm{Ad}{*}C$ orbits in $\hat{K}$ with representations of C. Hence take such an orbit 0, and some $\psi \in 0$. Let A be the isotropy group of ψ under $\mathrm{Ad}*$. With ψ, we may associate the bilinear form $f: K \times K \to T$ (the circle), defined by $f(x, y) = \psi([x, y])$.

LEMMA 1.1. *A is* e.e. *That is, if $I = \log A$, then I is a lattice, and $[I, I] \subseteqq \pi^\beta I$. Moreover, I is the radical of f, that is, the set* *of $x \in K$, such that $f(x, K) = 1$. If $A = C$, that is, if ψ is invariant, then it defines a character of C, unless $p = 2$, in which case ψ^2 does.*

Proof. The lemma follows by inspection of C.H. If $x \in K$, then $\operatorname{Ad}\exp x - 1 = \exp \operatorname{ad} x - 1 = \tau(x) \operatorname{ad} x$, where $\tau(x)$ is an automorphism of K as abelian group, since K is e.e. Hence, $\operatorname{Ad}*\exp x(\psi) = \psi$ if and only if $\psi([x, \tau(x)y]) = 1$ for $y \in K$ if and only if $f(x, y) = 1$, and conversely. If $x, y \in \log A$, then $\pi^{-\beta}[y, z] \in K$, if $z \in K$, so $f(\pi^{-\beta}[x, [y, z]]) = 1$. Exchanging x and y, and employing the Jacobi identity, we have $f(\pi^{-\beta}[[x, y], z]) = 1$, or $[x, y] \in \pi^{\beta} \log A$.

For the second assertion, observe that if K^{i+1} is that lattice generated by $[K, K^i]$, then $K^i \subseteqq \pi^{\beta(i-2)} K^2$, and the invariance of ψ means it annihilates K^2, by the above. Then the form of C.H. as in [2], and the choice of β yield the conclusion.

From now on, to avoid the complications foreshadowed in this lemma when $p = 2$, we will take p odd. The arguments can be mostly carried through when $p = 2$ also, but they are the more cumbersome, and the results are not so clean.

Again take ψ with isotropy group A. An e.e. group $B = \exp J$, with $A \subseteqq B \subseteqq C$ and $^{\#}(B/A) = {}^{\#}(C/B)$ (where $\#$ indicates cardinality) will be said to polarize ψ if $r(\psi) \in \hat{J}$ is $\operatorname{Ad}*B$ invariant. This is because then J/I forms "half of a polarization" of the nondegenerate antisymmetric form f on K/I. In this case, $r(\psi)$ is a character of B, and $\operatorname{Ad}*B(\psi)$ consists of all $\phi \in r^{-1}(r(\psi))$ as one sees by counting. Thus, if whenever g is a function on B, $\dot{g}$ is the function on C which is zero off B, then $r(\dot{\psi}) = {}^{\#}(C/B)^{-1} \sum_{\phi \in \operatorname{Ad}*B(\psi)} \phi$, by the orthogonality relations for abelian characters.

LEMMA 1.2. *Suppose B polarizes ψ. Then $U(O) = U^{r(\psi), B}$, the representation of C induced from $r(\psi)$ on B, is irreducible of degree $^{\#}(O)^{1/2}$ and character $^{\#}(O)^{-1/2} \sum_{\phi \in O} \phi$. In particular $U(O)$ depends only on O, not on ψ or B.*

Proof. The statement on the degree, and the dependence of $U(O)$ only on O follow from the character formula. The character formula follows from the usual character formula for induced representations, and the formula for $r(\dot{\psi})$. For if H is an open subgroup of a compact group G, and V is a representation of H, with character ψ, then the character of the induced representation $U^{V,H} = U^{\psi,H}$ of G is $\Sigma \operatorname{Ad}*g_i(\dot{\psi})$ where the $\{g_i\}$ are a set of coset representatives of H in G. In our case, $^{\#}(C/B) = {}^{\#}(O)^{1/2}$, and, if $\{c_i\}$ are coset representatives for B in C, $\Sigma \operatorname{Ad}*c_i(r(\dot{\psi})) = {}^{\#}(O)^{-1/2} \Sigma \operatorname{Ad}* c_i(\sum_{\phi \in \operatorname{Ad}*B(\psi)} \phi) = {}^{\#}(O)^{-1/2}(\sum_{\phi \in O} \phi)$.

It remains to show that $U(O)$ is irreducible. We see that since $B \supseteqq A$, if $c \in C - B$, $\mathrm{Ad}{*}c(r(\dot\psi))$, is a sum over a set of characters of $\hat{K}$, disjoint from $\mathrm{Ad}{*}(B)(\psi)$. Hence $\mathrm{Ad}{*}c(r(\dot\psi))$ is orthogonal (in $L_2(C)$) to $r(\dot\psi)$. We may conclude irreducibility from Lemma 6 of [1], which we give here verbatim.

LEMMA 1.3. *If G is a finite group, H_1, H_2 subgroups, V_i representations of H_i, with characters ψ_i, then the induced representations $U_i = U^{V_i, H_i}$ are disjoint (have no common subrepresentations) if and only if $\mathrm{Ad}{*}x(\dot\psi_1)$ is orthogonal (in $L_2(G)$) to $\mathrm{Ad}{*}y(\dot\psi_2)$ for all $x, y \in G$. U_1 is irreducible if and only if $\mathrm{Ad}{*}x(\dot\psi_1)$ is orthogonal to $\dot\psi_1$ for all $x \in G - H_1$.*

Now we attend to the existence of groups polarizing ψ. If G is a separable profinite group, and H is a group of automorphisms G, then we will say H acts pronilpotently if there is a neighborhood base $\{G_i\}$ of the identity of G, consisting of open normal subgroups stable by H, such that the quotient action on G_i/G_{i+1} is trivial for all i. In such a case H always has compact closure in the automorphism group of G, and if H is closed and G is pro-p, then H is pro-p. Conversely, if G is pro-p, and H is pro-p, then H acts pronilpotently. We note that to show that there are groups B polarizing ψ is the same as to show there are e.e. subalgebras J, with $I \subseteqq J \subseteqq K$, where I is the radical of f, the bilinear form attached to ψ, such that $^{\#}(J/I) = {}^{\#}(K/J)$, and such that f is trivial on J (J is subordinate to ψ).

LEMMA 1.4. *Let K be* e.e., *and suppose moreover that $\pi^{-\beta}K$ is a pronilpotent lie algebra. Suppose G is a group of R-linear (lie algebra) automorphisms of K acting pronilpotently on K. Suppose $\psi \in \hat{K}$ is fixed under G. Then there is an* e.e. *polarizing subalgebra J for ψ, which is stable under G.*

Proof. First, we will only assume K pronilpotent, and will show we can find J polarizing ψ, and G-stable. We proceed by induction on $^{\#}(K/L)$ where L is the largest ideal in ker ψ. If ψ is trivial, or if K/L is abelian, then the result is evident. (Note ker ψ and L are invariant by G.) $K/L = K'$ is a finite nilpotent lie algebra, and ψ, and the action of G factors to ψ', G' defined on K'. Let $\mathscr{Z}^{(i)}(K')$ be the ith member of the ascending central series of K'. Then ψ' is faithful on $\mathscr{Z}(K')$. Since G' acts nilpotently, there is $M \subseteqq \mathscr{Z}^2(K')$ such that $M/\mathscr{Z}(K')$ is cyclic, of order p, and M is G'-stable, and the action of G' on $M/\mathscr{Z}(K')$ is trivial. Let K_1' be the centralizer in K' of M. Then $M \subseteqq \mathscr{Z}(K_1')$ and K_1' is G'-stable, and $^{\#}(K'/K_1') = p$. Moreover, M is contained in the radical of $f'(=f$ factored to $K')$ restricted to K_1', but is not in the radical of f' since $[M, K_1'] = 0 \neq [M, K] \subseteq \mathscr{Z}(K')$, and ψ' is faithful on $\mathscr{Z}(K)$. We may assume by induction that there is, for f' restricted to K_1', a polarizing subalgebra J', stable by G'. Necessarily $M \subseteq J'$. Hence, by the considerations just above J' also polarizes f' on K'. Then lifting back to K, we obtain the J we wanted.

Now consider K e.e., with $\pi^{-\beta}K$ pronilpotent, and G acting R-linearly and pronilpotently on K. Then the action of G extends F-linearly to an action on L, the span of K. In particular we get a pronilpotent action on $\pi^{-\beta}K$. If $\psi \in \hat{K}$, define ψ'' in $\widehat{\pi^{-\beta}K}$ by $\psi''(k) = \psi(\pi^{2\beta}k)$. Then, if f'' is the associated antisymmetric form, the radical of f'' is $\pi^{-\beta}I$, where I is the radical of f, the form attached to ψ. Moreover, if G fixes ψ, it clearly also fixes ψ''. So now let J'' be a polarizing subalgebra for ψ''. Then $J = \pi^{\beta}J''$ is an e.e. subalgebra, subordinate to ψ, and $^{\#}(J/I) = {}^{\#}(J''/\pi^{-\beta}I) = {}^{\#}(\pi^{-\beta}K/J'') = {}^{\#}(K/J)$ so J polarizes ψ. This completes the proof.

REMARK. A useful example of K which is e.e. and such that $\pi^{-\beta}K$ is pronilpotent is K such that $[K, K] \subseteq \pi^{\beta+1}K$. In particular, for any K which is e.e., πK satisfies the conditions of the lemma.

Suppose now we have K satisfying the conditions of Lemma 4. Then by Lemma 2, we get for any Ad$*C$ orbit O in $\hat{K}$, an irreducible representation $U(O)$ of C, of dimension $^{\#}(O)^{1/2}$. Now for any m, $C_m = \exp \pi^m K \subseteq C$ is a normal subgroup, and C.H. shows that for $k \in K, \log(\exp kC_m) = k + \pi^m K$. Thus, if ψ is trivial on $\pi^m K$, it may be regarded as a function on C/C_m. Clearly the set of ψ trivial on $\pi^m K$ is a union of Ad$*C$ orbits, and has cardinality $^{\#}(C/C_m)$. Thus, we get a collection of representations of C/C_m, the sum of the squares of whose dimensions equals $^{\#}(C/C_m)$. By classical results for finite groups, this exhausts the collection of representations of C/C_m (note that representations attached to different orbits are distinct, since their characters are orthogonal in $L_2(C)$). Thus, letting m go to infinity, we find we have all the representations of C. Summing up the arguments to this point:

THEOREM 1.1. *Let K be a sublattice of a lie algebra $L \subseteq M_n(F)$, consisting of elements whose exponentials exist, and such that*

(a) $[K, K] \subseteq \pi^{\beta}K$, *and*

(b) $\pi^{-\beta}K$ *is pronilpotent (with β defined as above). Then $C = \exp K$ is a group, and the unitary representations of C correspond bijectively to* Ad$*C$ *orbits in $\hat{K}$. If $U(O)$ corresponds to O, then the character of $U(O)$ is given by $\psi(O) = {}^{\#}(O)^{-1/2} \sum_{\phi \in O} \phi$. Moreover, there is a nice way to realize $U(O)$ as an induced representation,* *as described above.*

Of course, the most interesting compact p-adic groups (e.g., $Gl_n(\boldsymbol{Z}_p)$ etc.) do not satisfy the hypotheses of Theorem 1.1. To deal with their representation theory more fully, we shall apply the theory of extensions of group representations. We do not by any means pretend to a complete solution. Such a solution would in particular entail finding the representations of the finite simple algebraic groups, an apparently very difficult problem. Nevertheless, there are some useful observations which can be made.

We recall briefly the basics of the extension theory. If G is a finite group, N a normal subgroup, then $\mathrm{Ad}*G/N$ operates on $\hat{N}$ (the "dual object" of N: the set of its irreducible representations) via conjugation. Any element U of $\hat{G}$ (e.g., an irreducible representation of G) breaks up when restricted to N into the direct sum of representations of N comprising an $\mathrm{Ad}*G/N$ orbit O in $\hat{N}$. Each representation occurs with the same multiplicity. We write $r(U)=0$. This defines a map $r\colon \hat{G} \rightarrow \hat{N}/\mathrm{Ad}*G/N$. To compute the elements in $\hat{G}$ lying above O (e.g., in $r^{-1}(O)$), we may proceed as follows. Take $V \in 0$, and let $H \subseteqq G$ be the group which becomes the isotropy group of V in $\mathrm{Ad}*G/N$. Then, if V' is any representation of H lying above V, $U^{V',H}$, the induced representation of G is irreducible, and this sets up a bijection between $r^{-1}(V) \subseteqq \hat{H}$, and $r^{-1}(O) \subseteqq \hat{G}$. Thus the problem is reduced to finding representations of H lying above V. In trying to do this, a certain cohomology class in $H^2(H/N; T)$ ($T =$ the circle) arises, which is the obstruction (called the Mackey obstruction in the case of locally compact groups) to extending V to a representation of H. This class defines a central extension of H/N, and the representations in $r^{-1}(V)$ correspond bijectively and may be constructed in terms of the representations of this extension of H/N with the specified central character. Of course, one is interested in the vanishing of the Mackey obstruction, since then life is simplest. In this regard, there are two general and useful facts:

(a) the order of the class always divides ${}^{\#}(H/N)$ (as does the order of all classes); and

(b) the order of the class always divides the dimension of V.

Now consider the case at hand. Suppose $C = \exp K$ is covered by Theorem 1.1, and that C is normal and of finite index in G. Then $\mathrm{Ad}*G$ operates naturally on $\hat{K}$, and the character formula implies that the factored action of G/C on $\hat{K}/\mathrm{Ad}*C$ coincides with $\mathrm{Ad}*G/C$ acting on $\hat{C}$. If $0 \subseteqq \hat{K}$ is an $\mathrm{Ad}*C$ orbit, and $H \subseteqq G$ is the isotropy group of $U(O)$, then if $\psi \in 0$, and $h \in H$, $\mathrm{Ad}*h(\psi) \in 0$. Thus there is $c \in C$, such that $hc \in H_0$, the isotropy group for ψ under

$\mathrm{Ad}{*}G$. If $A = H_0 \cap C$, we see therefore that $H_0/A = H/C$, so that we may, for purposes of extension, identify the isotropy groups of 0 and ψ. This statement can be made precise. Since the dimension of $U(O)$ is a power of p, since C is a pro-p group, the obstruction to extending $U(O)$ to H vanishes, if and only $U(O)$ may be extended to P, some Sylow subgroup of H. But P acts pronilpotently on K, so there is a subgroup B of C polarizing ψ and normalized by P_0, a Sylow subgroup of H_0. But $A = P_0 \cap C$, and ψ defines an $\mathrm{Ad}{*}P_0$ invariant character of A. If $B_1 = P_0B$, then the obstruction to extending ψ from B to B_1 is visibly the same as the obstruction to extending ψ from A to P_0 (and in these cases, is the image of ψ under the transgression $H^1(A, T)^{P_0} \longrightarrow H^2(P_0/A, T)$ or $H^1(B, T)^{B_1} \longrightarrow H^2(B_1/B, T)$). Moreover, if ψ may be extended to ϕ on B_1, then U^{ϕ, B_1} is an irreducible representation of $P = P_0C$ which extends $U(O)$. We summarize this.

PROPOSITION 1.1. *There is a canonical identification of the Mackey obstruction to extending* $U(O)$ *to* H, *with the Mackey obstruction to extending* ψ *to* H_0.

In general, whether the obstruction vanishes or not, the restriction of the representation $U^{\psi, B}$ of B_1 induced from ψ on B to P_0 clearly is naturally unitarily equivalent to the representation of P_0 induced from ψ on A, and the restriction sets up a bijection between irreducible components of the two representations. Then inducing up to P, we see that each irreducible component of $U^{\psi, B}$ determines an irreducible representation of P lying above $U(O)$, and inequivalent components determine inequivalent representations. We summarize this.

PROPOSITION 1.2. *There is a natural bijective correspondence between irreducible representations of* P_0 *lying above* ψ *on* A, *and irreducible representations of* P *lying above* $U(O)$.

Of course a similar result holds for any subgroup P' of P. The most interesting case would be when P' was normal in G. Proposition 2 is useful in carrying out a detailed analysis with regard to extending $U(O)$ to P, which leads to results sharpening and clarifying Theorem 1.1.

Now we discuss induced representations, particularly the computation of intertwining numbers. First, for completeness, we record the situation for groups covered by Theorem 1.1. The proof is a simple exercise using the character formula.

PROPOSITION 1.3. *If $C_1 \subseteqq C_2$ are two subgroups subject to Theorem* 1.1, *and $K_i = \log C_i$, and r: $\hat{K}_2 \to \hat{K}_1$ is the restriction map, and O_i are* Ad$*C_i$ *orbits in $\hat{K}_i$, then the multiplicity $m(O_1, O_2)$ with which $U(O_1)$ occurs in $U(O_2)$ restricted to C_1 is given by*

$$^{\#}(O_2 \cap r^{-1}(O_1))^{\#}(O_1)^{-1/2\#}(O_2)^{-1/2} .$$

There is another form of this which is useful. Since $O_2 \cap r^{-1}(O_1)$ is Ad$*C_1$ invariant, and r is Ad$*C_1$ equivariant, $^{\#}(O_2 \cap r^{-1}(O_1))$ is a multiple of $^{\#}(O_1)$. Write this multiple $c(O_1, O_2)$ and call it the covering number of O_1 by O_2. Then $m(O_1, O_2) = c(O_1, O_2)^{\#}(O_1)^{1/2\#}(O_2)^{-1/2}$.

We are principally interested in decomposing representations induced up to some larger group. To fix things, suppose K has as span over F a lie algebra L, which is the lie algebra of (the F-rational points of) some algebraic group G. Then via conjugation in G, the adjoint action of G on L, and the co-adjoint action on $L*$ are defined, although this action may not factor to an action on $\hat{K}$, because it may not preserve K or $K*$. Nevertheless, this action controls the multiplicity theory of representations induced from C. In a very real sense, the representations of C are "located" in $L*$.

Suppose C_1, C_2 are two open subgroups of G to which Theorem 1.1 applies, and put $K_i = \log C_i$. As we saw at the outset, we have well defined maps σ_i: $L* \to \hat{K}_i$. Let $O_i \subseteqq \hat{K}_i$ be Ad$*C_i$ orbits. We will, when convenient, identify O_i with $\sigma_i^{-1}(O_i) \subseteqq L*$. Let V^{O_i} denote the representation of G induced from $U(O_i)$ on K_i. We will say $g \in G$ intertwines the $U(O_i)$, if the double coset $C_1 g C_2$ supports an intertwining operator between the V^{O_i}.

PROPOSITION 1.4. *g intertwines the $U(O_i)$ if and only if* Ad$*g(\sigma_1^{-1}(O_1)) \cap \sigma_2^{-1}(O_2) \neq \varnothing$. *The dimension of the space of intertwining operators supported on $C_1 g C_2$ is then given by $^{\#}(O_1)^{-1/2\#}(O_2)^{-1/2} i($*Ad$*$ *$g(\sigma_1^{-1}(O_1)), \sigma_2^{-1}(O_2))$ where $i($*Ad$*g(\sigma_1^{-1}(O_1)), \sigma_2^{-1}(O_2))$ *is the number of* Ad$*g(\pi^m K_1^*) \cap \pi^m K_2^*$ *cosets contained in* Ad$*g(\sigma_1^{-1}(O_1)) \cap \sigma_2^{-1}(O_2)$.

Proof. We use the theorem of Mackey ([5]) on intertwining numbers. Let $C_3 = gC_1g^{-1} \cap C_2$. Then $K_3 = \log C_3 =$ Ad $g(K_1) \cap K_2$. Then Ad$*g$: $\hat{K}_1 \to$ Ad $g(K_1)$ sends $U(O_1)$ to $U($Ad$*g(O_1))$. By Mackey's result, g intertwines the $U(O_1)$ if and only if $U($Ad$*g(O_1))$ and $U(O_2)$ intertwine when restricted to C_3, and the dimension of the space of intertwining operators is just the intertwining number of these restrictions. So replacing C_1 by Ad $g(C_1)$, we reduce to the case $g = 1$.

Now we have $\hat{K}_i \simeq L*/\pi^m K_i^*$, and this identification is consistent with restriction maps. Since $K_3 = K_1 \cap K_2$, $K_3* = K_1* + K_2*$. If $U(O_3)$ is a representation of C_3, then according to Proposition 1.3,

$m(O_3, O_i)$ $(i = 1, 2)$ is equal to ${}^\#(O_3)^{-1/2\#}(O_i)^{-1/2}$ times the number of $\pi^m K_i*$ cosets contained in $\sigma_3^{-1}(O_3) \cap \sigma_i^{-1}(O_i)$. Since $\sigma_i^{-1}(O_i)$ is $\mathrm{Ad}*C_3$ invariant, this latter number is equal to ${}^\#(O_3)$ times the number of $\pi^m K_i*$ cosets of $\sigma_i^{-1}(O_i)$ contained in a given $\pi^m K_3*$ coset of $\sigma_3^{-1}(O_3)$ (the covering number, $c(O_3, O_i)$). Since $K_3* = K_1* + K_2*$, any $\pi^m K_{1*}$ coset intersects any $\pi^m K_{2*}$ coset contained in the same $\pi^m K_{3*}$ coset. Since the intertwining number of the restrictions is $\Sigma m(O_3, O_1)m(O_3, O_2)$, the result follows.

COROLLARY 1. *The V^{O_i} are disjoint if and only if* $\mathrm{Ad}*g(\sigma_1^{-1}(O_1))$ *is disjoint from $\sigma_2^{-1}(O_2)$ for all $g \in G$. The V^{O_i} have a finite intertwining number if and only if the set of g such that* $\mathrm{Ad}*g(\sigma_1^{-1}(O_1)) \cap \sigma_2^{-1}(O_2) \neq \varnothing$ *are a compact set. V^{O_1} is irreducible if and only if* $\mathrm{Ad}*g(\sigma_1^{-1}(O_1)) \cap \sigma_1^{-1}(O_1) = \varnothing$ *for $g \notin C_1$. V^{O_1} is finitely decomposable if and only if the set of g such that* $\mathrm{Ad}*g(\sigma_1^{-1}(O_1)) \cap \sigma_1^{-1}(O_1) \neq \varnothing$ *is compact.*

The proof is immediate from Proposition 1.4 and Mackey's theory [5] for intertwining numbers of representations induced from open compact subgroups.

COROLLARY 2. *Suppose every point $p \in \sigma_1^{-1}(O_1)$ lies on a closed* $\mathrm{Ad}*G$ *orbit, and has compact isotropy group. Then V^{O_1} is finitely decomposable. Conversely, if V^{O_1} is finitely decomposable, then the isotropy group of every $p \in \sigma_1^{-1}(O_1)$ is compact.*

Proof. Clear.

COROLLARY 3. *Suppose $G_0 \supseteq C_1$ is a compact subgroup of G, containing all $g \in G$ which intertwine $U(O_1)$ with itself. Then any irreducible representation of G_0 lying above $U(O_1)$ induces an irreducible representation of G_1 and V^{O_1} decomposes into a finite sum over these representations.*

Proof. Clear.

II. Retain the notation of Proposition 1.4. If G is semisimple, then the Killing form $(\, ,)$ on L is nondegenerate, and so defines an isomorphism of L with $L*$, and so eventually isomorphisms θ of L and $\hat{L}$, and, for $K \subseteq L$, of $\hat{K}$ and $L/\pi^m K*$, where now $K*$ is considered to be the set of $l \in L$ such that $(l, K) \subseteq R$. Suppose that for some i, $K* = \pi^i K$. Then $(\pi^j K)* = \pi^{i-j} K$, and in $\hat{K}$, $(\pi^j K)^\perp$ is identified to $\pi^{m+i-j} K/\pi^{m+i} K$. Thus if $j = m + i$, $(\pi^j K)^\perp \subseteq \hat{K}$ is identified to $K/\pi^j K$, and as this identification is $\mathrm{Ad}\, C$ equivariant ($C = \exp K$), this identifies Ad C orbits in $K/\pi^j K$ with Ad$*C$ orbits in $\hat{K}/(\pi^j K)^{\perp}$; that is, if $C_j = \exp \pi^j K$, then there is a canonical identification of conjugacy classes of C/C_j with representations of C/C_j. Thus C/C_j, which may be nonabelian, is in some sense self dual. (Note that since m is arbitrary, so is j.) If α_1, α_2 are two conjugacy classes, corresponding to representations $U(\alpha_i)$, with characters $\xi_0(\alpha_i)$, then $\xi(\alpha_i) = \dim U(\alpha_i)^{-1}\xi_0(\alpha_i)$ are the elementary idempotents corresponding to the representations and it may be verified that $\xi(\alpha_1)(\alpha_2) = \xi(\alpha_2)(\alpha_1)$. We thus have a symmetric function of two conjugacy classes (a "pairing" of conjugacy classes) which completely determines the harmonic analysis of C/C_j.

A more important application of the isomorphism $\theta: L \rightarrow \hat{L}$ consists in the following. Suppose $H \subseteq G$ is a subgroup, with subalgebra $M \subseteq L$ as lie algebra. Suppose (,) is nondegenerate on M, so we have also an isomorphism of M and $\hat{M}$, and so an injection $\lambda: \hat{M} \rightarrow \hat{L}$. If $J = K \cap M$, we get a consistent map $\lambda: \hat{J} \rightarrow \hat{K}$, and so a mapping $\tilde{\lambda}$ from the representations of $B(= \exp J)$ to those of $C(= \exp K)$ (since λ is Ad$*B$ equivariant) which has the property that the representation U of C corresponding to the representation V of B occurs in $U^{V,B}$. As we shall see, this mapping can in some cases be parlayed into a mapping from representations of H to representations of G. These mappings are related to the Cartan-Weyl highest weight theory for compact lie groups, and also seem to be related to the mappings discussed by Langlands in [4].

Now let $G = Gl_n(R) \cdot Sl_n(F)$. Let F' be an extension of F of degree n, with maximal order R'. Let $h: F' \rightarrow M_n(F)$ be a nontrivial homomorphism such that $h(R') \subseteq M_n(R)$. Let $H = G \cap h(F')$. Then H is compact and $H \subseteq h(R'^x)$.

LEMMA 2.1. *Of all maximal compact subgroups of G conjugate to $Gl_n(R)$, only $Gl_n(R)$ contains H and if $H' \subseteq G$ is conjugate to H, and $H' \subseteq Gl_n(R)$, then the conjugacy is accomplished by $g \in Gl_n(R)$.*

Proof. $h(R') \subseteq M_n(R)$ means R^n is invariant by $h(R')$. It follows that the totality of R-lattices in F^n invariant by $h(R')$ are those of the form $h(\pi')^m(R^n)$, π' a prime of F'. But a maximal compact subgroup of G may be specified as the set of $g \in G$ leaving fixed the R-lattices $\{\pi^m U\}$ for any fixed R-lattice $U \subseteq F^n$. Call this group $G_1(U)$. In order that $H \subseteq G_1(U)$, it is necessary and sufficient that $\{\pi^m U\} \subseteq \{h(\pi'^m)R^n\}$, that is, that $U = h(\pi'^l)R^n$ for some l. Thus, if e is the ramification index of F' over F, the maximal compact groups containing H are $G_1(h(\pi'^l)R^n)$ for $l = 0, 1, \cdots, e-1$. But since $h(\pi'^{l+1})R^n \subseteq h(\pi'^l)R^n$ strictly, none of these groups are conjugate in G.

Now take $g \in G$ such that $gHg^{-1} = H'$, with $H' \subseteqq Gl_n(R)$. Then $g(R^n)$ is H-invariant, and so equal to $h(\pi'^l)R^n$ for some l. Since g preserves volume $l = 0$, and $g \in Gl_n(R)$.

Now let L be the lie algebra of G: the traceless matrices, and $M = h(F')$, the lie algebra of H. Up to a constant, $x \times y \to \operatorname{tr}(xy)$ is the Killing form on L. It is nondegenerate on M. Also $M_n(R)* = M_n(R)$.

Theorem 1.1 applies to $C = (1 + \pi^{\beta+1}M_n(R))$. We have $K = \pi^{\beta+1}M_n(R)$, $\hat{K} \cong L/\pi^{m-2\beta-2}K$. Let $J = K \cap M$ and $B = h(F) \cap C$. Then, if we take $x \in M$, we have seen above that x determines a character ψ of B, and a representation $\tilde{\lambda}(\psi)$ of C. Let V be any representation of $Gl_n(R)$ which lies above $\tilde{\lambda}(\psi)$.

PROPOSITION 2.1. *Suppose that $x + \pi^{m-2\beta-2}K \subseteqq D = \operatorname{Ad} Gl_n(R)(M)$. Then U^V is an irreducible supercuspidal representation of G.*

Proof. We apply Corollary 3 of Proposition 1.4. Since $\operatorname{Ad} Gl_n(R)(M)(= \{gxg^{-1}\colon g \in Gl_n(R),\ x \in M)$ is $\operatorname{Ad} Gl_n(R)$-stable, the hypothesis of the proposition shows it contains the $\operatorname{Ad}{*}C$ orbit of $\theta(x)$. But now Lemma 2.1 shows V satisfies the conditions of Corollary 3, and since if U^V is irreducible, it is automatically supercuspidal, the proposition follows.

REMARK. D has nonempty interior, and is a sort of p-adic cone. The condition of the proposition is that x be well inside D, which is equivalent to asking that it be sufficiently far from any proper subfield of $h(F')$. Thus most x give rise to supercuspidal representations of G, and we obtain for each conjugacy class of maximal compact torus of G a series of representations, as one would expect. It is only near the walls that there are difficulties. It is of interest to know, how far from a subfield of $h(F')$ must x be to satisfy the proposition. If F is tamely ramified then estimates may be made showing that the necessary distance is very near that which guarantees that $x + \pi^{m-2\beta-2}J$ will intersect no subfield of $h(F')$. For F' wildly ramified, the necessary distance is larger, and depends on the different.

Reasoning analogous to that of § IV permits refinement of Proposition 2.1. I would not be surprised if a suitable refinement yielded all supercuspidal representations of Sl_n. It may be that this will be the main value of the Kirillov picture-to give an overall pattern which then may be filled in by other means to obtain complete results. Or it may happen that the Kirillov picture will itself be suitably adapted to encompass general groups.

III. Here we consider an algebraic group G defined over Q, and $\Gamma \subseteqq G$ an arithmetic subgroup. For each prime p, we have G_p, the points of G over Q_p, or the p-adic completion of the rational points of G, and the compact open subgroup Γ_p, the closure of Γ in G_p. The direct product $\prod_p \Gamma_p = \tilde{\Gamma}$ is a compact group and via the injection $\Gamma \to \tilde{\Gamma}$, the representation theory of $\tilde{\Gamma}$ can be used to construct finite dimensional representations of Γ. Since a representation of $\tilde{\Gamma}$ is just given by the tensor product of representations of a finite number of the Γ_p's, the results of § 1 may be applied to the solution of this problem. (We note also, by taking infinite products of representations of the local groups, infinite dimensional representations may be constructed.)

For some groups Γ, the above indicated construction exhausts all finite dimensional representations. Suppose that G contains a nonabelian unipotent subgroup N, defined over Q. Let U be a finite dimensional representation of Γ, and consider the restriction of U to $N \cap \Gamma = N_1$. N_1 is a discrete, nonabelian finitely generated, torsion free nilpotent group. According to Proposition one of [1], the kernel of U on N_1 is of finite index in $N_1^{(2)}$, the commutator subgroup.

Suppose Γ has the property that for a maximal rational unipotent subgroup N of G, the largest normal subgroup of Γ containing $(\Gamma \cap N)^{(2)}$ contains a congruence subgroup of G. Then clearly U factors through $\Gamma \cap G(q)$ for some q. ($G(q)$) consists of all rational elements of G whose entries are congruent to one modulo q), and so does arise from a representation of $\tilde{\Gamma}$. $Sl_n(\boldsymbol{Z})$ for $n \geqq 3$ may be verified to have this property, and in view of [6], it is possible any arithmetic subgroup of G, for G connected, simply connected, simple and split over Q does also.

IV. In this section, we find the representations of the multiplicative group D^x of a division algebra D of prime degree n over a local field F, of residual characteristic p, which may be the same as n. Although we shall not refer explicitly to the results of section one, the procedures here are essentially based on them, and it should not be too hard to see the connection.

Let $F_n \subseteqq D$ be an unramified splitting field. Let S, R_n, and R be the maximal orders of D, F_n, and F, and let Π be a prime of D, normalizing F_n, and such that $\Pi^n = \pi$, a prime of F (and of F_n). See Weil [8].

We denote the reduced trace of D by tr, and the bilinear form $\operatorname{tr}(xy)$ by τ. The reduced norm will be written det. The module of D, which defines an ultrametric norm on D will be denoted by vertical bars $|\ |$, and the corresponding valuation by ord_D. See [8] again.

We shall say that $d \in D$ is in general position if $|d| = \inf\{|d-f|: f \in F\}$. Clearly, given $d \in D$, there is $f \in F$, such that $d + f$ is in general position.

Given $d \in D - F$, d will generate a field F' over F, which is necessarily of degree n, n being prime. F' is therefore either totally ramified or unramified over F; and unless $n = p$, the residual characteristic, F' is tamely ramified. We denote by $F'^{\perp}$ the orthogonal complement of F' with respect to τ. Unless F' is inseparable, $D = F' \oplus F'^{\perp}$.

For $x, y \in D$, let $[x, y] = xy - yx$ be the commutator.

LEMMA 4.1. *If $d \in D$ is in general position, then* $[d, S] = F'^{\perp} \cap dS$.

Proof. Clearly $[d, S] \subseteqq dS$, and $[d, D] \subseteqq F'^{\perp}$, since if $d_1 \in F'$, then $\tau(d_1, (dx - xd)) = \mathrm{tr}\,(d_1xd) - \mathrm{tr}\,(d_1dx) = \mathrm{tr}\,(dd_1x) - \mathrm{tr}(dd_1x)$ since d_1 and d commute.

On the other hand $F'^{\perp}$ is an F' module, of dimension $n-1$ over F', and $[d, S]$ is an R' module (R' being the maximal order of F'). Therefore, to prove the proposition, it will suffice to find $n-1$ elements in $[d, S]$, which are linearly independent over F', and which generate over R' the intersection of dS with their span over F'. Write $d = \Pi^i u$, where $u \in S^x$. Then either $i = nj$, or i is prime to n. In the first case $u \notin F + \Pi^i S$ if d is to be in general position. Therefore, for $1 \leqq j \leqq n-1$, $\Pi^j u \Pi^{-j} - u \notin \Pi^i S$, and the set $[\Pi^j, d]$ will answer. If i is prime to n, then on $S/\Pi S$, conjugation by d induces a galois automorphism whose fixed points are just $R/\pi R$. Then if $u_1 \cdots u_{n-1}$ are units whose images in $S/\pi S$ span a complement to $R/\pi R$, it is not hard to see that the $[d, u_i]$ will answer.

Let G_i be the subgroup $1 + \Pi^i S$ of D^x, for $i \geqq 1$. G_i is a pro-p group, and $D^x = Z \cdot E \cdot G_1$, where Z is the cyclic group generated by Π and E is the group of mth roots of 1 in F_n^x, m prime to p.

Via the homomorphism det: $D^x \longrightarrow F^x$, we get for each character ψ of F^x, a one dimensional representation $\tilde{\psi}$ of D^x. Let U be any irreducible unitary representation of D. Let G_k be the smallest of the G_i's on which U is nontrivial. We will say U is in general position if the restriction of U to G_k does not coincide with a multiple of the restriction of some $\tilde{\psi}$. It is clear that U may always be written in the form $U = V \otimes \tilde{\psi}$, with V in general position. We may, therefore, restrict our attention to representations in general position.

If $2j > k$, then G_j/G_{k+1} is abelian, and the map $d \longrightarrow 1+d$ defines an isomorphism of $\pi^j S/\pi^{k+1} S$ onto G_j/G_{k+1}. Thus, if X is a basic character of F, whose conductor we will for simplicity take to be

R, then any character ϕ of G_j/G_{k+1} may be represented in the form $\phi(1+d) = X(\tau(h, d))$ for some $h \in D$. Since we may calculate that, with respect to τ, $S* = \Pi^{-n+1}S$, it follows that $\mathrm{ord}_D(h) = -l - n$, where l is the largest number such that ϕ is not trivial on G_l/G_{k+1}. In particular, if $j = k$, then $\mathrm{ord}_D(h) = -k - n$, if ϕ is nontrivial.

LEMMA 4.2. *If ϕ is not of the form $\tilde{\psi}|_{G_k}$, then h is in general position and conversely. In particular if k is prime to n, ϕ is not of the form $\tilde{\psi}|_{G_k}$ unless it is trivial.*

Proof. If k is prime to n, then h is already in general position. And since everything in G_k is already in the kernel of det modulo G_{k+1} this case is evident. If $k = k_0 n$, then if h is not in general position $|\Pi^i h \Pi^{-i} - h| < h$ for all i, so ϕ is $\mathrm{Ad}{*}D^x$ invariant. But now $G_k/G_{k+1} \cong H_{k_0}/H_{k_0+1}$ where $H_i = 1 + \pi^i R_n$, and under this isomorphism, $\mathrm{Ad}{*}D^x$ invariance translates to Galois-invariance, and the dual of Hilbert's Theorem 90 shows $\phi = \tilde{\psi}|_{G_k}$. On the other hand, if $\phi = \tilde{\psi}_{|G_k}$, ϕ is $\mathrm{Ad}{*}D^x$ invariant, and so $|\Pi^i h \Pi^{-i} - h| < |h|$, and h is not in general position.

Now let $j = (k/2) + 1$ or $(k+1)/2$, according as k is even or odd, and let ϕ be given on G_j/G_{k+1} by $\phi(1+d) = X(\tau(h, d))$, with h in general position. We intend to find all representations of D^x whose restrictions to G_j contain ϕ. There are four basic cases, according as k is even or odd, and prime to or divisible by n. The first step is to compute the isotropy group of ϕ under $\mathrm{Ad}{*}D^x$. Since $(\pi^j S)* = \pi^{-j-n+1}S$, the isotropy group, $A(\phi)$ of ϕ is the set of all $d \in D^x$ such that $\mathrm{ord}_D(dhd^{-1} - h) \geqq -j - n + 1$, or $\mathrm{ord}_D([d, h]) \geqq -j - n + 1 + \mathrm{ord}_D(d)$. Let F' be the field generated by h. Clearly $F'^{\times} \in A(\phi)$. If F' is unramified, then since h is in general position, $\Pi^i u$, with $u \in S^x$ cannot be in $A(\phi)$ unless $i = 0$. Similarly, if F' is ramified, only units in G_1 modulo F^x can possibly be in $A(\phi)$. In either case $A(\phi) = F'^x \cdot (A(\phi) \cap G_1)$, so we may take $d \in G_1$. Hence $\mathrm{ord}_D(d) = 0$. If $d = 1 + d'$, then $[d, h] = [d', h]$. If $d \in A(\phi)$, then by Lemma 4.1, we can choose $d'' \in \pi^{k-j+1}S$, such that $[d', h] = [d'', g]$. Conversely, if $\mathrm{ord}_D(d') \geqq k - j + 1$, $1 + d' \in A(\phi)$. Therefore $A(\phi) = F'^x \cdot G_{k-j+1}$. We see that if k is even, $A(\phi) = F'^x \cdot G_{j-1}$, while if k is odd, $A(\phi) = F'^x \cdot G_j$. In the latter case things are very simple, and are covered by the following lemma.

LEMMA 4.3. *Suppose $G = HN$, with N finite and normal and H abelian. Let ϕ be a one dimensional character of N which is $\mathrm{Ad}{*}G/N$ invariant. Let $\{\psi_\alpha\}$ be the characters of H which extend ϕ on $H \cap N$. Then ϕ is extendable to G, and the representations of G lying over ϕ are one dimensional characters ϕ_α such that $\phi_\alpha(hn) =$*

$\psi_\alpha(h)\phi(n)$.

The proof is obvious. Inducing up from $A(\phi)$ then gives all representations above ϕ.

We now describe the case when k is even. First, take F' to be unramified. Up to conjugacy, we may take $F' = F_n$. We have in this case $k = k_0 n$. If $n = 2$, so D is a quaternion algebra, then k_0 is arbitrary. Otherwise k_0 must be even. If k_0 is even, then $j - 1 = (k_0/2)n$, and $F_n^x \cdot G_{j-1} = F_n^x \cdot G_j$, so we are in the same situation as for k odd, and Lemma 4.3 applies again.

If $n = 2$ and k_0 is odd, then $k_0 = j - 1$. Consider the map $d_1 \times d_2 \rightarrow \phi(d_1 d_2 d_1^{-1} d_2^{-1})$ on G_{j-1}. This defines a nondegenerate antisymmetric bilinear form σ on G_{j-1}/G_j, with values in the circle. σ measures the Mackey obstruction of extending ϕ to G_{j-1}. In particular, ϕ may be extended to $G' \subseteq G_{j-1}$ if and only if σ is trivial on G'. If $\bar{F} = R/\pi R$ is the residue class field of F, then G_{j-1}/G_j can be identified with $\bar{F}_2$, the quadratic extension of $\bar{F}$, in such a way that conjugation by F_2^x becomes multiplication by the elements of $\bar{F}_2$ of norm one. Of course, this action preserves σ. Also, this action is clearly irreducible over $\bar{F}$. I claim it is actually irreducible over the prime field $\boldsymbol{Z}/p\boldsymbol{Z}$, so that no proper subgroup of G_{j-1}/G_j is invariant under conjugation by F_2^x. This will follow from this simple lemma.

LEMMA 4.4. *Let $\bar{F}_n \supseteq \bar{F}_m \supseteq \bar{F}$ be finite fields, with $\bar{F}_n$, $\bar{F}_m$ of degrees n, m over $\bar{F}$. Let N be the kernel the norm map from $\bar{F}_n^x$ to $\bar{F}_m^\times$. Then N generates $\bar{F}_n$ over $\bar{F}$.*

Proof. If $^\#(\bar{F}) = q$, then $^\#(\bar{F}_n) = q^n$, and $^\#(N) = (q^n - 1)/(q^m - 1) = q^{n-m} + q^{n-2m} + \cdots + 1 > q^{n/2}$, since $n = mn_0$, with $n_0 \geqq 2$. If N generates $\bar{F}_l$ over $\bar{F}$, then $^\#(\bar{F}_l) = q^l \leqq q^{n/2}$, unless $l = n$. (Of course, the fact for general fields follows from dimension considerations.)

It is now clear that ϕ cannot be extended to any subgroup of $A(\phi)$ large enough to induce an irreducible representation. The best procedure seems to be this. Choose a maximal subgroup G'_{j-1} of G_{j-1} on which σ is trivial. ϕ may then be extended to ϕ' on $Z \cdot F^x \cdot H_1 \cdot G'_{j-1}$, which will induce an irreducible representation V of $Z \cdot F^x \cdot G_1$. This representation will be $\mathrm{Ad}*(D^x/Z \cdot F^x \cdot G_1)$ invariant, and so upon inducing to D^x, $^\#(D^\#/Z \cdot F^x \cdot G_1) = {}^\#(F_n^x/F^x \cdot H_1)$ different irreducible representations $\{U_i\}$ will result. (The Mackey obstruction now vanishes trivially, since $D^x/Z \cdot F^x \cdot G_1$ is cyclic.) However, the U_i are apparently not monomial representations.

We proceed to the case when k is even and F' is ramified. Once again $d_1 \times d_2 \rightarrow \phi(d_1 d_2 d_1^{-1} d_2^{-1})$ is an antisymmetric form σ on G_{j-1}/G_j. σ now has radical $(F'^x \cap G_{j-1})/G_j$. If we identify G_{j-1}/G_j

with $\bar{F}_n$ over $\bar{F}$, by choosing a prime π' of F' and identifying $1 + \pi'^{j-1}$ to $1 \in \bar{F}_n$, then $(F'^x \cap G_{j-1})/G_j$ is identified to $\bar{F}$ and conjugation by F'^x becomes the galois action on $\bar{F}_n$ over $\bar{F}$. If, moreover, G_k/G_{k+1} is also identified to $\bar{F}_n$ by sending $1 + \pi'^k$ to $1 \in \bar{F}_n$, and if τ is the element of the galois group of $\bar{F}_n$ over $\bar{F}$ defined by conjugation by π'^{j-1} in G_{j-1}/G_j, then σ takes the form $\sigma(a, b) = \psi(\tau(a)b - a\tau(b))$, where ψ is a character of $\bar{F}_n$. The condition that $\bar{F}$ be in the radical of σ implies that ψ is trivial on T, the elements of $\bar{F}_n$ of trace zero. We must distinguish several cases.

First, suppose n equals the residual characteristic. Then the galois action is unipotent, and $\tau - 1$, where τ is the above defined element of the galois group, is nilpotent with kernel $\bar{F}$. Thus there is $x \in \bar{F}_n$ such that $\tau(x) = x + 1$. From this, it follows that $(\tau-1)^{k+1}(x^k) = 0$. Thus the image of $\tau - 1$, which is T, is spanned by $1, x, \cdots, x^{n-2}$. Then the span Y of $x, x^2, \cdots, x^{n-1/2}$, is galois invariant and σ is trivial on it. Let G'_{j-1}/G_j be the subgroup of G_{j-1}/G_j corresponding to Y. Then we see ϕ on G_j may be extended to ϕ' on $F'^x \cdot G'_{j-1}$, which is a group by construction of G'_{j-1}, and ϕ' will induce an irreducible representation of D^x. All representations of D^x lying over ϕ are obtained in this way, by standard theory.

If n is prime to the residual characteristic, T, the traceless elements, is a complement to $\bar{F}$ in $\bar{F}_n$, and is stable by the galois group $\mathrm{Gal}(\bar{F}_n/\bar{F})$. On T, τ satisfies the cyclotomic equation $\sum_{i=0}^{n-1} \tau^i = 0$. Let $\bar{F}_m$ be the field generated over $\bar{F}$ by the nth roots of unity. Then m divides $n - 1$, and the galois action splits into $(n - 1/m) = l$ irreducible submodules over $\bar{F}$, each corresponding to a set of nontrivial homomorphisms $h\colon \mathrm{Gal}(\bar{F}_n, \bar{F}) \longrightarrow \bar{F}_m^x$ which are mutually conjugate under the action of $\mathrm{Gal}(\bar{F}_m, F)$. If h is one such homomorphism, let $\bar{h}$ be the homomorphism such that $\bar{h}h = 1$, the trivial homomorphism. If m is even, then the unique element of order 2 in $\mathrm{Gal}(\bar{F}_m, F)$ conjugates h and $\bar{h}$. If m is odd, however, h and $\bar{h}$ are nonconjugate. In this case, we may select $l/2$ irreducible modules for the galois action, whose sum forms an isotropic subspace for σ, and so in this case too, the representations above ϕ on G_j are monomial. If m is even, though, no galois invariant $\bar{F}$-subspace of T is isotropic for σ. If $\bar{F} = \boldsymbol{Z}/p\boldsymbol{Z}$, subspace may be replaced by subgroup; in general, the situation is somewhat delicate. We will not pursue the matter further. In any case, one may select G'_{j-1}, a maximal σ-isotropic subgroup of G_{j-1}, extend ϕ to a character ϕ' of $F^x \cdot (F'^x \cap G_1) \cdot G'_{j-1}$, induce to an irreducible representation of $F_n^x \cdot G_1$ then finally induce to get n irreducible representations of D^x, corresponding to the n extensions of ϕ' on $F^x(F'^x \cap G_1) \cdot G_j$ to $F'^x \cdot G_j$. This finishes the last case. There remains the question of parametrizing the representations appropriately. This is quite easy for n prime to p, but presents difficulties when $n=p$. In either case, it necessitates considerations sufficiently different from what has gone before, that we defer it for fuller treatment later.

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