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ON ARC LENGTH SHARPENINGS

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This paper introduces two new sharpenings:

THEOREM. Let A denote a rectifiable arc (with length $l(A)$) of a metric space, let P denote a finite, normally-ordered subset of A, and let $l(T^*(P))$ denote the linear content of a mini-tree $T^*(P)$ spanning P. Then $\text{l.u.b.}_{P\subset A}\, l(T^*(P)) = l(A)$.

DEFINITION. If E is a nonempty subset of a set P that is spanned by tree T, then T is said to be on E.

THEOREM. Let $\sigma(E)$ denote the greatest lower bound of the linear contents of all trees on E. If A denotes a rectifiable arc of a finitely compact metric space, then $\text{l.u.b.}_{E\subset A}\, \sigma(E) = l(A)$, where E denotes any finite normally-ordered subset of A.

On arc length sharpenings.[1] It is convenient to call an unordered pair of distinct points p, q of a metric space M a *segment*, denoted by $\{p, q\}$. Each of the points p, q of the segment $\{p, q\}$ is an *endpoint* of the segment, and the *length* of $\{p, q\}$ is the distance pq of its endpoints.

A nonempty set S of distinct segments forms a *chain* C provided the end points of the segments may be labelled $a_0, a_1, \cdots, a_k$ (with all the a_i's representing pairwise distinct elements of M) so that the elements of S are $\{a_0, a_1\}, \{a_1, a_2\}, \cdots, \{a_{k-1}, a_k\}$. The chain is said to *join* a_0 and a_k; the points $a_0, a_1, \cdots, a_k$ are the *vertices* of the chain.

A nonempty set S of segments forms a *tree* T provided each two distinct points of the set of endpoints of the segments are joined by exactly one chain of its segments. The *vertices* of T are the endpoints of its segments. The segments of a tree are called *branches*, and the *linear content* of a tree is the sum of the lengths of its branches. If a tree T has set E as its vertex set, then T is said to *span* E. If E is a nonempty subset of a set P, and tree T spans P, then T is said to be *on* E.

A finite subset E (containing at least two points) of M is spanned by only a finite number of trees. Let $L(E)$ denote the minimum of the linear contents of the trees that span E and let $T^*(E)$ symbolize any tree spanning E whose linear content $l(T^*(E))$ equals $L(E)$. $T^*(E)$ is referred to as a *mini-tree spanning* E.

Denote by $\sigma(E)$ the greatest lower bound of linear contents of all trees that span P where $P \supset E$ (P is a finite subset of M); that

[1] From research for University of Missouri Dissertation (1973).

is, $\sigma(E)$ is the greatest lower bound of the linear contents of all trees on E. Clearly $0 < \sigma(E) \leqq L(E)$, and easy examples of subsets E of the Euclidean plane exist such that $\sigma(E) < L(E)$.

A subset A of a metric space is an *arc* provided it is the homeomorph of a line segment: that is, $A = f(I)$, where f is biuniform and bicontinuous in the line segment $I = [a, b]$, $a < b$. The points $\alpha = f(a)$, $\beta = f(b)$ are the *endpoints* of A. Calling α the initial point and β the terminal point of A serves to orient the arc, and a finite subset $P = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ of A is *normally ordered* provided these points are encountered in the order of their subscripts when the arc is traversed from α to β.

This paper furnishes the following two primary theorems, along with several other results in support of these theorems.

THEOREM 1. *For each rectifiable arc A of a metric space,*

$$\operatorname*{l.u.b.}_{P \subset A} l(T^*(P)) = l(A) ,$$

where P represents a finite normally ordered subset of A.

THEOREM 2. *If A denotes a rectifiable arc of a finitely compact metric space M, with length $l(A)$, then*

$$\operatorname*{l.u.b.}_{E \subset A} \sigma(E) = l(A) ,$$

where E denotes any finite normally ordered subset of A.

The literature refers to this kind of theorem as an arc length "sharpening." Theorem 1 is similar in nature to a "second sharpening" as presented by Blumenthal [1], and Theorem 2 partially answers the question of a possible "third sharpening" posed by the same author. Menger and Mimura [3] have previously proved a third sharpening for arcs of Euclidean space.

For arcs of the type of space prescribed by Theorem 2, points not on the arc may be allowed to enter into the computation of its length.

If p_i is a vertex of a tree T, let $B(p_i)$ denote the set of vertices of T, which paired with p_i, form branches of T. Call $B(p_i)$ the *set of vertices of T sending branches to p_i*. The cardinality of $B(p_i)$ is denoted by $o(p_i)$ and is called the *order* of p_i. A *subtree* of a tree T is a tree which has each of its branches a branch of T.

LEMMA 1. *Let T be a tree spanning vertex set $E = \{p_1, p_2, \cdots, p_n\}$. If $p_i \in E$ has order $o(p_i) = k$, then $E - \{p_i\}$ is the union of k non-*

empty mutually exclusive subsets, each of which is a vertex set of a subtree of T or is a single point.

Proof. To say that a vertex p_i of a tree T is of order k means that p_i is joined directly (paired with) exactly k other vertices of T by branches of T. So with respect to pairs of points of E which are branches of T, there are k distinct disjoint subsets of E which are joined only through point p_i by branches of T. For if the contrary be assumed, some two of them would have a vertex in common and the two would be connected by branches of T. Thus, in the tree T, there would exist two sequences of branches joining any point of the two subsets with p_i, contradicting the uniqueness of chains in a tree.

THEOREM A. *If T is a tree spanning a set E of n distinct points ($n \geqq 2$), then T has exactly $n - 1$ branches.*

Proof. The statement is clearly valid for $n = 2$. Suppose the result holds for trees with vertex sets E of cardinality at most $n - 1$, it will be shown valid for n. Let E be a set of n distinct points, T a tree spanning E, and $p \in E$ a vertex of T of order r. By Lemma 1, the set $E - \{p\}$ is the sum of a finite number of subsets $E_1, E_2, \cdots, E_r$, each of which admits a subtree T_i of T but ceases to do so after the adjunction of any point of $E - \{p\}$ not belonging to it, or is a single point. If E_i consists of n_i points, then $n_i \leqq n - 1$ $(i = 1, 2, \cdots, r)$. By the inductive hypothesis, if E_i admits a subtree T_i, then T_i consists of exactly $n_i - 1$ branches. If E_i consists of a single point, no branch of T is admitted by E_i. In either case, each E_i admits exactly $n_i - 1$ branches of T. Hence $E - \{p\}$ admits exactly

$$\sum_{i=1}^{r} (n_i - 1) = \sum_{i=1}^{r} n_i - r = (n - 1) - r$$

branches of T. Since T is a tree, there is exactly one point $p_i \in E_i$ such that $\{p_i, p\}$ is a branch of $T (i = 1, 2, \cdots, r)$. Hence T has exactly $((n - 1) - r) + r = n - 1$ branches.

A normally ordered subset $P = \{p_1, p_2, \cdots, p_n\}$ of an arc A with endpoints a', b' is a *homogeneous ε-chain* provided

(1) $a'p_1 < \varepsilon$, $p_n b' < \varepsilon$,

(2) $p_i p_j = \varepsilon$, for $|i - j| = 1$ $(i, j = 1, 2, \cdots, n)$,

(3) $p_i p_j \geqq \varepsilon$, for $|i - j| > 1$ $(i, j = 1, 2, \cdots, n)$.

If a normally ordered subset is such that $a'p_1 < \varepsilon$, $p_n b' < \varepsilon$, $p_i p_{i+1} \leqq \varepsilon$ for $i = 1, 2, \cdots, n - 1$, then P is called an *ε-chain*.

Three known lemmas needed in the proof of Theorem 1 will now be stated without proof (cf. [1]).

LEMMA 2. *For each positive number ε, there exists a homogeneous ε-chain in A.*

LEMMA 3. *For every positive number δ there exists a positive number ε such that every ε-chain in A is δ-dense in A.*

LEMMA 4. *Let $A = f(I)$ be a rectifiable arc of a metric space with length $l(A)$. Then any positive number η implies the existence of a positive δ such that for each finite normally ordered subset P of A which is δ-dense in A, $l(P) > l(A) - \eta$.*

Proof of Theorem 1. Since $l(T^*(P)) \leqq l(P)$, for $P \subset A$, then

$$\operatorname*{l.u.b.}_{P\subset A} l(T^*(P)) \leqq \operatorname*{l.u.b.}_{P\subset A} l(P) = l(A) \, .$$

It remains to show that $\text{l.u.b.}_{P\subset A}\, l(T^*(P)) \geqq l(A)$. By Lemma 4, $\eta > 0$ implies the existence of a positive δ such that for each P which is δ-dense in A, $l(P) > l(A) - \eta$. By Lemma 3, there corresponds to this δ a positive ε such that every ε-chain in A is δ-dense in A, and Lemma 2, there exists for this ε a homogeneous ε-chain $P = \{p_1, p_2, \cdots, p_n\}$ in A. Then P is δ-dense in A and $l(P) > l(A) - \eta$. Now P being a normally ordered set and a homogeneous ε-chain imply that $l(P) = (n-1)\varepsilon$. Since each two points of P has distance at least ε and tree T^* spanning P has $n-1$ branches, $l(T^*(P)) \geqq (n-1)\varepsilon$. So for each homogeneous ε-chain P, $l(P) = l(T^*(P))$. Hence corresponding to each $\eta > 0$ a subset P of A exists with $l(T^*(P)) > l(A) - \eta$; that is,

$$\operatorname*{l.u.b.}_{P\subset A} l(T^*(P)) \geqq l(A) \, .$$

The remainder of this paper is presented in support of Theorem 2. The following problem is similar to one found in [2].

Problem T_n. Given a set $E = \{p_1, p_2, \cdots, p_n\}$ of $n \geqq 3$ points of a metric space M. Find the mini-tree(s) on E.

If $p, q, r \in M$ are linear with q metrically between p and r, then a mini-tree on set $\{p, q, r\}$ is $\{\{p, q\}, \{q, r\}\}$; that is, no additional vertices are required to yield a mini-tree on $\{p, q, r\}$. Further, the trees $\{\{p, r\}\}$ and $\{\{p, q\}, \{q, r\}\}$ have the same linear content.

DEFINITION. A tree T is an R-tree on a subset $E = \{p_1, p_2, \cdots, p_n\}$ of $n \geqq 3$ points of M provided the set of all vertices $p_1, p_2, \cdots, p_n$,

$q_1, \cdots, q_k$ of T are elements of such that (1) $o(q_i) \geqq 3$ $(i = 1, 2, \cdots, k)$, and (2) $0 \leqq k \leqq n - 2$.

THEOREM B. *Let $E = \{p_1, p_2, \cdots, p_n\}$ be a finite subset of $n \geqq 3$ points of a metric space M. If a solution of the Problem T_n exists, then it is an R-tree on E or it can be replaced by an R-tree having the same linear content.*

Proof. The result is trivial, if $k = 0$, in which case $o(q_i) \geqq 3$ is vacuously satisfied. So suppose that $k > 0$ and that T is a mini-tree on E spanning $E^A = \{p_1, p_2, \cdots, p_n, q_1, \cdots, q_k\}$ (augmented vertex set). It is easy to see that $o(q_i) > 1$ $(i = 1, 2, \cdots, k)$. Suppose $o(q_i) = 2$ for some i and that $B(q_i) = \{r', r''\} \subset E^A$. If q_i is linear with vertices r', r'', then branches $\{r', q_i\}$, $\{q_i, r''\}$ can be replaced by $\{r', r''\}$ without increasing the linear content of T. On the other hand, if r', r'', q_i are not linear, the triangle inequality implies that $r'r'' < r'q_i + q_ir''$. Then the tree formed by replacing branches $\{r', q_i\}$, $\{q_i, r''\}$ by $\{r', r''\}$ would decrease the linear content of T, contradicting the minimal linear content assumption on T. Thus $o(q_i) \geqq 3$ for each i $(i = 1, 2, \cdots, k)$ or T can be replaced by a tree of equal linear content having this property.

Without loss of generality, it may be assumed that T has $o(q_i) \geqq 3$ for each i $(i = 1, 2, \cdots, k)$.

Then the number of branches leading from q-points is at least $3k/2$; the number from p-points at least $n/2$. But $n + k - 1 =$ (total number of branches of $T) \geqq 3k/2 + n/2$, from which it is easily deduced that $k \leqq n - 2$. This concludes that proof.

Let T, T' be R-trees with vertex sets $\{p_1, p_2, \cdots, p_n, q_1, q_2, \cdots, q_k\}$ and $\{p_1, p_2, \cdots, p_n, q_1', q_2', \cdots, q_k'\}$, respectively. The trees T and T' *have the same structure* provided the mapping f with $f(p_i) = p_i$, $f(q_j) = q_j'$ $(i = 1, 2, \cdots, n; j = 1, 2, \cdots, k)$ maps $B(p_i)$ onto $B'(p_i)$ and $B(q_i)$ onto $B'(q_j)$, where $B'(p_i)$ and $B'(q_j)$ are subsets of the vertex set of T'.

In general, there will exist infinitely many trees having the same structure. The relation "has the same structure as" on the class of all R-trees on a given finite set E is an equivalence relation. Because $0 \leqq k \leqq n - 2$ and a structure class is determined largely by the k additional vertices, it is further seen that the number of equivalence classes of R-trees on a given set of n points of M under the relation "has the same structure as" is finite. Let these equivalence classes be denoted by $C_1, C_2, \cdots, C_N$.

THEOREM C. *Let $E = \{p_1, p_2, \cdots, p_n\}$ be a subset of a finitely compact metric space M. There exists a mini-tree on E.* (*As in* [2], *p.* 447, *with modifications.*)

Proof. Since there are only a finite number of the aforementioned equivalence classes, it suffices to show that if $C \in \{C_1, C_2, \cdots, C_N\}$, then there exists on E a mini-tree in C.

The structure of trees in C stipulates which of the pairs $q_i p_j$, $q_i q_j$, $p_i p_j$ will be segments as branches of trees in C.

If $k = 0$, there are no q points. Since E is finite, there are only a finite number of ways to form pairs of elements of E. Hence a minimum length tree can be selected.

If $k \neq 0$, each R-tree in C has vertex set consisting of $\{p_1, p_2, \cdots, p_n\}$ together with k additional q-points. Let these be denoted $q_1, q_2, \cdots, q_k$. For $i = 1, 2, \cdots, k$, let $E_i = \{p_{i_1}, p_{i_2}, \cdots, p_{i_{\lambda_i}}\} = \{p_j : p_j \in E$ and $\{q_i, p_j\}$ is a branch of trace of trees in $C\}$. Let N_1, N_2 be sets of unordered pairs of natural numbers, defined as follows:

$N_1 = \{\{i, j\} : \{q_i, q_j\}$ is a branch of trees in $C\}$,

$N_2 = \{\{i, j\} : \{p_i, p_j\}$ is a branch of trees in $C\}$.

Then the linear content of a tree in C is

$$f(q_1, q_2, \cdots, q_k) = \sum_{i=1}^{k} \sum_{j=1}^{\lambda_i} q_i p_{i_j} + \sum_{\{i,j\} \in N_1} q_i q_j + \sum_{\{i,j\} \in N_2} p_i p_j \,.$$

Now suppose that $\{\bar{q}_1, \bar{q}_2, \cdots, \bar{q}_k\}$ represents a specific (fixed) set of q-points. Since $\bar{E}^A = \{p_1, p_2, \cdots, p_n, \bar{q}_1, \bar{q}_2, \cdots, \bar{q}_k\}$ is a finite set, there exists a mini-tree $\bar{T}(k)$ spanning set $\bar{E}^A$. Let $\bar{L}_k$ denote the linear content of this mini-tree. Let set $X = \{r : r \in M$ and $\min_i rp_i \leqq \bar{L}_k\}$. Then every q-point of a tree of minimal linear content in C is in X, for otherwise the length of such a tree would be greater than $\bar{L}_k$. Thus if $\{q_1, q_2, \cdots, q_k\}$ is a set of q-points of a mini-tree of C, then $\{q_1, q_2, \cdots, q_k\}$ is an element of the Cartesian product X^k. Since X is bounded and M is finitely compact, X is compact. Tychonoff's theorem implies that X^k is compact. But $f(q_1, q_2, \cdots, q_k)$ is continuous on X^k, and so assumes a minimum value on X^k. This proves the theorem.

Suppose that A denotes a rectifiable arc of a finitely compact metric space M. Denote by $l(A)$ the length of arc A and let $P_i = \{p_i^1, p_i^2, \cdots, p_i^{n(i)}\}$ $(i = 1, 2, \cdots)$ be an ε_i-chain in arc A and $\lim_{i \to \infty} \varepsilon_i = 0$. By the above theorem, there exists a mini-tree $T(P_i)$ spanning a set P_i^* containing P_i. Denote by $l(T(P_i))$ the linear content of such a tree. Then for each i, $l(T(P_i)) \leqq \sum_{j=1}^{n(i)-1} p_i^j p_i^{j+1} \leqq l(A)$. Hence $\text{l.u.b.}_i\, l(T(P_i)) \leqq l(A)$.

LEMMA 5. *If $T(P_i)$ is a mini-tree on an ε_i-chain P_i of A, then each branch of $T(P_i)$ has length at most ε_i.*

Proof. Suppose that the length of a branch of $T(P_i)$ were greater than ε_i. Then that branch could be deleted from $T(P_i)$, destroying the connectedness of $T(P_i)$; the connectedness could then be restored by pairing an appropriate two successive points of P_i as a branch. This would result in a tree $T'(P_i)$ of smaller linear content than that of $T(P_i)$, contradicting the mini-property of $T(P_i)$.

Note the following two possibilities:

Either (1) there exists $\delta > 0$ and for infinitely many i, point $p_i \in P_i^*$ such that distance $p_iA \geqq \delta$,

or

(2) for each $\delta > 0$, there exists a positive integer N such that $i > N$ implies $p_iA < \delta$, for each $p_i \in P_i^*$.

THEOREM D. *Let A be a rectifiable arc (with length $l(A)$) of a finitely compact metric space M and let P_i $(i = 1, 2, \cdots)$ be an ε_i-chain in arc A and $\lim_{i\to\infty} \varepsilon_i = 0$. Then statement (2) above holds.*

Proof. Suppose the contrary. Then there exists $\delta > 0$ and for infinitely many i, a point $p_i \in P_i^*$ such that p_i has distance from A greater than or equal to δ. The number of points of P_i^* which can be connected with p_i by a chain of $T(P_i)$ of length at least $\delta/2$ is at least $2^{[\delta/(2\varepsilon_i)]}$, since by the above lemma, $T(P_i)$ has at least $[(\delta/2)/\varepsilon_i]$ branches. Since P_i^* is connected by branches of $T(P_i)$ and P_i^* contains P_i, each of the above mentioned $2^{[\delta/(2\varepsilon_i)]}$ points is connected to each of the points of the ε_i-chain P_i on arc A by a chain of length greater than $\delta/2$. Since $T(P_i)$ is a tree, these chains have no vertices in common. Thus for infinitely many i, $l(T(P_i)) > (\delta/2)\cdot 2^{[\delta/2\varepsilon_i]}$, and hence $\lim_{i\to\infty} l(T(P_i)) = \infty$. But $\text{l.u.b.}_i\, l(T(P_i)) \leqq l(A) < \infty$.

Intuitively, the above result says that for each $\delta > 0$, almost all of the mini-trees $T(P_i)$ are contained within a "tube" of radius δ about the arc A.

THEOREM E (Continuity). *Let $l(T(E))$ denote the linear content of a mini-tree $T(E)$ on $E = \{p_1, p_2, \cdots, p_n\} \subset M$, and suppose that $E^* = \{p_1, p_2, \cdots, p_n, q_1, q_2, \cdots, q_k\}$ is the vertex set of $T(E)$. Let $\varepsilon > 0$ be given and consider points $p_1', p_2', \cdots, p_j'$ (for some j such that $1 \leqq j \leqq n$) with distances $p_ip_i' < \delta = \varepsilon/(3(2n-3))$ $(i = 1, 2, \cdots, j)$. Then a mini-tree on $E' = \{p_1', p_2', \cdots, p_j', p_{j+1}, \cdots, p_n\}$ has linear content $l(T(E'))$ such that $|l(T(E)) - l(T(E'))| < \varepsilon$.*

Proof. Consider a tree T' spanning $\{p_1', p_2', \cdots, p_j', p_{j+1}, \cdots, p_n, q_1, q_2, \cdots, q_k\}$ and having the same structure as mini-tree $T(E)$. Now $l(T(E)) = \sum p_rp_s + \sum p_rq_s + \sum q_rq_s$, with $l(T')$ possibly differing from $l(T(E))$ only in the distances of the types p_rp_s and p_rq_s. But

$$p_r'p_s' \leqq p_r'p_r + p_rp_s + p_sp_s' < p_rp_s + 2\delta ,$$

$$p'_r p_s \leqq p'_r p_r + p_r p_s < p_r p_s + \delta \text{ , and}$$
$$p'_r q_s \leqq p'_r p_r + p_r q_s < p_r q_s + \delta \text{ .}$$

In total, there are $n + k - 1$ branches in the tree T' and $k \leqq n - 2$. So there are at most $2n - 3$ branches in T'. Therefore,

$$\begin{aligned} l(T') &< \sum p_r p_s + \sum_{i=1}^{2n-3} 2\delta + \sum p_r q_s + \sum_{i=1}^{2n-3} \delta + \sum q_r q_s \\ &= \sum p_r p_s + \sum p_r q_s + \sum q_r q_s + \sum_{i=1}^{2n-3} 3\delta \\ &= l(T(E)) + (2n - 3)\cdot 3\varepsilon/(3(2n - 3)) \\ &= l(T(E)) + \varepsilon \text{ .} \end{aligned}$$

But a mini-tree on E' has linear content $l(T(E')) \leqq l(T') < l(T(E)) + \varepsilon$. Hence $-\varepsilon < l(T(E)) - l(T(E'))$.

Now it remains to show that $l(T(E)) - l(T(E')) < \varepsilon$. Suppose the contrary; that is, that there exists $\varepsilon_0 > 0$ such that for each $\delta > 0$ (and in particular, for $\delta = \varepsilon_0/(3(2n - 3))$ there are points $p'_i (i = 1, 2, \cdots, j)$ with $p'_i p_i < \delta$ $(i = 1, 2, \cdots, j)$ and $l(T(E')) \leqq l(T(E)) - \varepsilon_0$. But for ε_0, there is $\delta = \varepsilon_0/(3(2n - 3))$ such that $p''_i p'_i < \delta (i = 1, 2, \cdots, j)$ implies that $l(T(E'')) < l(T(E')) + \varepsilon_0$ (by the first part of the proof). (Here $l(T(E''))$ denotes the linear content of a mini-tree on $E'' = \{p''_1, p''_2, \cdots, p''_j, p_{j+1}, \cdots, p_n\}$). In particular, the $p''_i = p_i (i = 1, 2, \cdots, j)$ are such that $p''_i p'_i < \delta$. Then $l(T(E)) < l(T(E')) + \varepsilon_0 \leqq (l(T(E)) - \varepsilon_0) + \varepsilon_0 = l(T(E))$. This contradiction establishes the result.

Proof of Theorem 2. Let P be a finite normally ordered subset of a rectifiable arc A of a finitely compact metric space M and $l(T(P))$ denote the linear content of a mini-tree on P (and spanning vertex set $P^* \supset P$). Defining $l^*(A) = \text{l.u.b.}_{P \subset A}\, l(T(P))$, it suffices to show that $l^*(A) \geqq l(A)$. To do this let $l(A) = \text{l.u.b.}_{P \subset A} \sum_{i=1}^{n-1} p_i p_{i+1}$ and denote by A_i the subarc of A lying between p_i and p_{i+1}. Then $l^*(A_i) \geqq p_i p_{i+1}$ holds, and it remains to show that $l^*(A) \geqq \sum_{i=1}^{n-1} l^*(A_i)$. But this follows by complete induction on the statement $l^*(A_1 + A_2) \geqq l^*(A_1) + l^*(A_2)$ the proof of which follows.

For an arbitrary $\varepsilon > 0$, there exists two finite subsets $E'_1 \subset A_1$ and $E'_2 \subset A_2$ such that $l^*(A_1) - \varepsilon \leqq l(T(E'_1))$ and $l^*(A_2) - \varepsilon \leqq l(T(E'_2))$, where $T(E'_i)$ denotes a mini-tree on $E'_i (i = 1, 2)$. By Theorem E, the linear content $l(T(E'_i))$ varies an arbitrarily small amount in a sufficiently small neighborhood of a point of E'_i. It is thus possible to choose E'_1 and E'_2 such that the common endpoint p_2 of A_1 and A_2 lies neither in E'_1 nor E'_2, so the last point q_1 of E'_1 and the first point q_2 of E'_2 are different from p_2. Let A'_1 be the subarc of A between p_1 and q_1; A'_2 the subarc between q_2 and p_3.

Since a metric space is a normal topological space, there exist two open sets U_1 and U_2 which contain the subarcs A'_1 and A'_2, respectively, and which have disjoint closures. If d denotes the distance of the set $A'_1 + A'_2$ from the boundary of $U_1 + U_2$, then $d > 0$. Let δ be such that $0 < \delta < \min\{d, \varepsilon d/l^*(A_1 + A_2)\}$. There exist two normally ordered finite sets E_1 and E_2 such that $E'_1 \subset E_1 \subset A'_1$ and $E'_2 \subset E_2 \subset A'_2$, and such that each two successive points in E_1 and E_2 have distance less than δ. Let $E = E_1 + E_2$ and let $l(T(E))$ denote the linear content of a mini-tree on E (and having vertex set $E^* \supset E$).

Now decompose the finite sets E_i $(i = 1, 2)$ into finitely many classes such that two points of E_i are in the same class provided the set $E^* \cdot \bar{U}_i$ contains the end points of segments of a chain of $T(E)$ joining the two points. If a given point of E_i is not connected by such a chain to any other point of E_i, then the point is in a class by itself. Let $n_i (i = 1, 2)$ be the number of these classes. Since E^* is connected by chains of $T(E)$, every two classes of E_i are such that each point of one class is joined by a chain of $T(E)$ to each point of the other, but which does not have all of its vertices contained in $\bar{U}_i$. According tothe definition of these classes and because of the disjointness of $\bar{U}_1$ and $\bar{U}_2$, every two of these $n_1 + n_2$ chains are disjoint. On the other hand, each pair of these chains has distance apart at least d, since $E_1 + E_2$ as a subset of $A'_1 + A'_2$ has distance $\geqq d$ from the boundary of $U_1 + U_2$. Then $(n_1 + n_2)d \leqq l(T(E)) \leqq l^*(A_1 + A_2)$, and so $\delta < \varepsilon d/((n_1 + n_2)d) = \varepsilon/(n_1 + n_2)$. Hence $(n_1 + n_2)\delta < \varepsilon$.

Now for each class, except for the two which contain the last points of E_1 and E_2, introduce a segment consisting of the last point of the class and the next point of E_1 or E_2 (according as the class is in E_1 or E_2, respectively). Since each pair of successive points in E_1 and E_2 has distance $\leqq \delta$, then each of the above $n_1 + n_2 - 2$ segments has length $\leqq \delta$ and is therefore contained in $\bar{U}_1$ or $\bar{U}_2$, respectively. Augmenting $T(E)$ with these $n_1 + n_2 - 2$ segments yields a set S of segments containing $T(E)$ and such that $l(S) \leqq l(T(E)) + (n_1 + n_2 - 2)\cdot\delta < l(T(E)) + (n_1 + n_2)\cdot\delta < l(T(E)) + \varepsilon$.

Set S is such that each two points in E_i are joined by a unique chain (of S) which is contained entirely in $\bar{U}_i$. Since $\bar{U}_1$ and $\bar{U}_2$ are disjoint, S must contain two disjoint trees $T_1(E_1)$ on E_1 and $T_2(E_2)$ on E_2, such that $l(T_1(E_1)) + l(T_2(E_2)) \leqq l(S) < l(T(E)) + \varepsilon$. Therefore, for mini-trees $T(E_1)$ and $T(E_2)$ on E_1 and E_2, respectively, $l(T(E_1)) + l(T(E_2)) < l(T(E)) + \varepsilon$.

On the other hand, $E'_i \subset E_i$ implies that $l(T(E'_i)) \leqq l(T(E_i))$ and since E'_i was chosen so that $l^*(A_i) - \varepsilon \leqq l(T(E'_i))$ $(i = 1, 2)$,

$$\begin{aligned} l^*(A_1) + l^*(A_2) &\leqq l(T(E_1')) + \varepsilon + l(T(E_2')) + \varepsilon \\ &\leqq l(T(E_1)) + l(T(E_2)) + 2\varepsilon \\ &< (l(T(E)) + \varepsilon) + 2\varepsilon \\ &\leqq \sup l(T(E)) + 3\varepsilon \\ &= l^*(A_1 + A_2) + 3\varepsilon \,. \end{aligned}$$

Since this holds for each $\varepsilon > 0$, $l^*(A_1) + l^*(A_2) \leqq l^*(A_1 + A_2)$, and the theorem is proved.

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