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PROBABILITY MEASURES AND THE C-SETS OF SELIVANOVSKIJ

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Let X be a Borel space and $\mathscr{S}(X)$ the smallest σ-field containing the Borel subsets of X and closed under operation (A). Then $\mathscr{S}(X)$ is a sub-σ-field of the class of universally measurable subsets of X. Let $P(X)$ be the space of probability measures on the Borel subsets of X and equip $P(X)$ with the weak topology. It is proved that if $A \in \mathscr{S}(X)$, then $\{p \in P(X): P(A) \geqq \lambda\}$ is in $\mathscr{S}[P(X)]$ for every real λ. This allows us to shows that if X and Y are Borel spaces, f is an $\mathscr{S}(X \times Y)$ measurable extended-real-valued function, and $x \to q(\cdot \mid x)$ is $\mathscr{S}(X)$ measurable from X to $P(Y)$, then $x \to \int f(x, y) q(dy \mid x)$ is $\mathscr{S}(X)$ measurable on the set where it is defined. (Measurability is relative to the Borel σ-fields in the range spaces.)

1. Introduction. In the study of sequential optimization, it is often convenient to pose problems in Polish spaces or Borel subsets of Polish spaces (hereafter called *Borel spaces*). Such a space together with its Borel subsets is a well behaved measurable space, but there are certain simple operations which lead one out of the Borel sets into larger σ-fields. One of these is infimization: if $f: X \times Y \to R^*$ is Borel, where X and Y are Borel spaces and $R^* = R \cup \{\pm\infty\}$ is a Borel space with the usual topology, then

$$g(x) = \inf_{y \in Y} f(x, y) \tag{1.1}$$

may fail to be Borel, but will be analytically measurable, i.e., measurable with respect to the σ-field generated by the analytic sets in X [10]. Another such operation is selection: if $B \subset X \times Y$ is Borel and the projection of B onto X is X, there may fail to exist a Borel measurable function $\phi: X \to Y$ whose graph lies in B [2], but by a famous result due independently to Jankov [6] and von Neumann [11], a selector ϕ which is analytically measurable can always be found. (See also [8] and [3].) Another selection problem is this. Let g be as in (1.1) and $\varepsilon > 0$. Does there exist a measurable $\phi: X \to Y$ such that

$$f[x, \phi(x)] \leqq \begin{cases} g(x) + \varepsilon & \text{if} \quad g(x) > -\infty \\ -\dfrac{1}{\varepsilon} & \text{if} \quad g(x) = -\infty ? \end{cases} \tag{1.1}$$

Such a selector which is Borel measurable need not exist, but one which is analytically measurable will. However, if we define $I = \{x \in X \mid g(x) = f(x, y) \text{ for some } y \in Y\}$, and require in addition to (1.2) that

$$f[x, \phi(x)] = g(x) \quad \forall x \in I , \tag{1.3}$$

it is unknown if an analytically measurable selector exists, but one which is the composition of two analytically measurable functions does exist [4]. This question is of particular interest in dynamic programming [3], [15], [16].

Once the decision has been made to consider classes of sets and functions larger than the Borel class, three questions arise. The first is whether the sets in the larger class are universally measurable,[1] the second is whether the class of functions is closed under composition, and the third is to determine if the mapping

$$x \longrightarrow \int f(x, y) q(dy \mid x) \tag{1.4}$$

is in the class when X and Y are Borel spaces, the extended real-valued function f is in the class, and the mapping $x \rightarrow q(\cdot \mid x)$ is Borel measurable from X to $P(Y)$, the space of probability measures on the Borel subsets of Y. Note that $P(Y)$ with the weak topology is again a Borel space, and we will always understand it to be equipped with this topology. We will generalize the last question somewhat and ask about the measurability of (1.4) when both f and $q(\cdot \mid x)$ are measurable with respect to some σ-field larger than the Borel σ-field. The significance of the first two questions is apparent; in fact, we must look at the second to discuss the selector given by [4] mentioned above. The third is of interest in dynamic programming and other sequential optimization problems.

In this paper we deal with $\mathscr{S}(X)$, the smallest σ-field on a Borel space X which contains the Borel sets and is closed under operation (A). This σ-field was studied by Selivanovskij [14] and is known to be contained in the σ-field of universally measurable sets (see, e.g., Saks [13]). Furthermore, all the selection results mentioned above and even much stronger ones are true for $\mathscr{S}(X)$ measurable functions [5]. We say that a mapping f from a Borel space X to a Borel space Y is $\mathscr{S}(X)$ measurable if $f^{-1}(B) \in \mathscr{S}(X)$ for every Borel set $B \subset Y$ and show that this implies $f^{-1}(B) \in \mathscr{S}(X)$ for every $B \in \mathscr{S}(Y)$. This allows us to conclude that the composition of two $\mathscr{S}(X)$ measurable functions is again $\mathscr{S}(X)$ measurable, which is the result of Lemma 2 of [5]. Our method of proving this, however,

[1] A subset A of a Borel space X is said to be *universally measurable* if given any probability measure p on the Borel subsets of X, A is in the completion of the Borel σ-field with respect to p.

differs from that of [5] and shows that $\mathscr{S}(X)$ is the smallest σ-algebra which contains the analytic sets and has this property. We show also that if $A \in \mathscr{S}(X)$, then the mapping $p \to p(A)$ from $P(X)$ to $[0, 1]$ is $\mathscr{S}[P(X)]$ measurable, and consequently the mapping (1.4) is $\mathscr{S}(X)$ measurable on the set where the integral is defined. The method of proof shows in addition that if A is analytic in X and $0 \leqq \lambda \leqq 1$, then $\{p: p(A) \geqq \lambda\}$ is analytic in $P(X)$, which is a result obtained earlier by different methods in [3] and [10]. A proof of this fact by our method can be found in [17], [18].

2. Souslin schemes. Following Meyer [9], we define a *paving* $\mathscr{P}$ of a space X to be a collection of subsets of X. We denote by $\mathscr{P}_\sigma$ the collection of sets obtainable as countable unions of sets in $\mathscr{P}$, and by $\mathscr{P}_\delta$ the collection obtainable as countable intersections. The σ-field generated by $\mathscr{P}$ will be denoted $\sigma(\mathscr{P})$.

Let $\mathscr{N}$ be the set of infinite sequences and Σ the set of finite sequences of positive integers. For $s = (\sigma_1, \cdots, \sigma_n) \in \Sigma$ and $z = (\zeta_1, \zeta_2, \cdots) \in \mathscr{N}$, we write $s < z$ if $\sigma_1 = \zeta_1, \cdots, \sigma_n = \zeta_n$. A *Souslin scheme* S for $\mathscr{P}$ is a mapping from Σ into $\mathscr{P}$. The *nucleus of* S is

$$N(S) = \bigcup_{z \in \mathscr{N}} \bigcap_{s < z} S(s) .$$

The nucleus $N(S)$ is called the result of *operation* (A) acting on the sets $\{S(s): s \in \Sigma\}$. We denote by $\mathscr{A}(\mathscr{P})$ the collection of nuclei of Souslin schemes for $\mathscr{P}$. It is well known that

$$\mathscr{P} \subset \mathscr{A}(\mathscr{P}) , \tag{2.1}$$

$$\mathscr{A}(\mathscr{P})_\delta = \mathscr{A}(\mathscr{P})_\sigma = \mathscr{A}(\mathscr{P}) , \tag{2.2}$$

$$\mathscr{A}(\mathscr{P}) = \mathscr{A}[\mathscr{A}(\mathscr{P})] . \tag{2.3}$$

In general, $\mathscr{A}(\mathscr{P})$ is not a σ-field because it is not closed under complementation. However, if the complement of each set in $\mathscr{P}$ is in $\mathscr{A}(\mathscr{P})$, then $\sigma(\mathscr{P}) \subset \mathscr{A}(\mathscr{P})$. Finally, if X is a Borel space and $\mathscr{B}$ is the collection of Borel subsets of X, then the analytic subsets of X are the members of $\mathscr{A}(\mathscr{B})$.

3. The σ-field $\mathscr{S}(X)$. Let $\mathscr{F}$ be the collection of closed sets in a Borel space X and define by transfinite induction:

$$\mathscr{C}_0 = \mathscr{F} \tag{3.1}$$

$$\mathscr{C}_\alpha = \mathscr{A}[\sigma(\bigcup_{\beta < \alpha} \mathscr{C}_\beta)] , \tag{3.2}$$

where α ranges over the ordinals less than the first uncountable ordinal Ω. Note that $\mathscr{C}_1$ is the collection of analytic subsets of X. We set

(3.3) $$\mathscr{S}(X) = \bigcup_{\alpha<\Omega} \mathscr{C}_\alpha .$$

If S is a Souslin scheme for $\mathscr{S}(X)$, then since S has a countable range, there exists an $\alpha < \Omega$ such that S is a Souslin scheme for $\mathscr{C}_\alpha$. Therefore $N(S) \in \mathscr{C}_{\alpha+1}$, so $\mathscr{S}(X) = \mathscr{A}[\mathscr{S}(X)]$. Since $\mathscr{S}(X)$ is closed under complementation, $\mathscr{A}[\mathscr{S}(X)]$ must contain $\sigma[\mathscr{S}(X)]$, so $\mathscr{S}(X)$ is a σ-field; indeed, it is the smallest σ-field containing $\mathscr{F}$ and closed under operation (A). It is also true that if X is uncountable and $\beta < \alpha < \Omega$, then $\mathscr{C}_\beta$ is a proper subset of $\mathscr{C}_\alpha$. Although there seems to be no easily accessible proof of this in the literature, a discussion of the method of proof using universal sets or universal functions can be found in [1] and [7].

If $(X, \mathscr{D})$ is a measurable space and Y is a Borel space, we say $f: X \to Y$ is $\mathscr{D}$ measurable if $f^{-1}(B) \in \mathscr{D}$ for every Borel set B in Y. We are interested in the following two theorems.

THEOREM 1. *If X and Y are Borel spaces and $f: X \to Y$ is $\mathscr{S}(X)$ measurable, then for every $\mathscr{S}(Y)$ set B, we have $f^{-1}(B) \in \mathscr{S}(X)$. Equivalently, if X, Y, and Z are Borel spaces and $f: X \to Y$ and $g: Y \to Z$ are $\mathscr{S}(X)$ and $\mathscr{S}(Y)$ measurable respectively, then $g \circ f$ is $\mathscr{S}(X)$ measurable. Furthermore, if $\mathscr{D}$ is a σ-field on X containing the analytic sets such that the $\mathscr{D}$ measurability of $f: X \to X$ and $g: X \to X$ implies the $\mathscr{D}$ measurability of $g \circ f$, then $\mathscr{D}$ contains $\mathscr{S}(X)$.*

THEOREM 2. *If A is a $\mathscr{C}_\alpha$ set in a Borel space X and $0 \leqq \lambda \leqq 1$, then $\{p: p(A) > \lambda\}$ and $\{p: p(A) \geqq \lambda\}$ are $\mathscr{C}_\alpha$ sets in $P(X)$. In particular, when A is analytic, $\{p: p(A) > \lambda\}$ and $\{p: p(A) \geqq \lambda\}$ are analytic.*

COROLLARY. *Let X and Y be Borel spaces, let $f: X \times Y \to R^*$ be $\mathscr{S}(X \times Y)$ measurable, and let $x \to q(\cdot \mid x)$ be an $\mathscr{S}(X)$ measurable map into $P(Y)$. Then the set*

$$C = \left\{x \in X \,\middle|\, \int f^+(x, y) q(dy \mid x) = \int f^-(x, y) q(dy \mid x) = \infty \right\}$$

is $\mathscr{S}(X)$ measurable, and if for some $a \in R^*$ we define

$$I(x) = \begin{cases} \int f(x, y) q(dy \mid x) , & x \notin C , \\ a , & x \in C , \end{cases}$$

then I is $\mathscr{S}(X)$ measurable.

Proof of Theorem 1. We show first that if X and Y are Borel spaces, $f: X \to Y$ is $\mathscr{S}(X)$ measurable, and $B \in \mathscr{S}(Y)$, then $f^{-1}(B) \in \mathscr{S}(X)$. By definition, $B \in \mathscr{C}_\alpha$ for some $\alpha < \Omega$, and the proof is by transfinite induction on α. If $\alpha = 0$, then $f^{-1}(B) \in \mathscr{S}(X)$. If the result holds for all $\beta < \alpha$, then $f^{-1}(C) \in \mathscr{S}(X)$ for every $C \in \sigma(\bigcup_{\beta<\alpha} \mathscr{C}_\beta)$. Since $B = \bigcup_{z \in \mathscr{N}} \bigcap_{s<z} S(s)$, where S is some Souslin scheme for $\sigma(\bigcup_{\beta<\alpha} \mathscr{C}_\beta)$, we have

$$f^{-1}(B) = \bigcup_{z \in \mathscr{N}} \bigcap_{s<z} f^{-1}[S(s)] \in \mathscr{A}[\mathscr{S}(X)] = \mathscr{S}(X) .$$

It is straightforward to verify that the second statement of the theorem is equivalent to the first. Under the assumptions of the last part of the theorem, we show that $\mathscr{A}(\mathscr{D}) = \mathscr{D}$. Since $\mathscr{S}(X)$ is the smallest σ-field containing the analytic sets and closed under operation (A), this will imply $\mathscr{S}(X) \subset \mathscr{D}$. We may restrict attention to uncountable X, and since every such X is Borel isomorphic to the space of sequences of zeroes and ones with the topology of componentwise convergence [12], we may assume X has this form. Let S be a Souslin scheme for $\mathscr{D}$, let ψ be a one-to-one function from the positive integers onto Σ, and let $f: X \to X$ be defined so that the kth component of $f(x)$ is the indicator of $S[\psi(k)]$. Define $R(k) = \{(\xi_1, \xi_2, \cdots) \in X: \xi_k = 1\}$, and define $g(x) = x_1$ if $x \in \bigcup_{z \in \mathscr{N}} \bigcap_{s<z} R[\psi^{-1}(s)]$ and $g(x) = x_2 \neq x_1$ otherwise. It is easy to verify that f is $\mathscr{D}$ measurable, g is measurable with respect to the σ-field generated by the analytic sets, and $f^{-1}[g^{-1}(x_1)] = N(S)$. It follows that $N(S) \in \mathscr{D}$, so $\mathscr{D}$ is closed under operation (A) as claimed.

Proof of Theorem 2. We proceed by transfinite induction to show that if A is a $\mathscr{C}_\alpha$ set in X and $0 \leqq \lambda \leqq 1$, then $\{p \in P(X): p(A) \geqq \lambda\}$ is a $\mathscr{C}_\alpha$ set in $P(X)$. If $\alpha = 0$, this follows from Theorem II.6.1(c) of [12]. If the result holds for all $\beta < \alpha$, then for $C \in \sigma(\bigcup_{\beta<\alpha} \mathscr{C}_\beta)$, the mapping $p \to p(C)$ is $\sigma(\bigcup_{\beta<\alpha} \mathscr{C}_\beta)$ measurable by a Dynkin system argument. Now let S be a Souslin scheme for $\sigma(\bigcup_{\beta<\alpha} \mathscr{C}_\beta)$ (in X) and set $A = N(S)$. For $(\sigma_1, \cdots, \sigma_k) \in \Sigma$, define

$$M(\sigma_1, \cdots, \sigma_k) = \{(\zeta_1, \zeta_2, \cdots) \in \mathscr{N}: \tau_1 \leqq \sigma_1, \cdots, \tau_k \leqq \sigma_k\} , \tag{3.4}$$

$$R(\sigma_1, \cdots, \sigma_k) = \bigcup_{z \in M(\sigma_1, \cdots, \sigma_k)} \bigcap_{s<z} S(s) , \tag{3.5}$$

$$K(\sigma_1, \cdots, \sigma_k) = \bigcup_{\tau_1 \leqq \sigma_1, \cdots, \tau_k \leqq \sigma_k} \bigcap_{j=1}^{k} S(\tau_1, \cdots, \tau_j) . \tag{3.6}$$

As $\sigma_1 \uparrow \infty$, $R(\sigma_1) \uparrow A$, and as $\sigma_k \uparrow \infty$, $R(\sigma_1, \cdots, \sigma_k) \uparrow R(\sigma_1, \cdots, \sigma_{k-1})$.

Suppose $p \in P(X)$ is such that $p(A) \geqq \lambda$ and let n be a positive integer. Choose $\bar{\zeta}_1, \bar{\zeta}_2, \cdots$ such that

$$p(A) \leqq p^*[R(\bar{\zeta}_1)] + \frac{1}{2n}$$

$$p^*[R(\bar{\zeta}_1, \cdots, \bar{\zeta}_{k-1})] \leqq p^*[R(\bar{\zeta}_1, \cdots, \bar{\zeta}_{k-1}, \bar{\zeta}_k)] + \frac{1}{2^k n},$$

where p^* denotes p outer measure. Then

$$p(A) \leqq p^*[R(\bar{\zeta}_1, \cdots, \bar{\zeta}_k)] + \frac{1}{n} \quad \text{for all} \quad k,$$

and since $R(\bar{\zeta}_1, \cdots, \bar{\zeta}_k) \subset K(\bar{\zeta}_1, \cdots, \bar{\zeta}_k)$ and the latter set is p measurable, we have

$$\lambda \leqq p(A) \leqq p[K(\bar{\zeta}_1, \cdots, \bar{\zeta}_k)] + \frac{1}{n} \quad \text{for all} \quad k.$$

Thus we may write

$$\{p \in P(X) \mid p(A) \geqq \lambda\} \subset \bigcap_{n=1}^{\infty} \bigcup_{z \in \mathscr{N}} \bigcap_{s<z} \left\{p \in P(X)\colon p[K(s)] \geqq \lambda - \frac{1}{n}\right\}. \tag{3.7}$$

We show the reverse of set containment (3.7). We begin with a proof found in Saks [13] that

$$\bigcap_{k=1}^{\infty} K(\zeta_1, \cdots, \zeta_k) \subset A \quad \text{for all} \quad (\zeta_1, \zeta_2, \cdots) \in \mathscr{N}. \tag{3.8}$$

To show (3.8), choose $x \in \bigcap_{k=1}^{\infty} K(\zeta_1, \cdots, \zeta_k)$, i.e.,

$$x \in \bigcap_{k=1}^{\infty} \bigcup_{\tau_1 \leqq \zeta_1, \cdots, \tau_k \leqq \zeta_k} \bigcap_{j=1}^{k} S(\tau_1, \cdots, \tau_j). \tag{3.9}$$

An argument by contradiction will be used to show that for some $\bar{\tau}_1 \leqq \zeta_1$, we have $x \in S(\bar{\tau}_1)$ and

$$x \in \bigcap_{k=2}^{\infty} \bigcup_{\tau_2 \leqq \zeta_2, \cdots, \tau_k \leqq \zeta_k} \bigcap_{j=2}^{k} S(\bar{\tau}_1, \tau_2, \cdots, \tau_j). \tag{3.10}$$

If no such $\bar{\tau}_1$ existed, then for every $\tau_1 \leqq \zeta_1$, there would exist a positive integer $k(\tau_1)$ such that

$$x \notin S(\tau_1) \cap \left[\bigcap_{k=2}^{k(\tau_1)} \bigcup_{\tau_2 \leqq \zeta_2, \cdots, \tau_k \leqq \zeta_k} \bigcap_{j=2}^{k} S(\tau_1, \tau_2, \cdots, \tau_j)\right].$$

If $\bar{k} = \max_{\tau_1 \leqq \zeta_1} k(\tau_1)$, then

$$\begin{aligned} x \notin & \bigcup_{\tau_1 \leqq \zeta_1} \left\{S(\tau_1) \cap \left[\bigcap_{k=2}^{\bar{k}} \bigcup_{\tau_2 \leqq \zeta_2, \cdots, \tau_k \leqq \zeta_k} \bigcap_{j=?}^{k} S(\tau_1, \tau_2, \cdots, \tau_j)\right]\right\} \\ & \supset \bigcup_{\tau_1 \leqq \zeta_1, \cdots, \tau_k \leqq \zeta_{\bar{k}}} \bigcap_{j=1}^{\bar{k}} S(\tau_1, \tau_2, \cdots, \tau_j) \\ & = K(\zeta_1, \zeta_2, \cdots, \zeta_{\bar{k}}), \end{aligned}$$

and a contradiction is reached. Because (3.10) is of the same form as (3.9), we can apply this argument repeatedly to obtain $\bar{\tau}_1 \leqq \zeta_1$, $\bar{\tau}_2 \leqq \zeta_2, \cdots$ such that

$$x \in \bigcap_{k=1}^{\infty} S(\bar{\tau}_1, \cdots, \bar{\tau}_k) \subset A ,$$

and (3.8) is proved.

Now choose

$$p \in \bigcap_{n=1}^{\infty} \bigcup_{z \in \mathscr{N}} \bigcap_{s<z} \left\{q \in P(X) \colon q[K(s)] \geqq \lambda - \frac{1}{n}\right\} .$$

For each n there exists $(\zeta_1, \zeta_2, \cdots) \in \mathscr{N}$ such that

$$p[K(\zeta_1, \cdots, \zeta_k)] \geqq \lambda - \frac{1}{n} \quad \text{for all} \quad k . \tag{3.11}$$

Since the sets $K(\zeta_1, \cdots, \zeta_k)$ decrease with increasing k, (3.11) implies

$$p\left[\bigcap_{k=1}^{\infty} K(\zeta_1, \cdots, \zeta_k)\right] \geqq \lambda - \frac{1}{n} ,$$

and this together with (3.8) tells us that $p(A) \geqq \lambda - 1/n$. Since n is arbitrary, $p(A) \geqq \lambda$. This shows that equality holds in (3.7), i.e.,

$$\{p \colon p(A) \geqq \lambda\} = \bigcap_{n=1}^{\infty} N(S_n) ,$$

where S_n is the Souslin scheme defined by

$$S_n(s) = \left\{p \in P(X) \colon p[K(s)] \geqq \lambda - \frac{1}{n}\right\} .$$

Since $K(s) \in \sigma(\bigcup_{\beta<\alpha} \mathscr{C}_\beta)$ (in X) for each s, $S_n(s) \in \sigma(\bigcup_{\beta<\alpha} \mathscr{C}_\beta)$ (in $P(X)$) by the induction hypothesis, so $\{p \colon p(A) \geqq \lambda\} \in \mathscr{C}_\alpha$ by (2.2). Note that

$$\{p \colon p(A) > \lambda\} = \bigcup_{n=1}^{\infty} \left\{p \colon p(A) \geqq \lambda + \frac{1}{n}\right\} ,$$

and use (2.2) to conclude that this is also a $\mathscr{C}_\alpha$ set.

Proof of Corollary. Suppose f is the indicator of an $\mathscr{S}(X \times Y)$ set A. For $x \in X$, let $p_x \in P(X)$ be the unit point mass at x, and for $p \in P(X)$, $q \in P(Y)$, let $qp \in P(X \times Y)$ be defined on Borel sets B by

$$(qp)(B) = \int q\{y \colon (x, y) \in B\} p(dx) .$$

Then I is the composition of the mappings $x \to (p_x, q(\cdot \mid x))$, $(p, q) \to qp$ and $r \to r(A)$, where the last mapping is from $P(X \times Y)$ to $[0, 1]$ and is $\mathscr{S}(X \times Y)$ measurable by Theorem 2. The $\mathscr{S}(X)$ measurability of I follows from Theorem 1. The extension to general f is straightforward.

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