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A RADON-NIKODYM THEOREM FOR *-ALGEBRAS

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A noncommutative Radon-Nikodym theorem is developed in the context of *-algebras. Previous results in this direction have assumed a dominance condition which results in a bounded "Radon-Nikodym derivative". The present result achieves complete generality by only assuming absolute continuity and in this case the "Radon-Nikodym derivative" may be unbounded. A Lebesgue decomposition theorem is established in the Banach *-algebra case.

1. **Definitions and Examples.** Although there is a considerable literature on noncommutative Radon-Nikodym theorems, all previous results have needed a dominance, normality or other restriction [1-4, 7, 8, 12, 15-18]. Moreover, most of these results are phrased in a von Neumann algebra context. In this paper, we will obtain a general theorem on a *-algebra with no additional assumptions.

Let $\mathscr{A}$ be a *-algebra with identity I. A **-representation* of $\mathscr{A}$ on a Hilbert space H is a map π from $\mathscr{A}$ to a set of linear operators defined on a common dense invariant domain $D(\pi) \subseteq H$ which satisfies:

(a) $\pi(I) = I$;

(b) $\pi(AB)x = \pi(A)\pi(B)x$ for all $x \in D(\pi)$ and $A, B \in \mathscr{A}$;

(c) $\pi(\alpha A + \beta B)x = \alpha\pi(A)x + \beta\pi(B)x$ for all $x \in D(\pi)$, $\alpha, \beta \in C$ and $A, B \in \mathscr{A}$;

(d) $\pi(A^*) \subset \pi(A)^*$ for all $A \in \mathscr{A}$.

The *induced topology* on $D(\pi)$ is the weakest topology for which all the operations $\{\pi(A): A \in \mathscr{A}\}$ are continuous [13]. A *-representation π is *closed* if $D(\pi)$ is complete in the induced topology. A *-representation π is *strongly cyclic* if there exists a vector x_0 such that $\pi(\mathscr{A})x_0 = \{\pi(A)x_0: A \in \mathscr{A}\}$ is dense in $D(\pi)$ in the induced topology [13]. We then call x_0 a *strongly cyclic vector*. Denoting the set of bounded linear operators on H by $\mathscr{L}(H)$, the *commutant* $\pi(\mathscr{A})'$ of π is

$$\pi(\mathscr{A})' = \{T \in \mathscr{L}(H): \langle T\pi(A)x, y\rangle = \langle Tx, \pi(A^*)y\rangle A \in \mathscr{A}, x, y \in D(\pi)\} .$$

Let v and w be positive linear functionals on $\mathscr{A}$. A sequence $A_i \in \mathscr{A}$ is called a (v, w) *sequence* if

$$\lim_{i\to\infty} v(A_i^* A_i) = \lim_{i,j\to\infty} w[(A_i - A_j)^*(A_i - A_j)] = 0 .$$

We now generalize various forms and strengthened forms of the classical concept of absolute continuity.

(i) w is *v-dominated* if there exists an $M > 0$ such that $w(A^*A) \leqq Mv(A^*A)$ for all $A \in \mathscr{A}$.

(ii) w is *strongly v-absolutely continuous* if for any (v, w) sequence $A_i \in \mathscr{A}$ we have $\lim_{i\to\infty} w(A_i^*A_i) = 0$.

(iii) w is *v-absolutely continuous* if $v(A^*A) = 0$ implies that $w(A^*A) = 0$.

It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). The following examples show that the reverse implications need not hold.

EXAMPLE 1. Let (Ω, Σ) be a measurable space and let $\mathscr{A}$ be the C^*-algebra of bounded measurable functions on (Ω, Σ) with $\|f\|_\infty = \sup\{|f(\omega)|: \omega \in \Omega\}$. Let v_1 and w_1 be probability measures on (Ω, Σ) and define the states $v(f) = \int f dv_1$ and $w(f) = \int f dw_1$ on $\mathscr{A}$. It is easy to see that w is v-absolutely continuous if and only if $w_1 \ll v_1$ (i.e., w_1 is absolutely continuous relative v_1). Now let $H = L^2(\Omega, \Sigma, v_1)$ and let $\pi: \mathscr{A} \to \mathscr{L}(H)$ be the *-representation with $D(\pi) = H$ defined by $[\pi(f)g](\omega) = f(\omega)g(\omega)$. Clearly, π is closed and strongly cyclic with strongly cyclic vector 1.

Now suppose that w is v-absolutely continuous and let W be the positive self-adjoint operator on H with domain

$$D(W) = \left\{g \in H: \left(\frac{dw_1}{dv_1}\right)^{1/2} g \in H\right\}$$

and defined by $Wg(\omega) = (dw_1/dv_1)^{1/2}(\omega)g(\omega)$, $g \in D(W)$. Notice that $\mathscr{A} \subseteqq D(W)$ since $(dw_1/dv_1) \in L^1(\Omega, \Sigma, v_1)$. Moreover,

$$w(f) = \int f dw_1 = \int \frac{dw_1}{dv_1} f dv_1 = \langle W\pi(f)1, W1\rangle \tag{1.1}$$

for all $f \in \mathscr{A}$. The expression $w(f) = \langle W\pi(f)1, W1\rangle$ is equivalent to the Radon-Nikodym theorem. It is this expression which we shall generalize to the noncommutative case. We now show that w is strongly v-absolutely continuous. Suppose $f_i \in \mathscr{A}$ is a (v, w) sequence. Then $f_i \to 0$ in H and from (1.1) we have

$$\begin{aligned}\lim_{i,j\to\infty} \|Wf_i - Wf_j\|^2 &= \lim_{i,j\to\infty} \langle W(f_i - f_j), W(f_i - f_j)\rangle \\ &= \lim_{i,j\to\infty} \langle W\pi[(f_i - f_j)^*(f_i - f_j)]1, W1\rangle \\ &= \lim_{i,j\to\infty} w[(f_i - f_j)^*(f_i - f_j)] = 0 .\end{aligned}$$

Hence, Wf_i converges and since W is closed, we conclude that $Wf_i \to 0$ in H. It follows from (1.1) that $w(f_i^*f_i) \to 0$. We thus see that (ii) and (iii) are equivalent in this case.

Finally, suppose w is v-dominated. Then there exists an $M > 0$ such that

$$\int_A \frac{dw_1}{dv_1} dv_1 = w_1(A) = w(\chi_A^*\chi_A) \leqq Mv(\chi_A^*\chi_A) = Mv_1(A) = \int_A M dv_1$$

for every $A \in \Sigma$. Hence $dw_1/dv_1 \leqq M$ almost everywhere. Since the converse easily holds, we see that w is v-dominated if and only if $w_1 \ll v_1$ and dw_1/dv_1 is bounded. In this case we have $W \in \pi(\mathscr{A})'$. This shows that (ii) need not imply (i) and (iii) need not imply (i).

Our results in §§2 and 3 will generalize the above considerations.

EXAMPLE 2. Let $\mathscr{A}$ be the C^*-algebra of continuous functions on the unit interval $[0, 1]$ with the supremum norm and let μ be Lebesque measure on $[0, 1]$. Let v and w be the states on $\mathscr{A}$ defined by $v(f) = \int f d\mu$ and $w(f) = f(0)$. Clearly, w is v-absolutely continuous. We now show that w is not strongly v-absolutely continuous. Let $f_n \in \mathscr{A}$ be the function $f_n(x) = 1 - nx$ for $x \in [0, 1/n]$ and $f_n(x) = 0$ for $x \in [1/n, 1]$. Then

$$\lim_{n\to\infty} v(f_n^* f_n) = \lim_{n\to\infty} \frac{1}{3n} = 0$$

and $w[(f_n - f_m)^*(f_n - f_m)] = 0$. Hence, f_n is a (v, w) sequence. But $w(f_n^* f_n) = 1$, so $\lim w(f_n^* f_n) \neq 0$. Thus (iii) need not imply (ii).

2. A Radon-Nikodym Theorem. If v is a positive linear functional on a *-algebra $\mathscr{A}$, then the GNS construction [10, 13] provides a unique (to within unitary equivalence) closed *-representation π_v of $\mathscr{A}$ on a Hilbert space H_v with a strongly cyclic vector $x_0 \in H_v$ such that $v(A) = \langle \pi_v(A)x_0, x_0 \rangle$ for all $A \in \mathscr{A}$. We now give our main result.

THEOREM 1. *If v and w are positive linear functionals on a *-algebra $\mathscr{A}$, then there exists a positive self-adjoint operator W on H_v and a (v, w) sequence $A_i \in \mathscr{A}$ such that*

$$w(A) = \langle W\pi_v(A)x_0, Wx_0 \rangle + \lim w(A_i^* A)$$

for every $A \in \mathscr{A}$.

(a) *w is v-absolutely continuous if and only if $v(A^*A) = 0$ implies $w(A_i^* A^* A) = 0$ for every $i = 1, 2, \cdots$.*

(b) *w is strongly v-absolutely continuous if and only if $w(A) = \langle W\pi_v(A)x_0, Wx_0 \rangle$ and*

$$\langle W\pi_v(A)x, Wy \rangle = \langle Wx, W\pi_v(A^*)y \rangle \tag{2.1}$$

for every $A \in \mathscr{A}$ and $x, y \in \pi(\mathscr{A})x_0$.

(c) *w is v-dominated if and only if $w(A) = \langle W\pi_v(A)x_0, Wx_0 \rangle$ for every $A \in \mathscr{A}$, and $W^2 \in \pi(\mathscr{A})'$.*

Proof. Let $H = H_v$, $\pi = \pi_v$, x_0 and K_1, π_1, x_1 be the Hilbert spaces, closed *-representations and strongly cyclic vectors of the GNS constructions corresponding to the positive linear functionals v and $v + w$ on $\mathscr{A}$, respectively. Let J be the unique contractive linear map from K_1 into H satisfying $J\pi_1(A)x_1 = \pi(A)x_0$ for every $A \in \mathscr{A}$. Let P be the projection from K_1 onto $K = (\ker J)^{\perp}$. Let $T: H \to H$ be the positive self-adjoint operator defined by $T = JJ^*$. Then $\ker T = (\text{range } J)^{\perp} = \{0\}$ and hence $S = T^{-1}$ exists as a positive self-adjoint operator on H. Since J is contractive, $J \leqq I$ and hence $S \geqq I$. Let $W = (S - I)^{1/2}$. Then

$$D(W) = D(S^{1/2}) = T^{1/2}H = JK .$$

(The first and second equality follows by the spectral theorem and the third equality follows by the polar decomposition theorem.) By the polar decomposition theorem, $(S^{1/2}J)^*S^{1/2}J = P$ and hence $P - J^*J = (WJ)^*(WJ)$. Therefore,

$$\begin{aligned} w(A) &= \langle \pi_1(A)x_1, x_1\rangle - \langle \pi(A)x_0, x_0\rangle \\ &= \langle \pi_1(A)x_1, (I-P)x_1\rangle + \langle \pi_1(A)x_1, Px_1\rangle - \langle J\pi_1(A)x_1, Jx_1\rangle \\ &= \langle \pi_1(A)x_1, (I-P)x_1\rangle + \langle WJ\pi_1(A)x_1, WJx_1\rangle \\ &= \langle \pi_1(A)x_1, (I-P)x_1\rangle + \langle W\pi(A)x_0, Wx_0\rangle . \end{aligned} \tag{2.2}$$

Since $\{\pi_1(A)x_1: A \in \mathscr{A}\}$ is dense in K_1, there exists a sequene $A_i \in \mathscr{A}$ such that $\pi_1(A_i)x_1 \to (I - P)x_1$. Hence,

$$\pi(A_i)x_0 = J\pi_1(A_i)x_1 \longrightarrow J(I - P)x_1 = 0$$

and

$$v(A_i^*A_i) = \langle \pi(A_i)x_0, \pi(A_i)x_0\rangle \longrightarrow 0 .$$

Since $\pi_1(A_i)x_1$ is Cauchy in K_1 we have

$$\begin{aligned} \mathrm{w}[(A_i - A_j)^*(A_i - A_j)] &= \|\pi_1(A_i)x_1 - \pi_1(A_j)x_1\|^2 \\ &\quad - \|\pi(A_i)x_0 - \pi(A_j)x_0\|^2 \longrightarrow 0 . \end{aligned} \tag{2.3}$$

Therefore, A_i is a (v, w) sequence. Moreover, since $|v(A_i^*A)| \leqq v(A_i^*A_i)^{1/2}v(A^*A)^{1/2}$ we have $\lim v(A_i^*A) = 0$ for all $A \in \mathscr{A}$. Hence,

$$\begin{aligned} w(A) &= \langle W\pi(A)x_0, Wx_0\rangle + \lim \langle \pi_1(A)x_1, \pi_1(A_i)x_1\rangle \\ &= \langle W\pi(A)x_0, Wx_0\rangle + \lim w(A_i^*A) . \end{aligned}$$

(a) For sufficiency, if $v(A^*A) = 0$, then

$$\|\pi(A)x_0\|^2 = v(A^*A) = 0 \quad \text{and} \quad \lim w(A_i^*A^*A) = 0$$

and hence, $w(A^*A) = 0$. For necessity, if w is v-absolutely continuous and $v(A^*A) = 0$, then

$$|W(A_i^* A^* A)| \leqq w[(AA_i)^* AA_i]^{1/2} w(A^* A)^{1/2} = 0 .$$

(b) For sufficiency, let $A_i \in \mathscr{A}$ be a (v, w) sequence. Then $\pi(A_i)x_0 \to 0$ and hence,

$$\begin{aligned} &\| W\pi(A_i)x_0 - W\pi(A_j)x_0 \|^2 \\ &= \langle W\pi(A_i - A_j)x_0, W\pi(A_i - A_j)x_0 \rangle \\ &= \langle W\pi[(A_i - A_j)^*(A_i - A_j)]x_0, Wx_0 \rangle \\ &= w[(A_i - A_j)^*(A_i - A_j)] \longrightarrow 0 . \end{aligned}$$

Hence, $W\pi(A_i)x_0$ is Cauchy and since W is closed, $W\pi(A_i)x_0 \to 0$. It follows that

$$w(A_i^* A_i) = \langle W\pi(A_i^* A_i)x_0, Wx_0 \rangle = \| W\pi(A_i)x_0 \|^2 \longrightarrow 0$$

and w is strongly v-absolutely continuous.

For necessity, suppose w is strongly v-absolutely continuous. We first show that $J: K_1 \to H$ is injective. Suppose $x \in K_1$ and $Jx = 0$. Let $A_i \in \mathscr{A}$ be a sequence satisfying $\pi_1(A_i)x_1 \to x$. Then

$$\pi(A_i)x_0 = J\pi_1(A_i)x_1 \longrightarrow Jx = 0 .$$

Hence, $v(A_i^* A_i) = \| \pi(A_i)x_0 \|^2 \to 0$. Since $\pi_1(A_i)x_1$ is Cauchy as in (2.3) we have $w[(A_i - A_j)^*(A_i - A_j)] \to 0$. Thus, A_i is a (v, w) sequence and $w(A_i^* A_i) \to 0$. Hence

$$\| \pi_1(A_i)x_1 \|^2 = w(A_i^* A_i) + v(A_i^* A_i) \longrightarrow 0$$

so that $\pi_1(A_i)x_1 \to 0$ and $x = 0$. It follows that $\ker J = \{0\}$ and hence, $P = I$. Applying (2.2) we obtain $w(A) = \langle W\pi(A)x_0, Wx_0 \rangle$. To prove (2.1), applying (2.2) we have

$$\begin{aligned} \langle W\pi(AB)x_0, Wx_0 \rangle &= w(AB) \\ &= \langle \pi_1(B)x_1, \pi_1(A^*)x_1 \rangle - \langle \pi(B)x_0, \pi(A^*)x_0 \rangle \\ &= \langle (I - J^*J)\pi_1(B)x_1, \pi_1(A^*)x_1 \rangle \\ &= \langle (WJ)^*(WJ)\pi_1(B)x_1, \pi_1(A^*)x_1 \rangle \\ &= \langle W\pi(B)x_0, W\pi(A^*)x_0 \rangle . \end{aligned}$$

If $x = \pi(B)x_0$, $y = \pi(C)x_0 \in \pi(\mathscr{A})x_0$ we obtain

$$\begin{aligned} \langle W\pi(A)x, Wy \rangle &= \langle W\pi(AB)x_0, W\pi(C)x_0 \rangle \\ &= \langle W\pi(C^*AB)x_0, Wx_0 \rangle = \langle W\pi(B)x_0, W\pi(A^*C)x_0 \rangle \\ &= \langle Wx, W\pi(A^*)y \rangle . \end{aligned}$$

(c) The following proves sufficiency

$$\begin{aligned} w(A^*A) &= \langle W\pi(A^*A)x_0, Wx_0 \rangle = \langle W\pi(A)x_0, W\pi(A)x_0 \rangle \\ &= \| W\pi(A)x_0 \|^2 \leqq \| W \|^2 \| \pi(A)x_0 \|^2 = \| W \|^2 v(A^*A) . \end{aligned}$$

For necessity, suppose w is v-dominated. Then w is strongly v-absolutely continuous so (b) holds. Applying (2.1) there is an $M > 0$ such that

$$\begin{aligned} \|W\pi(A)x_0\|^2 &= \langle W\pi(A)x_0, W\pi(A)x_0\rangle \\ &= \langle W\pi(A^*A)x_0, Wx_0\rangle = w(A^*A) \leqq Mv(A^*A) \\ &= M\|\pi(A)x_0\|^2 \end{aligned}$$

for every $A \in \mathscr{A}$. Hence, W is bounded on $\pi(\mathscr{A})x_0$ and since W is self-adjoint, $W \in \mathscr{L}(H)$. It follows from (2.1) that

$$\langle W^2\pi(A)x, y\rangle = \langle W^2x, \pi(A^*)y\rangle \tag{2.4}$$

for all $A \in \mathscr{A}$, $x, y \in \pi(\mathscr{A})x_0$. Since $D(\pi)$ is the completion of $\pi(\mathscr{A})x_0$ in the induced topology [10], if $y \in D(\pi)$ there exists a net $y_\alpha \in \pi(\mathscr{A})x_0$ such that $y_\alpha \to y$ in the induced topology. Hence,

$$\begin{aligned} \langle W^2\pi(A)x, y\rangle &= \lim \langle W^2\pi(A)x, y_\alpha\rangle \\ &= \lim \langle W^2x, \pi(A^*)y_\alpha\rangle = \langle W^2x, \pi(A^*)y\rangle \end{aligned}$$

for every $y \in D(\pi)$, $x \in \pi(\mathscr{A})x_0$. Reasoning in a similar way for x, we conclude that (2.4) holds for all $x, y \in D(\pi)$. Hence, $W^2 \in \pi(\mathscr{A})'$.

3. Banach *-algebras. In this section we apply the material of §2 to obtain much stronger results on Banach *-algebras. When we speak of a *-representation π of a Banach *-algebra on a Hilbert space H we always mean a bounded representation; that is, $\pi: \mathscr{A} \to \mathscr{L}(H)$. The commutant of $\pi(\mathscr{A})$ now satisfies

$$\pi(\mathscr{A})' = \{T \in \mathscr{L}(H): T\pi(A) = \pi(A)T \quad \text{for all} \quad A \in \mathscr{A}\} .$$

If v and w are positive linear functionals on a *-algebra $\mathscr{A}$, we say that w is *v-semisingular* if there exists a (v, w) sequence $A_i \in \mathscr{A}$ such that $w(A) = \lim w(A_i^*A)$ for every $A \in \mathscr{A}$. Notice that if $A_i \in \mathscr{A}$ is a (v, w) sequence, then $\lim w(A_i^*A)$ automatically exists for every $A \in \mathscr{A}$.

COROLLARY 2. *If v and w are positive linear functionals on a Banach *-algebra $\mathscr{A}$ with identity then there exists a positive self-adjoint operator W on H_v which is affiliated with $\pi_v(\mathscr{A})'$ and a (v, w) sequence $A_i \in \mathscr{A}$ such that*

$$w(A) = \langle \pi_v(A)Wx_0, Wx_0\rangle + \lim w(A_i^*A)$$

for every $A \in \mathscr{A}$.

(a) *w is v-absolutely continuous if and only if the positive linear functional $A \mapsto \lim w(A_i^*A)$ is v-absolutely continuous.*

(b) *w is strongly v-absolutely continuous if and only if* $w(A) = \langle \pi_v(A) W x_0, W x_0 \rangle$ *for every* $A \in \mathscr{A}$.

(c) *w is v-dominated if and only if* $w(A) = \langle \pi_v(A) W x_0, W x_0 \rangle$ *and W is bounded.*

Proof. For the first statement of the theorem we need only prove that W is affiliated with $\pi_v(\mathscr{A})'$ and apply Theorem 1. From the proof of Theorem 1, J intertwines the representations π_1 and π and hence $T \in \pi(\mathscr{A})'$. Since $W = (T^{-1} - I)^{1/2}$, it follows that W is affiliated with $\pi(\mathscr{A})'$. Parts (a), (b), and (c) are a straightforward application of Theorem 1.

Corollary 2 (c) is a classical result [5, 9, 11]. We next prove a noncommutative analogue of the Lebesque decomposition theorem.

COROLLARY 3. *Let v and w be positive linear functionals on a Banach *-algebra $\mathscr{A}$ with identity. Then w admits a decomposition $w = w_a + w_s$ where w_a is strongly v-absolutely continuous and w_s is v-semisingular. Moreover, w is v-absolutely continuous if and only if w_s is v-absolutely continuous.*

Proof. Let $w_a(A) = \langle \pi_v(A) W x_0, W x_0 \rangle$ and $w_s(A) = \lim w(A_i^* A)$ for all $A \in \mathscr{A}$ as in Corollary 2. Then w_a and w_s are positive linear functionals and $w = w_a + w_s$. It follows from Corollary 2 (b) that w_a is strongly v-absolutely continuous. We now show that w_s is v-semisingular. Since $A_i \in \mathscr{A}$ is a (v, w) sequence and $w_a, w_s \leqq w$, we conclude that A_i is both a (v, w_a) and (v, w_s) sequence. Since w_a is strongly v-absolutely continuous we have

$$|w_a(A_i^* A)| \leqq w_a(A_i^* A_i)^{1/2} w_a(A^* A)^{1/2} \longrightarrow 0$$

for all $A \in \mathscr{A}$. Hence

$$w_s(A_i^* A) = w(A_i^* A) - w_a(A_i^* A) \longrightarrow w_s(A)$$

for all $A \in \mathscr{A}$ so w_s is v-semisingular.

We have not been able to prove uniqueness for the above decomposition. However, if $w = w_1 + w_2$ where w_1 is strongly v-absolutely continuous, w_2 is v-semisingular and w_2 has the same "support" as w_s (that is, $w_2(A) = \lim w_2(A_i^* A)$ for all $A \in \mathscr{A}$), then $w_1 = w_a$, $w_2 = w_s$. Indeed, then A_i is a (v, w) sequence and hence, $w_1(A_i^* A) \to 0$ for all $A \in \mathscr{A}$. Therefore,

$$w_2(A) = \lim w_2(A_i^* A) = \lim w(A_i^* A) = w_s(A)$$

for all $A \in \mathscr{A}$. Thus, $w_2 = w_s$ and $w_1 = w - w_2 = w - w_s = w_a$.

The v-semisingular functional $w_s(A) = \lim w(A_i^* A)$ in Corollary 2 can be put in the form $w_s(A) = \lim w(A_i^* A A_i)$ which exhibits its positivity directly. The reason for this is that $\pi_1(A)\colon \ker J \to \ker J$ in the notation of Theorem 1. Indeed, suppose $Jy = 0$ and let $B_i \in \mathscr{A}$ satisfy $\pi_1(B_i)x_1 \to y$. Then

$$\pi(B_i)x_0 = J\pi_1(B_i)x_1 \longrightarrow Jy = 0 .$$

Hence,

$$\begin{aligned} J\pi_1(A)y &= \lim J\pi_1(AB_i)x_1 = \lim \pi(AB_i)x_0 \\ &= \pi(A) \lim \pi(B_i)x_0 = 0 . \end{aligned}$$

It follows that $P\pi_1(A) = \pi_1(A)P$ for all $A \in \mathscr{A}$. Applying (2.2) we have

$$w_s(A) = \langle \pi_1(A)(I - P)x_1, (I - P)x_1 \rangle .$$

Hence,

$$w_s(A) = \lim \langle \pi_1(A)\pi_1(A_i)x_1, \pi_1(A_i)x_1 \rangle = \lim w(A_i^* A A_i) .$$

Example 2 of §1 gives an illustration of Corollary 3. In this example, w is v-absolutely continuous. Even though w is v-absolutely continuous, w is quite singular relative to v. In fact, $w(f) = \int f d\mu_0$ where μ_0 is the probability measure concentrated at 0, and μ_0 and μ are mutually singular measures. We showed in Example 2 that f_i is a (v, w) sequence. Moreover, $w(f) = \lim w(f_i^* f)$ for all $f \in \mathscr{A}$. Hence, in this case $w = w_s$ and $w_a = 0$.

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REFERENCES

1. H. Araki, *Bures Distance function and a generalization of Sakai's non-commutative Radon-Nikodym theorem*, Publ. RIMS, Kyoto Univ., **8** (1972), 335–362.

2. ———, *Some properties of the modular conjugation operator of von Neumann algebras and a non-commutative Radon-Nikodym theorem with a chain*, Pacific J. Math., **50** (1974), 309–354.

3. ———, *One-parameter family of Radon-Nikodym theorem for states of a von Neumann algebra*, Publ. RIMS, Kyoto Univ., **10** (1974), 1–10.

4. A. Connes, *Sur le theoreme de Radon-Nikodym pour les poids normaux fideles simifinis*, Bull. Sci. Math., **97** (1974), 253–258.

5. J. Dixmier, *Les C*-Algebras et leurs Representations*, Gauther-Villars, Paris (1964).

6. N. Dunford and J. Schwartz, *Liner Operators, Part II*, Wiley-Interscience, New York (1963).

7. H. Dye, *The Radon-Nikodym theorem for finite rings of operators*, Trans, Amer. Math Soc., **72** (1952), 243–280.

8. G. Elliott, *On the Radon-Nikodym derivative with a chain rule in von Neumann algebras*, Canad. Math. Bull., **18** (1975), 661-669.
9. G. Emch, *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*, Wiley-Interscience, New York (1972).
10. S. Gudder and W. Scruggs, *Unbounded representations of *-algebras*, Pacific J. Math., **70** (1977), 369-382.
11. M. Naimark, *Normed Rings*, Noordhoff, Groningen, The Netherlands (1964).
12. G. Pedersen and M. Takesaki, *The Radon-Nikodym theorem for von Neumann algebras*, Acta Math., **130** (1973), 53-88.
13. R. Powers, *Self-adjoint algebras of unbounded operators*, Commun. Math. Phys., **21** (1971), 85-124.
14. F. Riesz and B. Nagy, *Functional Analysis*, Frederick Ungar, New York (1955).
15. S. Sakai, A *Radon-Nikodym theorem in W*-Algebras*, Bull. Amer. Math. Soc., **71** (1965), 149-151.
16. I. Segal, *A non-commutative extension of abstract integration*, Ann. Math., **57** (1953), 401-457.
17. M. Takesaki, *Tomita's Theory of Modular Hilbert Algebras and its Applications*, Springer-Verlag, Berlin (1970).
18. A. van Daele, *A Radon-Nikodym theorem for weights on von Neumann algebras*, Pacific J. Math., **61** (1975), 527-542.

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