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NONFACTORIZATION IN COMMUTATIVE, WEAKLY SELF-ADJOINT BANACH ALGEBRAS

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A Banach algebra A is said to have "(weak) factorization" if for each $f \in A$, there exist $g, h \in A$ (resp. $n \geqq 1$ and $g, {}_1h_1, \cdots, g_n, h_n \in A$) such that $f = gh\,(f = \Sigma g_j h_j)$. Cohen's factorization theorem says that if A has bounded approximate identity, then A has factorization. The converse is false in general. This paper investigates various implications of factorization and weak factorization for commutative algebras that are weakly self-adjoint. (Defined below; these algebras include self-adjoint algebras.) The main result is Theorem 1.3: If the weakly self-adjoint commutative Banach algebra A of functions on the locally compact space X has weak factorization, then there exists $K > 0$ such that, for all compact subsets E of X, there exists $f \in A$ such that $\|f\| \leqq K$ and $f \geqq 1$ on E. Applications of 1.3 are given. In particular it is shown that a proper character Segal algebra on $L^1(G)$, (G a LCA group) cannot have weak factorization.

We say that a Banach algebra B of complex-valued continuous functions on a topological space is *weakly self-adjoint* if there exists $K_0 > 0$ such that for each $f \in B$

$$(0.1) \qquad |f|^2 \in B \quad \text{and} \quad \||f|^2\|_B \leqq K_0 \|f\|_B^2 .$$

Obviously (0.1) is satisfied if B is self-adjoint ($f \in B$ implies $\bar{f} \in B$), or if B is a Banach ideal of a self-adjoint Banach algebra A of complex-valued continuous functions. Recall that B is called a *Banach ideal* of a Banach algebra A if $\|f\|_B \geqq \|f\|_A$ and $fg \in B$, with $\|fg\|_B \leqq \|f\|_B \|g\|_A$ for all $f \in B, g \in A$. Dense Banach ideals have been called A-Segal algebras by Burnham [1] and others. It is worth mentioning that a self-adjoint Banach algebra B has weak factorization if and only if each element of B can be written as a linear combination of elements of the form $|f|^2$, $f \in B$ (this follows easily from the identity $f\bar{g} = 1/4 \sum_{k=0}^{3} f^k |f + i^k g|^2$).

Our work was motivated by an attempt to give a converse to Cohen's theorem for Segal algebras on abelian groups (defined in §2), and thereby extend the results of Burnham [1], Leinert [7], Wang [15], Yap [17] and others. Since such a Segal algebra cannot have bounded approximate units, that would prove that a proper Segal algebra cannot have weak factorization. We cannot prove that in full generality, but we are able to improve earlier results substantially.

Moreover, our results practically apply to all existing examples of Segal algebras. Special cases of some of the results of §2 appeared already in [4].

1. Weak factorization. In this section we shall only deal with weakly self-adjoint Banach algebras of complex-valued continuous functions on a locally compact space X. To avoid trivialities we assume throughout that B does not have an identity.

It is our aim to derive in this section certain properties for such a Banach algebra, if it has weak factorization. Since (weak) factorization is preserved under homomorphisms (in particular under the Fourier or Gelfand transform) these results can be used to prove nonfactorization for a number of subalgebras of $L^1(G)$ of a locally compact abelian group, in particular for most Segal algebras (cf. §2).

For later use let us denote the set $\{x \mid x \in X, f(x) \neq 0$ for some $f \in B\}$ by $S(B)$. We begin now with a lemma, abstracted from Lakien [4]:

LEMMA 1.1. *If B is a Banach algebra with weak factorization, then there exist positive integers N and K such that*

$$B(N, K) := \left\{ f \in B : f = \sum_{i=1}^{N} \prod_{j=1}^{4} f_i^j \quad \text{with} \quad \sum_{i=1}^{N} \prod_{j=1}^{4} \|f_i^j\| \leqq K \|f\| \right\}$$

is dense in B.

Proof. The assumption of weak factorization directly implies that $B = \bigcup_{N=1}^{\infty} \bigcup_{K=1}^{\infty} B(N, K)$. By the Baire category theorem, the closure of some $B(N', K')$ has nonempty interior. Thus, there exists some $f_0 \in B(N', K')$ and some $\delta > 0$ such that $B(N', K')^-$ contains the ball of radius δ centered at f_0.

Let $f \in B$ be any function such that $\delta/2 \leqq \|f\| < \delta$. Then for each $\delta/4 \geqq \varepsilon > 0$ there exist $(f_i^j)_{i=1}^{N'}$, $j = 1, \cdots, 4$ such that

$$\left\| f + f_0 - \sum_{i=1}^{N'} \prod_{j=1}^{4} f_i^j \right\| > \varepsilon$$

and

$$\sum_{i=1}^{N'} \prod_{j=1}^{4} \|f_i^j\| < K'(\|f - f_0\| + \varepsilon) \leqq K'(\varepsilon_1 \delta_1 \|f_0\|) \leqq K'(2 + \|f_0\|) .$$

Since

$$f_0 = \sum_{i=1+N'}^{2N'} \prod_{j=1}^{4} f_i^j$$

with

$$\sum_{i=1+N'}^{2N'} \prod_{j=1}^{4} ||f_i^j|| \leqq K' ||f_0||$$

we see that

$$\left\| \sum_{i=1}^{2N'} \sum_{j=1}^{4} f_i^j - f \right\| < \varepsilon$$

and

$$\sum_{i=1}^{2N'} \prod_{j=1}^{4} ||f_i^j|| \leqq \cdots 2K'(1 + ||f_0||) .$$

Note that

$$\left\| \sum_{1}^{2N'} \prod f_i^j \right\| \geqq \delta/2 - \varepsilon \geqq \delta/4 .$$

We set $N = 2N'$ and $K = \delta K'(1 + 1||f_0||)\delta^{-1}$: then $f \in B$ and $\delta/2 \leqq ||f|| \leqq \delta$ imply $f \in B(N, K)$. Since $tB(N, K) = B(N, K)$ for all $t > 0$ we see that $B = B(N, K)^-$.

The next lemma provides the key inductive step required in the proof of Theorem 1.3.

LEMMA 1.2. *Let B be a weakly self-adjoint Banach algebra with weak factorization. Then there exists a positive constant K_1 such that for every compact set $M \subset S(B)$ and every $f_1 \in B$ vanishing nowhere on M, there exists $f_2 \in B$ such that $f_2 \geqq 0$*

$$f_2 \geqq |f_1|^{1/2} , \quad \textit{and} \quad ||f_2|| \leqq K_1 ||f_1||^{1/2} .$$

Proof. According to 1.1 we can choose $f_1' \in B(N, K)$ such that $||f_1 - f_1'|| \leqq 1/4 \inf_{x \in M} |f_1(x)|$ (so necessarily $||f_1 - f_1'|| \leqq ||f_1||$). Write $f_1' = \sum_{i=1}^{N} \prod_{j=1}^{4} f_i^j$ with $\sum_{i=1}^{N} \prod_{j=1}^{4} ||f_i^j|| \leqq K||f_1'||$. Without loss of generality, we may assume that

$$||f_i^1|| = \cdots = ||f_i^4|| \leqq (K||f_1'||)^{1/4} \leqq (2K||f_1||)^{1/4} .$$

Set

$$f_2' = 1/4 \sum_{i=1}^{N} \sum_{j=1}^{4} |f_i^j|^2 ,$$

so $f_2' \geqq 0$ on X. By the weak self-adjointness, we have $f_2' \in B$ and $||f_2'|| \leqq (2K)^{1/2} N K_0 ||f_1||^{1/2}$.

Recall that (i) the geometric mean of n nonnegative numbers does not exceed their arithmetic mean and (ii) the square root of the sum of n nonnegative numbers does not exceed the sum of their square roots. We conclude that

$$f_2' \geqq \sum_{i=1}^{N} \prod_{j=1}^{4} |f_i^j|^{1/2} \geqq \Big(\sum_{i=1}^{N} \prod_{j=1}^{4} |f_i^j|\Big)^{1/2} \geqq |f_1'|^{1/2} .$$

Now $|f_1(x) - f_1'(x)| \leqq ||f_1 - f_1'|| \leqq 1/4|f_1(x)|$ for all $x \in M$, so $|f_2'| \geqq (3/4)^{1/2}|f_1|^{1/2}$ on M. Therefore $f_2 = 4/3f_2'$ satisfies the required conditions with $K_1 = 2K^{1/2}NK_0$, and the proof of the lemma is complete.

We now state the basic result of this paper.

THEOREM 1.3. *Let B be a weakly self-adjoint Banach algebra of complex-valued continuous functions. Suppose that B has weak factorization. Then there exists $K_2 > 0$ such that for each compact subset $M \subseteq S(B)$ there is an element $f \in B$ such that $f \geqq 1$ on M, $f \geqq 0$ on $S(B)$, and $||f|| \leqq K_2$.*

Proof. We shall apply 1.2 inductively. First note that the weak self-adjointness of B and the compactness of M imply the existence of $f_1 \in B$ such that $f_1 \geqq 1$ on M. By 1.2, there exist $f_2, f_3, \cdots \in B$ such that for $j = 1, 2, \cdots |f_{j+1}(x)| \geqq |f_j(x)|^{1/2}$ for $x \in M$ and $||f_{j+1}|| \leqq K_1||f_j||^{1\backslash 2}$. Hence, $f_{j+1} \geqq 1$ on M and $||f_{j+1}|| \leqq K_1^{\sum_{i=0}^{j} 2^{-i}}||f_1||2^{-j}$. Choose now $K_2 = 2K_1^2$ and j so large that $||f_1||^{2^{-j}} \leqq 2$. Then $f = f_{j+1}$ will do, and the theorem is proved.

COROLLARY 1.4. *Let B be a weakly self-adjoint Banach algebra. Let μ be an unbounded, regular Borel measure on $S(B)$. If B has weak factorization, then $B \nsubseteq \bigcup_{0<p<\infty} L^p(\mu)$.*

This corollary is an immediate consequence of the above theorem, together with part (ii) of the following lemma.

LEMMA 1.5. *Let Ω be a space, Σ a σ-algebra of subsets of Ω, and μ a measure defined on Σ. We write L^p in place of $L^p(\Omega, \Sigma, \mu)$.*

(i) *Let $B \subseteq L^1$ be a Banach space, continuously embedded in L^1. If $B \subseteq \bigcup_{1<p\leqq\infty} L^p$, then $B \subseteq L^{p_0}$ for some $p_0 > 1$.*

(ii) *Let $B \subseteq L^\infty$ be a Banach space, continuously embedded in L^∞. If $B \subseteq \bigcup_{0<p<\infty} L^p$, then $B \subseteq L^{p_0}$ for some $p_0 < \infty$.*

Proof. (i) Since $B \subseteq L^1$, the hypothesis implies $B \subseteq \bigcup_{n=1}^{\infty} L^{1+1/n} = \bigcup_{k,n=1}^{\infty} k(B, L^{1+1/n})$, with $(B, L^p) = \{f \mid f \in B, ||f||_p \leqq 1\}$. Since, by the reflexivity of $L^p, 1 < p < \infty$, (B, L^p) is closed in B, one of the sets $(B, L^p), 1 < p < \infty$, has to have interior in B. Consequently $(B, L^p) - (B, L^p) \subseteq 2(B, L^p)$ contains a neighborhood of the identity of B, and, hence, $B = L^p$. The proof of (ii) is similar.

Another consequence of Theorem 1.3 is that no proper Banach ideal of $C_0(X)$ can have factorization; but more is true.

COROLLARY 1.6. *Let B be a weakly self-adjoint Banach algebra with $X = S(B)$. Suppose that there exists a proper Banach ideal N of $C_c(X)$ such that $N \supseteqq B$ and $S(N) = X$. Then B does not have weak factorization.*

Proof. We argue by contradiction, and suppose that B has weak factorization. Then, by 1.3 and the closed graph theorem (applied to the inclusion of B into N), there is $L > 0$ such that, for each compact subset M of X, there exists $f_M \in B$ such that $f_M \geqq 1$ on M and $\|f_M\|_N \leqq L$. Let g be such that $g = f_M^{-1}$ on M and $\|g\|_\infty \leqq 1$. Then $\|h\|_\infty \leqq \|h\|_N \leqq \|g\|_\infty \|h\|_\infty \|f_M\|_N \leqq L\|h\|_\infty$ for all $h \in C_0(X)$ with compact support contained in M. This is not possible when M is a proper Banach ideal of $C_0(X)$, vanishing nowhere on X.

2. Applications to Segal (and other) algebras. Let G be a non-discrete locally compact abelian group. A *Segal algebra* on G is a dense subspace of $L^1(G)$ that is a Banach space with a norm $\| \ \|$ such that $\|f\| \geqq \|f\|_1$ for all $f \in B$ and such that for all $y \in G$, $f \in B$, $L_y f \in B (L_y f(x) = f(y^{-1}x))$, $\|L_y f\| = \|f\|$ and $\lim_{y\to 1} \|L_y f - f\| = 0$ ([12], §4, or [11]). We say that a Segal algebra B is *character invariant* if $\chi \in \hat{G}$, $f \in B$ imply $\chi f \in B$. The most important property of a Segal algebra is the fact that it is a Banach ideal in $L^1(G)$. It is thus obvious that any Segal algebra on a locally compact abelian group can be considered as a weakly self-adjoint Banach algebra on $X = S(B) = \hat{G}$ (via Fourier transform).

The main result of this section is Theorem 2.2. Proposition 2.1 is almost immediate from 1.4 and 1.6 and is stated here because it improves the main results of Wang [15] as well as of Burnham [1], Leinert [7], and Yap [17].

PROPOSITION 3.1. *Let B be a Segal algebra on G. Then B does not have weak factorization if one of the following conditions is satisfied:*

(a) $B \subseteqq \bigcup_{1<p\leqq\infty} L^p(G)$; (b) $\hat{B} \subseteqq \bigcup_{0<p<\infty} L^p(\hat{G})$;

(c) $\sup \hat{f}(\gamma)M(\gamma) < \infty$ *for all $f \in B$, with M being an unbounded, continuous strictly positive function on $\hat{G}$.*

Proof. (b) and (c) follow from §1. If condition (a) is satisfied we use 1.5(i). Let p_0 be the number given in 1.5(1). Then if $1<p_0<2$, the Hausdorff-Young theorem implies $\hat{B} \subseteqq L^2(G)$ and if $p_0 \geqq 2$, the Plancherel theorem implies $\hat{B} \subseteqq L^2(G)$, and (b) applies.

THEOREM 2.2. *Let B be a character invariant Segal algebra on* *the locally compact abelian group G. Then B has weak factorization if and only if $B = L^1(G)$.*

REMARKS. Wang [15] and Leinert [7] have proved this result under additional hypotheses. For example, among others Wang needs $\|\chi f\|_B = \|f\|_B$ for all $\chi \in \hat{G}$, $f \in B$. (In this case, B is called a *character Segal algebra.*) For character Segal algebras on compact abelian groups the result has been proved by Lakien [4]. It is his method that we extend below.

To prove 2.2, we introduce a new algebra, $\tilde{B}$, defined as follows. $\tilde{B}$ consists of those measures $\mu \in M(G)$ such that there exists a net $\{f_\alpha\} \subseteqq B$ with

(a) $\sup_\alpha \|f_\alpha\|_B < \infty$ and (b) (weak-$*$) $\lim f_\alpha = \mu$.

The norm $\|\mu\|_{\tilde{B}}$ is the infimum of the numbers (a) subject to (b). Obviously, B is an ideal in $\tilde{B}$, and $\tilde{B}$ is an ideal $M(G)$, with $\|\mu * \nu\|_{\tilde{B}} \leqq \|\mu\|_{M(G)} \|\nu\|_{\tilde{B}}$, for all $\mu \in M(G)$, $\nu \in \tilde{B}$. If G is compact, B is closed in $\tilde{B}$. (This is easily proved.) If G is not compact, B may not be closed in $\tilde{B}$. (See [3], §4, Remark E) for an example.) The following lemma is Lemma 3.8 of [3].

LEMMA 2.3. *Let B be a character invariant Segal algebra. Then B is closed in $\tilde{B}$.*

LEMMA 2.4. *Let G be a locally compact group and B a Segal algebra on G. If B is closed in $\tilde{B}$ and $\delta_0 \in \tilde{B}$, then $B = L^1(G)$.*

Proof. Let $C > 0$ be such that $C\|f\|_{\tilde{B}} \geqq \|f\|_B$ for all $f \in B$. Choose now now $\{g_\alpha\} \subseteqq B$ such that (weak-$*$) $\lim g_\alpha = \delta_0$ and $\sup_\alpha \|g_\alpha\|_B \leqq 2\|\delta_0\|_{\tilde{B}}$. Then $f =$ (weak-$*$) $\lim g_\alpha * f$, $f \in B$. This implies $\|f\|_{\tilde{B}} \leqq \sup_\alpha \|g_\alpha * f\|_B \leqq 2\|\delta_0\|_{\tilde{B}} \|f\|_1$. Consequently, B- and L^1-norms are equivalent on B and $B = L^1(G)$.

Proof of 2.2. One direction is obvious. We prove the other. Suppose then that B has weak factorization. Then by 1.3 there is a bounded net $\{f_\alpha\} \subseteqq B$ such that for any compact subset $K \subseteqq \hat{G}$ there is some α_0 with $\hat{f}_\alpha(t) \geqq 1$ for all $t \in K$ and $\alpha \geqq \alpha_0$. Without loss of generality, we may suppose that there is a bounded measure μ with $\mu =$ (weak-$*$) $\lim f_\alpha$, i.e., $\mu \in \tilde{B}$. We claim that $\hat{\mu}(t) \geqq 1$ for all $t \in \hat{G}$.

To see this, take $t_0 \in \hat{G}$ and consider any continuous, nonnegative, compactly supported function k on $\hat{G}$ with $\|k\|_1 = 1$. For sufficiently large α we have

$$\hat{f}_\alpha(t_0 s^{-1}) \geqq 1$$

for every s in the support of k and hence $k*\hat{f}_\alpha(t_0) \geqq 1$. Since $\check{k} \in C_0(G)$, the weak-$*$ convergence of $\{f_\alpha\}$ to μ gives

$$k*\hat{f}_\alpha(t_0) = (\check{k}f_\alpha)\hat{}(t_0) \longrightarrow (\check{k}\mu)\hat{}(t_0) = k*\hat{\mu}(t_0) ,$$

so $k*\hat{\mu}(t_0) \geqq 1$ for every such k. Hence $\hat{\mu}(t_0) \geqq 1$, as required.

It is a well-known result of Wiener (see, e.g., Rudin [14], 5.6.9) that μ must have a nonzero discrete part. We may assume (using the translation-invariance of $\tilde{B}$) that $\mu(\{0\}) \neq 0$. Let $g \in L^1(\hat{G})$ be such that g has compact support, $\hat{g}(0) = 1$, and

$$\int_{G\setminus\{0\}} |\hat{g}|\, d|\mu| < |\mu(\{0\})|/4 .$$

The character invariance of $\tilde{B}$ implies that $\hat{g}\mu \in \tilde{B}$ (cf. [3, proof of 3.8]). But $\hat{g}\mu = \mu(\{0\})\delta_0 + v$, where $||v|| < |\mu(\{0\})|/4$. Therefore $\hat{g}\mu$ is invertible in $M(G)$. Since $\tilde{B}$ is an ideal in $M(G)$, this implies $\delta_0 \in \hat{B}$ and the conclusion follows from 2.4 and 2.3.

The above result can be extended in the following way.

COROLLARY 2.5. *Let B a Banach algebra which is dense and continuously embedded in $L^1(G)$ of a locally compact abelian group. Suppose there exists $K_0 > 0$ such that*

$$(0.1') \qquad f^{*}*f \in B \quad \text{and} \quad ||f^{*}*f|| \leqq K_0||f||^2 \quad \text{for all} \quad f \in B .$$

If B has weak factorization, then $L^1(G)$ is the only character invariant Segal algebra on G containing B.

Proof. Suppose B is contained in a character invariant Segal algebra B_1. It then follows by the closed graph theorem that B is continuously embedded in B_1. Since we may identify B via Fourier transform with a weakly self-adjoint Banach algebra $\hat{B}$ on $\hat{G}$ with $S(\hat{B}) = \hat{G}$ it follows from 1.3 and the proof of 2.2 that $\tilde{B}_1$ contains δ_0 and thus $B_1 = L^1(G)$. This completes the proof.

We now turn to some other applications of our methods.

THEOREM 2.6. *If $I = \{f \in L^1(R), \hat{f}(t) = 0$ for $0 \leqq t \leqq 1\}$, then I does not have weak factorization.*

Proof. Suppose that I has weak factorization. Then, by 1.3, there exists $K > 0$ such that, for each compact subset $M \subseteq \boldsymbol{R}$ that is disjoint from $[0, 1]$, there exists $f \in L^1(\boldsymbol{R})$ with $||f||_1 \leqq K$, $\hat{f} = 0$ on $[0, 1]$ and $\hat{f} \geqq 1$ on M. For $n > 0$ let $\mu = \mu_n$ be the measure on $\boldsymbol{R}$ that assigns mass $1/k$ to the point $-k/n$ and mass $-1/k$ to the point k/n for $1 \leqq k \leqq n$. Then $\sup_{r \in R} |\hat{\mu}(r)| \leqq K_1$, where K_1 is independent of the integer n. Let $f \in I$ be such that $\|f\|_1 < K$ and $\hat{f} \geqq 1$ on $\{-k/n: 1 \leqq k \leqq n\}$. Then $\hat{f}\mu > 0$ and, therefore, $\|\hat{f}\mu\|_{M(T)} \geqq \sup_{r \in R} |(\hat{f}\mu)\hat{}(r)| \geqq \delta \log n$. Since $\sup_R |(\hat{f}\mu)\hat{}| \leqq \|f\|_1 \sup_R |\hat{\mu}|$, we have a contradiction.

That completes the proof of the theorem.

We conclude this section with a result concerning closed ideals of $L^1(G)$ (or any other Segal algebra on a compact abelian group G). A subset $E \subseteq \hat{G}$ is called a *small* M_0*-set* if $\hat{\mu} \in C_0(\hat{G})$ whenever $\hat{\mu}$ vanishes outside E. It is clear that any Sidon set or $\Lambda(p)$-set for $1 < p < \infty$ (see [8] for definitions) is of this type. Moreover it is trivial that any Rajchman set in the sense of Pigno, [10], is a small M_0-set. Thus for example $-P \cup E$ is a Rajchman set if E is a Rider set in $\hat{G}$ and $\hat{G}$ is an ordered group with positive cone P.

THEOREM 2.7. *Let G be a compact abelian group and $E \subseteq \hat{G}$ a small M_0-set. Then the closed ideal*

$$I_{E^c} = \{f \mid \hat{f} = 0 \text{ off of } E\}$$

has weak factorization if and only if E is finite.

Proof. If I_{E^c} has weak factorization then there exists $K > 0$ such that for any finite subset $F \subseteq E$ there is some $f_F \in I_{E^c}$ with $\hat{f}_F \geqq 1$ on F and $\|f\|_{L^1} \leqq K$. Taking a weak-$*$ limit, we obtain $\mu \in M(G)$ such that $\hat{\mu}(t) = 0$ on $\hat{G}\backslash E$ and $\hat{\mu} \geqq 1$ on E. This contradicts our assumptions concerning E, except the case that E is finite. Since I_{E^c} has a unit if E is finite, the proof of the theorem is complete.

In view of the above remarks Theorem 2.7 includes Theorem 2.5 of [15] as well as the following result.

COROLLARY 2.8. (cf. Theorem 8.16 of [16])

$$H^1 = \{f \in L^1(T) \mid \hat{f}(n) = 0 \quad \text{for} \quad n < 0\}$$

does not have weak factorization.

Proof. That $\{0, 1, 2 \cdots\}$ is a small M_0-set follows from the conjunction of the F. and M. Riesz theorem and the Riemann-Lebesgue lemma.

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