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IRREDUCIBLE LENGTHS OF TRIVECTORS OF RANK SEVEN AND EIGHT

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We determine the irreducible length of complex trivectors of rank less than or equal to eight. The irreducible length is invariant under the action induced by the general linear group on the underlying complex vector space. A classification under this action is available where representatives are explicitly given for each equivalence class and it is the lengths of these representatives which are determined.

In their paper [1], Busemann and Glassco consider the problem of determining the maximal irreducible length (called *length* from now on) $N(F, n, r)$ of r-vectors in $\Lambda^r U$ where U is an n-dimensional vector space over the field F. The length of an r-vector is the number of decomposable summands (blades in J. Schouten's book [5] and paper [6]) in a shortest possible representation of that r-vector. In [1] Busemann and Glassco state that "The values $N(C, 7, 3) = 5$, $N(C, 8, 3) = 7$, and $N(C, 9, 3) = 10$ have been claimed but questioned, see Schouten [3, p. 27] and [1]." The purpose of this paper is to show that $N(C, 8, 3) = 5$ (and not 7 as claimed) by determining the lengths of each of the representatives of the Gurevich classification in [2]. For sake of completeness the lengths of the rank 7 trivectors are also included. That $N(C, 7, 4) = 4$ (and not 5) is shown in [7].

Let U be a fixed 8-dimensional vector space over the complex numbex field and $\Lambda^3 U$ the space of trivectors considered. If $X \in \Lambda^3 U$ then $[X]$ denotes the intersection of all subspaces W of U for which $X \in \Lambda^3 W$. Then $\dim [X]$ is the rank of X. The letters a, b, c, q, r, s, p, t appearing in the Gurevich classification may be assumed to be independent vectors in U. Note that in [2] the "Λ" has been suppressed.

The equivalence classes are as follows.

I:	0
II:	$[abc]$
III:	$[aqp] + [brp]$
IV:	$[aqr] + [brp] + [cpq]$
V:	$[abc] + [pqr]$
VI:	$[aqp] + [brp] + [csp]$
VII:	$[qrs] + [aqp] + [brp] + [csp]$
VIII:	$[abc] + [qrs] + [aqp]$

IX: $[abc] + [qrs] + [aqp] + [brp]$

X: $[abc] + [qrs] + [aqp] + [brp] + [csp]$

XI: $[brp] + [csp] + [aqp] + [crt]$

XII: $[qrs] + [brp] + [csp] + [aqp] + [crt]$

XIII: $[abc] + [qrs] + [aqp] + [crt]$

XIV: $[abc] + [qrs] + [aqp] + [brp] + [crt]$

XV: $[abc] + [qrs] + [aqp] + [brp] + [csp] + [crt]$

XIV: $[aqp] + [bst] + [crt]$

XVII: $[aqp] + [brp] + [bst] + [crt]$

XVIII: $[qrs] + [aqp] + [brp] + [bst] + [crt]$

XIX: $[aqp] + [brp] + [csp] + [bst] + [crt]$

XX: $[qrs] + [aqp] + [brp] + [csp] + [bst] + [crt]$

XXI: $[abc] + [qrs] + [aqp] + [bst] + [crt]$

XXII: $[abc] + [qrs] + [aqp] + [brp] + [bst] + [crt]$

XXIII: $[abc] + [qrs] + [aqp] + [brp] + [csp] + [bst] + [crt]$

Since rank and length remain constant in an equivalence class the terms will be used on the equivalence class itself. There are five equivalence classes of trivectors with rank 7. Three of them, namely VI, VIII, and IX, have length 3 and the other two, VII and X, have length 4. There are thirteen equivalence classes with rank 8. Two of them, XVI and XIX have length 3; one of them, XV has length 5; and the remaining 10 classes have length 4. The results are proved as follows.

Consider first a trivector X of rank 7. If it has length 3 then $X = X_1 + X_2 + X_3$ where each X_i is decomposable. If $[X_1] \cap [X_2] \neq 0$ then we may write $X_1 + X_2 = x_1 \wedge (x_2 \wedge x_3 + x_4 \wedge x_5)$ where $x_1, \cdots, x_5$ are independent vectors in U. Then $X_3 = u \wedge x_6 \wedge x_7$ where $u \in \langle x_1, \cdots, x_5 \rangle$ and $x_1, \cdots, x_7$ are independent vectors in U. If u is a multiple of x_1 then X has the form VI where $p = x_1$. If u is not a multiple of x_1 then by rewriting $X_1 + X_2$ we may assume that $u = x_2$. (For if $u = \alpha x_1 + \beta x_2 + w$ where $\beta \neq 0$ and $w \in \langle x_3, x_4, x_5 \rangle$ then $X_1 + X_2 = x_1 \wedge (u \wedge \beta^{-1} x_3 - \beta^{-1} w \wedge x_3 + x_4 \wedge x_5)$ and $-\beta^{-1} w \wedge x_3 + x_4 \wedge x_5$ is decomposable.) Then X has the form VIII. If $[X_i] \cap [X_j] = 0$ for all pairs $i \neq j$ then we may write $X + X_2 = x_1 \wedge x_2 \wedge x_3 + x_4 \wedge x_5 \wedge x_6$ and $X_3 = u \wedge v \wedge x_7$ where $u, v \in \langle x_1, \cdots, x_6 \rangle$ and $x_1, \cdots, x_7$ are independent. Then $u = u_1 + u_2$, $v = v_1 + v_2$ where $u_1, v_1 \in [X_1]$ and $u_2, v_2 \in [X_2]$ so by refactoring X_1 and X_2 we may assume that $u = x_1 + x_4$ and $v = x_2 + x_5$. Then X has the form XIX which is equal to $[b(c + p)a] + [qr(x + p)] + [(q - b)p(a + r)]$ where $x_1 = -b$, $x_2 = a$, $x_3 = c + p$, $x_4 = q$, $x_5 = r$, $x_6 = s + p$, $x_7 = -p)$.

Since all possibilities for rank 7 length 3 trivectors have been considered the remaining classes have length at least 4. Equivalence class VII then has length 4 and X, which is equal to $([aqp] + [(b+s)(r-c)p] + [(a+p)bc] + [(p+q)rs])$ has length 4 also. This takes care of the rank 7 trivectors.

Suppose $X = X_1 + X_2 + X_3$ is a trivector of length 3 and rank 8, where each X_i is decomposable. If $[X_1] \cap [X_2] \neq 0$ then $X_1 + X_2 = x_1 \wedge (x_2 \wedge x_3 + x_4 \wedge x_5)$ and for X to have rank 8 it follows that $[X_3] \cap ([X_1] + [X_2]) = 0$. Therefore X has the form $x_1 \wedge (x_2 \wedge x_3 + x_4 \wedge x_5) + x_6 \wedge x_7 \wedge x_8$ or XVI in the Gurevich notation. If $[X_i] \cap [X_j] = 0$ for $i \neq j$ then $X_1 + X_2 = x_1 \wedge x_2 \wedge x_3 + x_4 \wedge x_5 \wedge x_6$ and $X_3 = u \wedge x_7 \wedge x_8$ where $u \in \langle x_1, \cdots, x_6 \rangle$ and $x_1, \cdots, x_8$ are independent. By refactoring X_1 and X_2 we may assume that $u = x_1 + x_4$ and so X is of type XIX which is equal to $([aqp] + 1/2[(b+c)(r+s)(p+t)] + 1/2[(b-c)(r-s)(p-t])$. This takes care of trivectors of rank 8 and length 3. For each of the remaining equivalence classes except XV we exhibit a length 4 representation.

Types XI, XIII, and XVII are already in the form advertised.

$$\mathrm{XII} = [(a-s)qp] + [(q-c)(p+r)s] + [(t+s)cr] + [brp]\,.$$
$$\mathrm{XIV} = [ab(c-p)] + [(a-r)(b+q)p] + [rq(p-s)] + [crt]\,.$$
$$\mathrm{XVIII} = [(t-r)bs] + [crt] + [r(p-s)(b-q)] + [(a-r)qp]\,.$$
$$\mathrm{XX} = [(r+s)(t-r)b] + [(r+s)(r+p)(c-q)] + [(a-r-s)qp] + [r(b-c)(s-p+t)]\,.$$
$$\mathrm{XXI} = [(b-r)(c+s)t] + [(a-t)bc] + [aqp] + [rs(q+t)]\,.$$
$$\mathrm{XXII} = [(a+r)(b+2q)(p-c+1/2s)] + [cr(t-3b-2q)] + [bs(1/2a+3/2r+t)] + [(b+q)(a+2r)(p-2c+s)]\,.$$
$$\begin{aligned}\mathrm{XXIII} = {} & [(a+1/2s+1/2r)q(b+c+r-s+1/2p+1/2t)] \\ & + [(b+c+1/2q)(r+s)(b+c+1/2p+1/2t)] \\ & + [a(-1/2b+1/2c+q)(-b-c-r+s+1/2p-1/2t)] \\ & + [(b-c)(-1/2a+r-s)(-r+s+1/2p-1/2t)]\,.\end{aligned}$$

The only item that remains to be justified is that XV has length 5. We write the representative in the form $X = X_1 + [crt]$ where $X_1 = [abc] + [qrs] + [aqp] + [brp] + [csp]$. We note that X_1 is of type X, has rank 7, length 4, from which it follows that X has length at most 5. We will show that $(t-u) \wedge X$ has length at least 4 for all $u \in \langle a, b, c, q, r, s, p \rangle$. This will complete the proof because if Y is any rank 8 trivector in $\bigwedge^3 U$ with length 4 then at least one of the terms in any representation of Y as a sum of 4 decomposable trivectors must contain a factor of the form $t-u$ for some $u \in \langle a, b, c, q, r, s, p \rangle$, and for this u, the length of $(t-u) \wedge Y$ is at most 3.

Since $(t-u) \wedge X = t \wedge (X_1 + [ucr]) - u \wedge X_1$ it is sufficient to prove that $X_1 + [ucr]$ has length at least 4 for all $u \in \langle a, b, c, q, r, s, p \rangle$. Let

$$u = \alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 q + \alpha_5 s + \alpha_6 r + \alpha_7 p .$$

After the substitution

$$\begin{aligned} s &\longrightarrow s - \alpha_7 r + \alpha_4 c \\ b &\longrightarrow b + \alpha_1 r \end{aligned}$$

with the other letters remaining unchanged we obtain

$$X_1 + [cr(\alpha_2 b + \alpha_5 s)] .$$

If $\alpha_2 = \alpha_5 = 0$ we have X_1 which has length 4. If $\alpha_2 \neq 0$ then $X_1 + [cr(\alpha_2 b + \alpha_5 s)] = [(a + \alpha_2 r)b(c - \alpha_2^{-1}p)] + [(q + \alpha_5 c)rs] + [a(q + \alpha_2^{-1}b)p] + [csp] \sim X_1$ under the substitution

$$\begin{aligned} a &\longrightarrow \alpha_2^{-1/2}(a - r) \\ b &\longrightarrow \alpha_2^{3/2} b + \alpha_5(c - p) \\ c &\longrightarrow \alpha_2^{-1} c \\ q &\longrightarrow \alpha_2^{1/2} q - \alpha_2^{-1}\alpha_5 c \\ s &\longrightarrow \alpha_2(s - p) \\ r &\longrightarrow \alpha_2^{-3/2} r \\ p &\longrightarrow p . \end{aligned}$$

If $\alpha_2 = 0$ and $\alpha_5 \neq 0$ then

$$\begin{aligned} X_1 + \alpha_2[crs] = [abc] + [brp] + [s(q + \alpha_5 c)(r - \alpha_5^{-1}p)] \\ + [(a + \alpha_5^{-1}s)qp] \sim X_1 \end{aligned}$$

under the substitution

$$\begin{aligned} a &\longrightarrow -\alpha_5 a \\ b &\longrightarrow -\alpha_5(b + p) \\ c &\longrightarrow \alpha_5^{-2} c \\ q &\longrightarrow -\alpha_5^{-1}(q + c) \\ r &\longrightarrow -\alpha_5^{-1} r \\ s &\longrightarrow \alpha_5^2 s \\ p &\longrightarrow p . \end{aligned}$$

Lastly, we point out also that $N(C, 9, 3) \leqq 9$. This follows from (2.5) of [1] since $N(C, 8, 3) = 5$. The bound 9 is not likely to be the best one however.

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