PACIFIC JOURNAL OF MATHEMATICS
Vol. 80, No. 2, 1979

NON-MINIMAL ROOTS IN HOMOTOPY TREES

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Let π be a finite group which does not satisfy the Eichler condition and let M be a π-module. A π-module M' is a noncancellation example of M if $M \oplus (Z\pi)^2 \cong M' \oplus (Z\pi)^2$ but $M \not\cong M'$. This note classifies the set $\mathcal{NC}_M(\pi)$ of isomorphism classes of noncancellation examples for $M = Z \oplus Z\pi$, where Z is the trivial π-module, $M = A(\pi)$, the augmentation ideal, and $M = Z\pi/(N)$, where (N) is the ideal generated by the norm element $N = \sum_{x \in \pi} x$. It is shown that these noncancellation examples yield nonminimal roots of the homotopy tree $HT(\pi, m)$ of (π, m)-complexes.

1. **Introduction.** Let π be a finite group. We say that a π-module M satisfies the *Eichler condition* if the endomorphism ring $\mathrm{End}(QM)$ has no simple component which is a totally definite quaternion algebra over its center (see [11, page 176] for a definition). A finitely generated, Z-torsion free (left) π-module M has the *cancellation property* (CP) iff for any π-module M' such that $M \oplus (Z\pi)^2 \cong M' \oplus (Z\pi)^2$ we have $M \cong M'$. If $M \oplus (Z\pi)^2 \cong M' \oplus (Z\pi)^2$, we say that M and M' are *stably isomorphic.* Note that this is completely general, for by Bass' cancellation [1, Corollary 10.2], $M \oplus (Z\pi)^2 \cong M' \oplus (Z\pi)^2$ iff $M \oplus (Z\pi)^n \cong M' \oplus (Z\pi)^n (n \geqq 2)$. If M has the Eichler condition, and $M \cong N \oplus Z\pi$, then M has the cancellation property [7], [11, Theorem 19.8].

In this paper we are interested in noncancellation examples. A module M' is a *noncancellation example for* M iff M' is stably isomorphic to, but not isomorphic to M. We determine in § 2 the set $\mathcal{NC}_M(\pi)$ of isomorphism classes on noncancellation examples of certain modules M. In § 3, we show that the Swan counterexample [10, Theorem 3] for the generalized quaternion group of order 32 gives rise to noncancellation examples.

We apply this to the homotopy classification of (π, m)-complexes. A (π, m)-complex is a finite, connected, m-dimensional CW-complex with $\pi_1 X \cong \pi$ and $\pi_i X = 0$ for $1 < i < m$. A (π, m)-conplex X is called a *root* if there is *no* other (π, m)-complex Y such that $Y \vee S^m \simeq X$; *a minimal root* if the number $(-1)^m \chi(X)$ is minimal over all (π, m)-complexes; otherwise a *nonminimal root.* In § 4, we show that the Swan counterexample gives rise to nonminimal roots for (GQ(32), 4i-1)-complexes.

For π a finite group, a recent theorem of W. Browning [2] (generalizing the Jacobinski cancellation theorem to the category of pointed modules) shows that such nonminimal roots occur very rarely. In fact, for π finite, nonminimal roots for (π, m)-complexes oncur only if π is periodic and $m = k - 1$, where k is a period of π. The situation for infinite groups is much less clear. However, M. J. Dunwoody [3] has constructed an example of a nonminimal root for $(T, 2)$-complexes, where T is the trefoil knot group.

I would like to thank R. Swan for his proof of the crucial Lemma 3.4 in this paper and the referee for simplifying the hypotheses in Lemmas 2.4 and 2.8.

2. **Noncancellation.** Let π be a finite group of order n. For each integer p prime to n, let (p, N) denote the ideal of the integral group ring $Z\pi$ generated by p and the norm element $N = \sum_{x \in \pi} x$. Each (p, N) is projective [8, Lemma 6.1]. If Z_n^* denotes the units of the ring of integers modulo n and $\bar{p}$ is the residue class of an integer p modulo n, then the correspondence $\bar{p} \to$ class $[(p, N)]$ of (p, N) in the (reduced) projective class group $\tilde{K}_0 Z\pi$ of $Z\pi$ defines a homomorphism

$$\partial\colon Z_n^* \longrightarrow \tilde{K}_0 Z\pi$$

(see [8, Lemma 6.1]).

Note. For any $p \in Z_n^*$, we will abuse the notation and write (p, N). This is well-defined up to isomorphism because if $r \equiv s(\mathrm{mod}\, n)$, then $(r, N) \cong (s, N)$.

Let $\mathscr{P}'(\pi)$ denote the set of isomorphism classes of projective (left) ideals in the integral group ring $Z\pi$ of π. By Theorem A of [9], $\mathscr{P}'(\pi)$ is also the set of isomorphism classes of rank 1 projective π-modules. Let $\{P\}$ denote the isomorphism class of the projective ideal P. Let $SF(\mathscr{P}')$ (respectively $SW(\mathscr{P}')$) denote the subset of $\mathscr{P}'(\pi)$ consisting of those isomorphism classes $\{P\}$ such that the element $[P]$ in $\tilde{K}_0 Z\pi$ is zero (respectively, $[P] \in \operatorname{im} \partial$). Furthermore, let $F(\pi) = \{p \in Z_n^* \mid (p, N) \cong Z\pi\}$ and $SF(\pi) = \ker \partial = \{p \in Z_n^* \mid (p, N) \oplus Z\pi \cong (Z\pi)^2\}$.

We may identify the groups $SF(\pi)/F(\pi) \hookrightarrow Z_n^*/F(\pi)$ as subgroups of the set $\mathscr{P}'(\pi)$ via $p \to \{(p, N)\}$. The group action is given by $\{(p, N)\} \cdot \{(q, N)\} = \{(p, N) \otimes_\pi (q, N)\} = \{(pq, N)\}$. Thus

$$\begin{array}{ccc} \dfrac{SF(\pi)}{F(\pi)} & \subset & \dfrac{Z_n^*}{F(\pi)} \\ \cap & & \cap \\ SF(\mathscr{P}) & \subset & SW(\mathscr{P}^1) \end{array} \subset \mathscr{P}^1(\pi)\,.$$

Furthermore, the group $Z_n^*/F(\pi)$ (respectively $SF(\pi)/F(\pi)$) acts on the set $SW(\mathscr{P}^1)$ (respectively $SF(\mathscr{P}^1)$) as follows: for each projective ideal P and $p \in Z_n^*$, define $P_p = (p, N) \otimes_\pi P$. Then let $p \cdot \{P\} =$

$\{P_p\}$. In order to define the above tensor product, we note that (p, N) is a 2-sided ideal, hence a right π-module ((p, N) is also an invertible bimodule). Then P_p has a left π-module structure using the left module structure of P, because $(p, N)\otimes_\pi P \cong Z\pi q\cdot P + Z\pi N\cdot P (q \in p$ is an integer) and hence is the left ideal generated by $(q\cdot P, N\cdot P)$.

DEFINITION. Let M be a π-module. Let $*_M$ denote the class of modules isomorphic to M. Let $\mathscr{N}\mathscr{C}_M(\pi)$ be the set whose elements consist of $*_M$ together with the set of isomorphism classes of non-cancellation examples of M. Thus

$$\mathscr{N}\mathscr{C}_M(\pi) = \{*_M\} \cup \{\{M'\} \mid M' \oplus (Z\pi)^2 \cong M \oplus (Z\pi)^2 \text{ but } M' \neq M\}.$$

In this section we will compute $\mathscr{N}\mathscr{C}_M(\pi)$ for $M = Z \oplus Z\pi$, where Z is the trivial π-module, $M = A(\pi)$, the augmentation ideal in $Z\pi$, and $M = Z\pi/(N)$, where (N) is the ideal generated by the norm element N. If a group G acts on a set S (on the left) as a group of permutations, we denote the set of orbits by S/G.

THEOREM 2.1. *The following sets are isomorphic*:
(a) $\mathscr{N}\mathscr{C}_{Z\oplus Z\pi}(\pi) \cong SW(\mathscr{P}^1(\pi))/(Z_n^*/F(\pi))$
(b) $\mathscr{N}\mathscr{C}_{A(\pi)}(\pi) \cong \mathscr{N}\mathscr{C}_{Z\pi/(N)}(\pi) \cong SF(\mathscr{P}^1(\pi))/(SF(\pi)/F(\pi))$.

Note 2.2. (a) It follows from [11, Theorem 9.7], [4, Propositions 5.3, 5.4, 5.5] that if M is $Z \oplus Z\pi$, $A(\pi)$, or $Z\pi/(N)$, then M' is stably isomorphic to M iff $M' \oplus Z\pi \cong M \oplus Z\pi$.

(b) Lemma 6.2 of [8] and Proposition 5.5 of [4] show that

$$M' \oplus Z\pi \cong (Z \oplus Z\pi) \oplus Z\pi$$

iff $M' \cong Z \oplus P$ where P is a projective ideal and $[P] \in \operatorname{im} \partial$ in $\tilde{K}_0 Z\pi$.

We will prove Theorem 2.1 after a series of propositions and lemmas.

LEMMA 2.3. *For any $\bar{q} \in Z_n^*$, and any projective (left) ideal $P \subset Z\pi$, $P_{\bar{q}}/P_{\bar{q}}^\pi \cong P/P\pi$, where $P_{\bar{q}} = (q, N)\otimes_\pi P$ and $P^\pi = \{p \in P \mid xp = p,\ \forall x \in \pi\}$.*

Proof. Let $N\cdot P = t_P\cdot Z\cdot N \subset (N) \cap P = s_P\cdot Z\cdot N = P^\pi$, where t_P and s_P are positive integers such that s_P divides t_P. Then

$$\begin{aligned} P_{\bar{q}}/P_{\bar{q}}^\pi &= \frac{(q, N)\otimes_\pi P}{((q, N)\otimes_\pi P)^\pi} \cong \frac{q\cdot P + P\cdot N}{(qs_pZ + t_pZ)N} \\ &\cong \frac{1\cdot P + P\cdot N}{(s_PZ + t_PZ)N} \cong P/(s_PZ)N = P/P^\pi . \end{aligned}$$

The second isomorphism is given by carrying $q\alpha + \alpha' N \to \alpha + \alpha' N$ for any $\alpha, \alpha' \in P$.

PROPOSITION 2.4. *Let P be a projective (left) ideal in $Z\pi$ and n be the order of π. Then* $\mathrm{Ext}^1_{Z\pi}(P/P^\pi, P^\pi) \cong Z_n$. *Furthermore, the projective extensions of $P^\pi \cong Z$ by P/P^π are* $Z_n^* = \{0 \to \underset{\underset{Z}{\|}}{P_q^\pi} \to P_q \to P_q/P_q^\pi \to 0 \mid q \in Z_n^*\}$, *where* $P/P^\pi \cong P_q/P_q^\pi$.

Proof. To prove the first statement we localize: $\mathrm{Ext}^1_{Z\pi}(P/P^\pi, P^\pi) \cong \bigoplus_{p \mid n} \mathrm{Ext}_{Z_{(p)}\pi}(P_{(p)}/P^\pi, P^\pi_{(p)})$. [6, Corollary 3.12, page 16.] Theorem 4.4 of [6] yields that $Z_{(p)}\pi$ is isomorphic to $P_{(p)}$. Thus $\mathrm{Ext}^1_{Z\pi}(P/P^\pi, P^\pi) \cong \mathrm{Ext}^1_{Z\pi}(Z\pi/Z\pi^\pi, Z\pi^\pi) \cong Z_n$.

The projective extensions are necessarily the units Z_n^* of Z_n [5, 1.1] and hence are given by the diagram below. Choose $s \in q \in Z_n^*$.

$$
\begin{array}{ccccccccc}
 & & Z & & & & & & \\
 & & \| & & & & & & \\
0 & \longrightarrow & P^\pi & \longrightarrow & P & \longrightarrow & P/P^\pi & \longrightarrow & 0 \\
 & & \downarrow{\cdot s} & & \downarrow{\cdot s} & & \| & & \\
0 & \longrightarrow & P_q^\pi & \longrightarrow & P_q & \longrightarrow & (P_q/P_q^\pi \xrightarrow{\cong})\, P/P^\pi & \longrightarrow & 0 \\
 & & & & \| & & & & \\
 & & & & \{sP + N \cdot P\} & & & &
\end{array}
$$

Thus P_q represents the element $q \in \mathrm{Ext}\,(P/P^\pi, P^\pi) = Z_n$.

Note. We observe that the function $Z = \mathrm{End}\, P^\pi \to \mathrm{Ext}^1_{Z\pi}(P/P^\pi, P^\pi)$ given by pushouts is surjective because P is projective.

LEMMA 2.5. *If* $h: P \oplus Z\pi \xrightarrow{\cong} (Z\pi)^2$, *then* $P/P^\pi \oplus Z\pi \cong Z\pi/(N) \oplus Z\pi$.

Proof. It is easy to see that $P^\pi = (N) \cap P$ and that $(P \oplus Z\pi)^\pi = ((N) \cap P) \oplus (N)$. Consider $\bar{h} = h \mid (P \oplus Z\pi)^\pi$. $\bar{h}$ is an automorphism of $Z \oplus Z$. By diagonalizing the (integer) matrix of $\bar{h}$, one may obtain a basis $\{e_1, e_2\}$ for $Z\pi^2$ with respect to which $h((N) \cap P) = N \cdot Z\pi \cdot e_1$. Thus $P/P^\pi \oplus Z\pi \cong Z\pi/(N) \oplus Z\pi$.

LEMMA 2.6. *For each* $p \in Z_n^*$, $P_p \oplus Z\pi \cong P \oplus (p, N)$.

Proof. Choose an integer $q \in p \in Z_n^*$ and consider the following commutative diagram with exact rows:

$$(*)\qquad \begin{array}{ccccccccc} 0 & \longrightarrow & (N) & \longrightarrow & Z\pi & \longrightarrow & Z\pi/(N) & \longrightarrow & 0 \\ & & \downarrow g & & \downarrow f & & \| & & \\ 0 & \longrightarrow & (N) & \longrightarrow & (q, N) & \xrightarrow{h} & (Z\pi/(N) & \longrightarrow & 0 \\ & & \| & & & & & & \\ & & Z & & & & & & \end{array}$$

where $h: (q, N) \to Z\pi/(N)$ carries $\alpha q + \alpha' N \mapsto \alpha + (N)$ and $f(\alpha) = q \cdot \alpha (\alpha, \alpha' \in Z\pi)$. Then g is multiplication by q, also. By tensoring the above diagram (*) on the right by P we obtain

$$\begin{array}{ccccccccc} & & Z & & & & & & \\ & & \| & & & & & & \\ 0 & \longrightarrow & (N) \otimes_\pi P & \longrightarrow & P & \longrightarrow & Z\pi/(N) \otimes_\pi P & \longrightarrow & 0 \\ & & \downarrow \bar{g} & & \downarrow \bar{h} & & \| & & \\ 0 & \longrightarrow & (N) \otimes_\pi P & \longrightarrow & (q, N) \oplus_\pi P & \longrightarrow & Z\pi/(N) \otimes_\pi P & \longrightarrow & 0 . \end{array}$$

Thus by Schanuel's lemma [8, §1], $Z \oplus P_p \cong Z \oplus P$ by a map of degree q (multiplication by q on the left factor). By Lemma 6.4 of [8], $[P_p] = [P] + [(p, N)]$ in $\tilde{K}_0 Z\pi$ and hence $P_p \oplus Z\pi \cong P \oplus (p, N)$ follows from Bass' cancellation theorem [11, Theorem 9.7].

LEMMA 2.7. *If* $[P]$ *is a member of* $\operatorname{im} \partial$, *then* $P/P^\pi \oplus Z\pi \cong Z\pi/(N) \oplus Z\pi$.

Proof.

$$\begin{aligned} P \oplus Z\pi \cong (p, N) \oplus Z\pi &\Longrightarrow P \oplus Z\pi \oplus (q, N) \cong (Z\pi)^3 (q = p^{-1}) \\ &\Longrightarrow P_q \oplus (Z\pi)^2 \cong (Z\pi)^3 \ (2.6) \\ &\Longrightarrow P_q \oplus Z\pi \cong (Z\pi)^2 \ \text{(Bass cancellation)} \\ &\Longrightarrow P_q/P_q^\pi \oplus Z\pi \cong Z\pi/(N) \oplus Z\pi \ (2.5) \\ &\Longrightarrow P/P^\pi \oplus Z\pi \cong Z\pi/(N) \oplus Z\pi \ (2.3). \end{aligned}$$

PROPOSITION 2.8. *If P and Q are projective ideals in $Z\pi$, then* $Z \oplus P \cong Z \oplus Q$ *iff* $Q \cong P_p$ *for some* $p \in Z_n^*$.

Proof. If $Q \cong P_p$, then $Z \oplus P \cong Z \oplus P_p$ follows from the proof of Lemma 2.6. $Z \oplus P \cong Z \oplus Q$ implies that $P/P^\pi \cong Q/Q^\pi$. Since $\operatorname{Ext}(Q/Q^\pi, Z) = Z_n$, there is an extension $0 \to Z \to R \to P/P^\pi \to 0$ such that $R \cong Q$. R is projective implies that $R \cong P_p$ for some $p \in Z_n^*$.

The following proposition follows easily from Lemma 6.1 of [8].

PROPOSITION 2.9. *$Z \oplus P \oplus Z\pi \cong Z \oplus (Z\pi)^2$ iff $[P] \in \operatorname{im} \partial \subset \tilde{K}_0 Z\pi$.*

PROPOSITION 2.10. *$Z\pi/(N) \oplus Z\pi \cong M \oplus Z\pi$ iff there exists a projective ideal P_M such that*

(a) $P_M|(P_M)^\pi \cong M$

(b) $[P_M] = 0$ *in* $\tilde{K}_0 Z\pi$.

Furthermore, let $Z\pi/(N) \oplus Z\pi \cong M' \oplus Z\pi$. Then $M \cong M'$ iff $P_{M'} \cong (P_M)_p$ for some $p \in Z_n^*$.

Proof. ($\Leftarrow$) By 2.7, $P_M \oplus Z\pi \cong (Z\pi)^2$ implies $Z\pi/(N) \oplus Z\pi \cong P_M/(P_M)^\pi \oplus Z\pi \cong M \oplus Z\pi$.

($\Rightarrow$) Consider the exact sequence

$$0 \longrightarrow \underset{\substack{\| \\ ((N),\,0)}}{Z} \xrightarrow{\ i\ } (Z\pi)^2 \xrightarrow{\ j\ } Z\pi/(N) \oplus Z\pi \longrightarrow 0\,.$$

Since α: $Z\pi/(N) \oplus Z\pi \cong M \oplus Z\pi$, we have

$$0 \longrightarrow Z \xrightarrow{\ i\ } Z\pi^2 \xrightarrow{\ \alpha\circ j\ } M \oplus Z\pi \longrightarrow 0$$

is exact. $Z\pi$ is a projective π-module implies that there exists a projective ideal P_M such that $\beta\colon P_M \oplus Z\pi \cong Z\pi \oplus Z\pi$ and

$$0 \longrightarrow Z \longrightarrow P_M \oplus Z\pi \underset{\substack{\| \\ j' \oplus id}}{\xrightarrow{\ \alpha j \beta\ }} M \oplus Z\pi \longrightarrow 0\,.$$

Thus $0 \to Z \xrightarrow{i'} P_M \xrightarrow{j'} M \to 0$ is exact. M is torsion free and $M^\pi = 0$ implies that $i'(Z) = P_M^\pi$. Thus $P_M/(P_M)^\pi \cong M$ and $[P_M] = 0$ in $\tilde{K}_0 Z\pi$.

For the second part, suppose that $P_{M'} \cong (P_M)_p$ for some $p \in Z_n^*$. Then $M' \cong P_{M'}/(P_{M'})^\pi \cong (P_M)_p/(P_M)_p^\pi \cong P_M/P_M^\pi \cong M$ by 2.3.

If $M \cong M'$, then $0 \to Z \to P_{M'} \to M \to 0$ is exact. By 2.4, $\operatorname{Ext}(M, Z) \cong Z_n$ and the set of projective extensions is given by

$$\{0 \longrightarrow P_M^\pi \longrightarrow (P_M)_p \longrightarrow (P_M)_p/(P_M)_p^\pi \cong P_M/P_M^\pi \cong M \longrightarrow 0 \mid p \in Z_n^*\}\,.$$

Thus $P_{M'} \cong (P_M)_p$ for some $p \in Z_n^*$.

The following proposition has a proof which is similar to that of 2.10. For any projective ideal $P \subset Z\pi$, let $\varepsilon\colon P \to Z$ be the augmentation.

PROPOSITION 2.11. *$A(\pi) \oplus Z\pi \cong M \oplus Z\pi$ iff there exists a projective ideal P_M such that*

(a) $0 \to \bar{M} \to P_M \xrightarrow{\varepsilon} Z \to 0$ *with* $\bar{M} \cong M$, *and*

(b) $[P_M] = 0$ *in* $\tilde{K}_0 Z\pi$.

Furthermore, let $A(\pi) \oplus Z\pi \cong M' \oplus Z\pi$. *Then* $M \cong M'$ *iff* $P_{M'} \cong (P_M)_p$ *for some* $p \in Z_n^*$.

We point out that the proof of 2.11 is not quite "dual" to that of 2.10, for it uses the relative injectivity of $Z\pi$ and the fact that, for any projective ideal P in $Z\pi$, $\mathrm{Ext}(Z, P/\ker \varepsilon) \cong Z_n$, etc.

Proof of Theorem 2.1. We prove only (a), as (b) is similar. Define a function $\nu: SW(\mathscr{P}'(\pi)) \to \mathscr{N}\mathscr{C}_{Z\oplus Z\pi}$ by $\nu(\{P\}) = \{Z \oplus P\}([P] \in \operatorname{im} \partial)$, where P is a projective ideal in $Z\pi$. Clearly ν is onto by 2.2(b). If $Z \oplus P \cong Z \oplus P'$, then (2.8) implies that $P' \cong P_p$ for some $p \in Z_n^*$.

3. **Nontrivial** $\mathscr{N}\mathscr{C}_N(\pi)$. In this section we show that both $\mathscr{N}\mathscr{C}_{Z\oplus Z\pi}(\pi)$ and $\mathscr{N}\mathscr{C}_{Z\oplus Z\pi}(\operatorname{Aut} \pi)$ are nontrivial for $\pi = GQ(32)$, the generalized quaternion group of order 32.

DEFINITION. Let θ be an automorphism of π. Two π-modules M, M' are *θ-isomorphic* ($M \cong {}_\theta M'$) if there is a function $\beta: M \to M'$ which is bijective such that $\beta(x \cdot m) = \theta(x)\beta(m)$ for all $x \in \pi$, $m \in M$. β is called a *θ-isomorphism*. Let $\bar{*}_M$ denote the class of all modules stably isomorphic to M *and* θ-isomorphic to M for some $\theta \in \operatorname{Aut} \pi$. Clearly $*_M \subset \bar{*}_M$. Furthermore, let $\mathscr{N}\mathscr{C}_M(\operatorname{Aut} \pi)$ denote the set which is the union of $\bar{*}_M$ with the set of $\operatorname{Aut} \pi$-isomorphism classes of π-modules M' such that

(a) $M' \oplus (Z\pi)^2 \cong M \oplus (Z\pi)^2$

and

(b) M' is not θ-isomorphic to M for any $\theta \in \operatorname{Aut} \pi$.

DEFINITION. A π-module M is *full* if for each $\theta \in \operatorname{Aut} \pi$, there is a θ-isomorphism $M \to M$.

For example, it is clear that $Z \oplus Z\pi$, $A(\pi)$, and $Z\pi/(N)$ are full π-modules.

PROPOSITION 3.1. *If M is a full π-module, then* $*_M = \bar{*}_M$.

Proof. We must show that $M \cong M'$ if $M \cong {}_\theta M'$. Suppose $\beta: M \to M'$ is an θ-isomorphism. Let $\alpha: M \to M$ be a θ^{-1}-isomorphism. Then the composite $\beta \cdot \alpha: M \to M'$ is an *id*-isomorphism.

COROLLARY 3.2. *If M is a full π-module, then* $\mathscr{N}\mathscr{C}_M(\pi) \neq *_M$ *yields* $\mathscr{N}\mathscr{C}_M(\operatorname{Aut} \pi) \neq \bar{*}_M$.

Now let $G = GQ(32)$, the generalized quaternion group of order 32, and let P be the projective ideal in ZG defined in [10]. P has the following properties:

3.3 (a) $P \oplus ZG \cong (ZG)^2$

but

3.3 (b) $P \not\cong ZG$.

The proof of the following lemma was shown to me by R. Swan. It generalizes (3.3(b)).

LEMMA 3.4. *For any* $p \in Z_{32}^*$, $(p, N) \not\cong P$.

Proof. Suppose $P \cong (p, N)$ for some $p \in Z_{32}^*$. Then $P \oplus Z \cong ZG \oplus Z$. Let Λ be the order considered in [10] and apply $\Lambda \otimes_{ZG} -$ to the above obtaining

$$(3.5) \qquad (\Lambda \otimes_{ZG} P) \oplus (\Lambda \otimes_{ZG} Z) \cong \Lambda \oplus (\Lambda \otimes_{ZG} Z) .$$

The module $\mathscr{P} = \Lambda \otimes_{ZG} P \not\cong \Lambda$ [10, Lemma 1]. Now $\Lambda \otimes_{ZG} Z$ is a torsion module because $QG \cong Q\Lambda \times Q \times \cdots$, so $Q\Lambda \otimes_{QG} Q = 0$. Factoring out the torsion in (3.5) gives $\mathscr{P} \cong \Lambda$, which is contradiction.

COROLLARY 3.6. *For* $G = GQ(32)$ *and* $M = Z \oplus ZG$, $A(G)$, *or* $ZG/(N)$, $\mathscr{N}\mathscr{C}_M(G) \neq *_M$.

Proof. $P \not\cong (p, N)$ for any $p \in Z_{32}^*$ implies that $Z \oplus ZG \not\cong Z \oplus P$, by 2.8. Clearly $Z \oplus P \oplus ZG \cong Z \oplus (ZG)^2$ by 3.3(a). If $A' = \ker\{\varepsilon: P \to Z\}$, then by 2.11, $A(G) \oplus ZG \cong A' \oplus ZG$, but $A' \not\cong A(G)$. Letting $B = P/P^G$, 2.10 shows that $B \oplus ZG \cong ZG/(N) \oplus ZG$, but $B \not\cong ZG/(N)$, by 3.4.

4. **Roots in homotopy trees.** Let (π, m) be fixed, where π is a group and m an integer greater than or equal to two. Let $\chi_{\min} = \chi_{\min}(\pi, m) = \min\{(-1)^m \chi(X) \mid X \text{ is a } (\pi, m)\text{-complex}\}$. The *level* of a (π, m)-complex X is the number $(-1)^m\chi(X) - \chi_{\min}$. For π finite, it is known that roots occur only at levels 0 (minimal roots) or 1. In this section we give an example of a (π, m)-complex which is a root at level one. As pointed out in the introduction, these level one roots are rare (for π finite), occurring only when π is periodic and $m = k - 1$, where k is a period of π. Dunwoody's example is also at level one [3].

Question. Do roots occur at levels other than 0 or 1?

DEFINITION. The homotopy tree $HT(\pi, m)$ is a directed tree whose vertices $[X]$ consist of the homotopy classes of (π, m)-complexes X; a vertex $[X]$ is connected by an edge to vertex $[Y]$ iff Y has the homotopy type of the sum $X \vee S^m$ of X and the m-sphere S^m.

COROLLARY 3.7. *Let* $G = GQ(32)$. *Then each homotopy tree* $HT(G, 4i\text{-}1)(i > 0)$ *has nonminimal roots* (*at level one*).

Proof. Consider $\ker\{\partial: Z_{32}^* \to \tilde{K}_0 ZG\}$ (see § 2). Recent computations of S. Ullom [12, Prop. 3.5] show that

$$\ker \partial = \pm (Z_{32}^*)^2 .$$

Let $\mathcal{NC} = \mathcal{NC}_{Z \oplus ZG}(\operatorname{Aut} G)$. For each $\alpha \in \mathcal{NC}$, choose a representative $Z \oplus P_\alpha \in \alpha$. It follows from Theorem 9.1 of [4] that the number of distinct homotopy classes of $(G; 4i\text{-}1)$-complexes at level one is given by order of the set

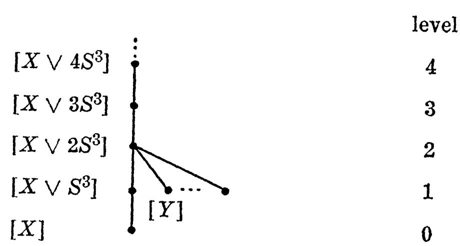
$$\dot{\bigcup_{\alpha \in \mathcal{NC}}} \{\ker \partial / Q_{4i-1}(Z \oplus P_\alpha)\} .$$

For a definition of the subgroup $Q_{4i-1}(Z \oplus P_\alpha)$ of $\ker \partial$, see [4, page 272]. The number of distinct classes of *roots* is given by the order of the *nonempty* set

$$\dot{\bigcup_{\alpha \in \mathcal{NC} - *}} \{\ker \partial / Q_{4i-1}(Z \oplus P_\alpha)\} .$$

We note that $\mathcal{NC}_M(G) \neq *$ for $M = A(G)$ or $ZG/(N)$ implies that the homotopy trees $HT(G, 4i\text{-}2)$ or (respectively) $HT(G, 4i)$ have nontrivial minimal roots, with the possible exception of $HT(G, 2)$.

Finally, the computations of Ullom [12, 3.5] allow one to show that the homotopy tree $HT(G, 3)$ looks like:



where X is the unique $(G, 3)$-complex (up to homotopy type) having Euler characteristic zero and Y is the $(G, 3)$-complex at level 1 having $\pi_3(Y) \cong Z \oplus P$. It follows that $Q_3(Z \oplus P_\alpha) = \ker \partial$ for all $\alpha \in \mathcal{NC}$ and hence the number of homotopy types of $(G, 3)$-complexes at level one is given by the order of the set

$$\mathscr{N}\mathscr{C}_{Z\oplus ZG}(\mathrm{Aut}\,G) .$$

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Received November 13, 1975 and in revised form February 1, 1978.

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