PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 2, 1979

MODULES WITH SUPPLEMENTS

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Let M be an R-module and $N \subset M$. Any $H \subset M$ satisfying $H + N = M$...... (i) and $H' \subset H, H' + N = M \Rightarrow H' = H$...... (ii) will be referred to as a supplement of N in M. In general N need not have a supplement in M. A module M will be said to have property (P_1) if every $N \subset M$ has a supplement in M. If for every $A \subset M, N \subset M$ with $A+N=M$, there exists a supplement H of N in M satisfying $H \subset A$, we say that M has property (P_2). Modules with property (P_2) play an important role in our study of dual Goldie dimension. In the present paper we determine the class of rings R with the property that every $M \in R$-mod possesses property (P_2). These ture out to be left perfect rings. Also the results obtained here throw more light on the differences between corank and P. Fleury's spanning dimension.

Introduction. While attempting to dualize the concept of Goldie dimension, Patrick Fleury [3] introduced the class of modules with finite spanning dimension. A module M is said to have finite spanning dimension if for every infinite, strictly decreasing chain $N_0 \supsetneqq N_1 \supsetneqq N_2 \supsetneqq \ldots\ldots$ of submodules of M, there exists an integer j (depending on the sequence) such that N_i is small in M for all $i \geqq j$. A module H is said to be hollow if $H \neq 0$ and any $X \subsetneqq H$ is small in H. One of the main results of [3] states that any module of finite spanning dimension can be expressed as an irredundant sum $M = \sum_{i=1}^{n} H_i$ of finitely many hollow submodules H_i of M and that their number n depends only on M. This number is referred to as the spanning dimension (abbreviated as $s \cdot d$) of M. In our earlier paper entitled Dual Goldie dimension [6] we indicated a completely different way of dualizing Goldie dimension and gave ample evidence to show that our approach has distinct advantages over Fleury's approach. Let k be an integer $\geqq 1$. In [6] we defined M to have corank $\geqq k$ if there was an epimorphism $f: M \to \prod_{i=1}^{k} N_i$ with each $N_i \neq 0$. For $M \neq 0$ we defined corank M to be k if corank $M \geqq k$, but corank $M \ngeqq (k+1)$. If corank $M \geqq n$ for all $n \geqq 1$, we set corank $M = \infty$. When $M = 0$ we set corank $M = 0$. In [6] we showed that the invariant corank had many more desirable properties to be termed dual Goldie dimension than the invariant $s \cdot d$ introduced by P. Fleury. Every module with finite spanning dimension possesses preperty (P_2). [3]. In our study of corank also [6], modules with property (P_2) played a special role, though the theory of corank works very well for all modules. In the present paper we characterise completely the rings R with the property that every $M \in R$-mod possesses property (P_2).

1. **Modules with property (P_2).** Throughout this paper all the rings considered are associative rings with an identity element. Unless otherwise mentioned all the modules considered are unitary left modules and all the concepts are left sided concepts. Let R be a ring and $M \in R$-mod. If M satisfies (P_2) it is clear that M satisfies (P_1). In [6] we showed that if every submodule of M has (P_1) then M itself has (P_2). In particular every $M \in R$-mod has $(P_2) \Leftrightarrow$ every $M \in R$-mod has property (P_1).

An epimorphism $\varepsilon: M \to N$ is said to be minimal if Ker ε is small in M.

PROPOSITION 1.1. *Let $N \subset M$ and $\varepsilon: P \to M/N$ a minimal epimorphism. Suppose there exists a lift $f: P \to M$ of ε. Then $H = f(P)$ is a supplement of N in M.*

Proof. Let $\eta: M/N$ denote the canonical quotient map and $K =$ Ker ε. Then $\eta(H) = \eta \circ f(P) = \varepsilon(P) = M/N$. Hence $H + N = M$. Suppose $H' \subset H$ satisfies $H' + N = M$. Let $P' = f^{-1}(H')$. Then $f(P') = H'$ and $\varepsilon(P') = \eta \circ f(P') = \eta(H') = M/N$. This yields $P' + K = P$. Since K is small in P we get $P' = P$. Hence $H' = f(P) = H$.

Before proceeding further recall the definition of an M-projective module due to G. Azumaya [1].

DEFINITION 1.2. Let M be a fixed R-module. An R-module P is called M-projective, if for any exact sequence $M \xrightarrow{\varphi} M'' \to 0$, the sequence $\operatorname{Hom}_R(P, M) \xrightarrow{\varphi^*} \operatorname{Hom}_R(P, M'') \to 0$ is exact.

PROPOSITION 1.3. *Let $A \subset M$, $N \subset M$ satisfy $A + N = M$. Suppose there exists a minimal epimorphism $\varepsilon: P \to M/N$ with P an M-projective module. Then there exists a supplement H of N in M with $H \subset A$.*

Proof. Let $\alpha = \eta/A$ where $\eta: M \to M/N$ is the canonical quotient map. Then $A + N = M \Rightarrow \alpha(A) = M/N$. Hence $A \xrightarrow{\alpha} M/N \to 0$ is exact. Since $A \subset M$ and P is M-projective, it follows that P is A-projective [1]. Hence, there exists a map $f: P \to A$ with $\alpha \circ f = \varepsilon$. Lemma 1.1 shows that $f(P) = H$ is a supplement of N in M. Clearly $H \subset A$.

Since any projective R-module is M-projective for any $M \in R$-mod, we get the following as a corollary of Proposition 1.3.

COROLLARY 1.4. (a) *If R is left perfect then every $M \in R$-mod possesses property (P_2).*

(b) *If R is semi-perfect then every finitely generated $M \in R$-mod possesses property (P_2).*

Before proceeding further we state a result of F. Kasch and E. Mares [4] which we need.

PROPOSITION 1.5. (*F. Kasch and E. Mares*). *Let P be a projective module, $N \subset P$. Suppose H is a supplement of N in P and that H itself has a supplement L in P. Then $H \cap L = 0$, hence $P = H \oplus L$. In particular H is projective.*

THEOREM 1.6. *The following conditions on a ring R are equivalent.*

(1) *Every free R-module F has property (P_1).*

(2) *Every $M \in R$-mod has property (P_2).*

(3) *R is left-perfect.*

Proof. (1) $\Rightarrow$ (3). Assuming (1) we will prove that any $M \in R$-mod has a projective cover. Let $F \xrightarrow{f} M \to 0$ be exact with F free. Let $K = \operatorname{Ker} f$. Let H be a supplement of K in F. Since H has a supplement in F, from Proposition 1.5 we see that H is projective. Also $H \cap K$ is small in H [4, or Lemma 2.7 of [6]]. It follows that $f|H: H \to M$ is a projective cover of M.

(3) $\Rightarrow$ (2) Immediate from Corollary 1.4 (a).

(2) $\Rightarrow$ (1) Trivial.

The arguments used in the above proof also yield the following result:

THEOREM 1.7. *The following conditions on a ring R are equivalent.*

(1) *$R \in R$-mod has property (P_1).*

(2) *R is semi-perfect.*

(3) *Every finitely generated $M \in R$-mod has property (P_2).*

2. **Surjective endomorphisms with small kernels.** Let P be a projective module, $N \subset P$. Suppose N has a supplement H in P. If H also has a supplement in P, then the result of Kasch and Mares asserts that H is a direct summand of P. In general, when

H need not possess a supplement in P, we do not know whether H will be projective. The results in the present section throw some light on this question.

It is well-known that any surjective endomorphism $f: M \to M$ of a noetherian module is an isomorphism. A result of Vasconcelos [7] asserts that if R is a commutative ring and M a finitely generated module over R, then any surjective endomorphism $f: M \to M$ is an isomorphism. In [5] M. Orzech obtained a generalization of the above results. Let R be a ring, $M \in R$-mod, $N \subset M$. Assume either M is noetherian or that R is commutative and M is finitely generated. Then any epimorphism $f: N \to M$ is an isomorphism. In this section we will first obtain a useful modification of the above result of Orzech. For any $M \in R$-mod, we denote the set of small submodules of M by $\Gamma(M)$ and partially order $\Gamma(M)$ under inclusion. The class of finitely generated R-modules will be denoted by $M(R)$.

LEMMA 2.1. *Let $M \in R$-mod. Then $\Gamma(M)$ satisfies $a \cdot c \cdot c \Leftrightarrow J(M)$ is noetherian.*

Proof. The implication $\Leftarrow$ follows from the well-known fact that any $N \in \Gamma(M)$ satisfies $N \subset J(M)$. Conversely, suppose $J(M)$ is not noetherian. Let $X_1 \subsetneqq X_2 \subsetneqq X_3 \subsetneqq \cdots\cdots$ be an infinite ascending chain of submodules of $J(M)$. Let $x_1 \in X_1$ and $x_j \in X_j - X_{j-1}$ for each $j > 1$. For any $k \geqq 1$ let $N_k = \sum_{j=1}^{k} Rx_j$. Then $N_k \in M(R)$ and $N_k \subset J(M)$. Hence $N_k \in \Gamma(M)$. It is clear that $N_1 \subsetneqq N_2 \subsetneqq N_3 \subsetneqq \cdots\cdots$. Hence $\Gamma(M)$ fails to satisfy $a \cdot c \cdot c$. This proves the implication $\Rightarrow$.

LEMMA 2.2. *Let $M \xrightarrow{f} M''$ be an epimorphism with* Ker f *small in M. Then $f(J(M)) = J(M'')$.*

Proof. Since $f(J(M)) \subset J(M'')$, to prove the equality $f(J(M)) = J(M'')$ we have only to show that $f^{-1}(J(M'')) \subset J(M)$. Let $y \in J(M'')$. Then Ry is small in M''. Since $f: M \to M''$ is an epimorphism with Ker f small in M, it follows that $f^{-1}(Ry)$ is small in M and hence $f^{-1}(Ry) \subset J(M)$. This proves $f^{-1}(J(M'')) \subset J(M)$.

PROPOSITION 2.3. *Let $M \in R$-mod, $N \subset M$ and $f: N \to M$ an epimorphism with* Ker f *small in N. Assume that either $J(M)$ is noetherian or that R is commutative and $J(M)$ is finitely generated. Then f is an isomorphism.*

Proof. Let $K = \text{Ker } f$. Then $K \subset J(N) \subset J(M)$. By Lemma 2.2, $g = f \mid J(N): J(N) \to J(M)$ is an epimorphism. From Orzech's result we see that g is an isomorphism. Hence $\operatorname{Ker} g = \operatorname{Ker} f = K = 0$. This proves that $f: N \to M$ is an isomorphism.

Let R be any ring and E any injective module over R, $N \subset E$ and L a relative complement of N in E (namely $N \cap L = 0$, $N \cap L' \neq 0$ for any $L \subsetneqq L' \subset E$). Since E is injective we can assume that the injective hull $E(L)$ of L is a submodule of E. From $N \cap L = 0$ we get $N \cap E(L) = 0$. Hence $L = E(L)$. Thus any relative complement in E is injective. If P is a projective R-module, $K \subset P$ admitting a supplement H in P, in general we do not know whether H will be projective. The following is a result in this direction. For any $M \in R$-mod let $S(M) = \{H \mid H \subset M, H \text{ a supplement of some } N \subset M\}$.

PROPOSITION 2.4. *Let P be a projective R module and $H \in S(P)$. If either $J(H)$ is noetherian or R is commutative and $J(H)$ is finitely generated over R then H is a direct summand of P and hence H is projective.*

Proof. Let H be a supplement of N in P and $\eta: P \to P/N$ denote the canonical quotient map. From Lemma 2.7 of [6] we see that $H \cap N$ is small in H. From $H + N = P$ we get $\eta(H) = P/N$. Let $\alpha = \eta \mid H$. Then $H \xrightarrow{\alpha} P/N \to 0$ is exact. Since P is projective, there exists a $\varphi: P \to H$ with

$$\begin{tikzcd} & P \arrow[dl, dashed, "\varphi"'] \arrow[d, "\eta"] & \\ H \arrow[r, "\alpha"] & P/N \arrow[r] & 0 \end{tikzcd}$$

commutative. From $\alpha(\varphi(H)) = \eta(H) = P/N$ we get $\varphi(H) + \operatorname{Ker} \alpha = H$. But $\operatorname{Ker} \alpha = H \cap N$. Since $H \cap N$ is small in H we get $\varphi(H) = H$. Also $\operatorname{Ker} \varphi \subset \operatorname{Ker} \eta$. If $f = \varphi \mid H: H \to H$, then $\operatorname{Ker} f \subset N \cap H$ and f is an epimorphism. From Proposition 2.3 we see that f is an isomorphism. Let $\theta = j \circ f^{-1}: H \to P$ where j is the inclusion of H in P. Then $\varphi \circ \theta = \theta \circ j \circ f^{-1} = f \circ f^{-1} = Id_H$. Thus θ is a splitting of φ. If $h = f^{-1} \circ \varphi: P \to H$ we have $h \circ j = f^{-1} \circ \varphi \circ j = f^{-1} \circ f = Id_H$. This proves that H is a direct summand of P.

3. Corank versus spanning dimension.

REMARKS.

(1) In [5] we proved that if $s \cdot dM < \infty$, then corank $M = s \cdot dM$. Also in the same paper we proved that if R is any local ring, and

$M = R \times R$ in R-mod then corank $M = 2$. But if R is nonartinian, then $s \cdot dM = \infty$. Since any local ring is semi-perfect, it follows form Corollary 1.4(b) that M satisfies property (P_2).

(2) For any M with property (P_2) and having corank $M = n < \infty$ we have proved the following two assertions in [6].

(a) $M = \sum_{i=1}^{n} H_i$ an irredundant sum with each H_i hollow.

(b) If $M = \sum_{j=1}^{k} H'_j$ is an irredundant sum with each H'_j hollow then $k = n$.

Thus P. Fleury's main result [Theorem 3.1 of [3]] was extended by us to the class of modules M with corank $M < \infty$ and satisfying (P_2). The example $M = R \times R$ in R-mod, with R a nonartinian local ring shows that the above class is strictly larger than the class of modules with finite spanning dimension.

(3) Let $S(M) = \{H/H \subset M, H$ a supplement of some $N \subset M\}$. If M is a module with property (P_2), we proved in [5] that corank $M = \text{Sup}\ \{k \mid$ there exists a strict increasing chain $H_0 \subsetneqq H_1 \subsetneqq \cdots \subsetneqq H_k$ of length k with each $H_i \in S(M)\}$. For the example $M = R \times R$ in R-mod with R a nonartinian local ring we have corank $M = 2$. Thus there exist strict increasing chains of supplement submodules in M of length 2 and not more. However $s \cdot dM = \infty$. Thus in this example $\text{Sup}\ \{k \mid$ there exists a strict increasing chain $H_0 \subsetneqq \cdots \subsetneqq H_k$ with each $H_i \in S(M) = 2 < s \cdot dM\}$. This is one instance where corank is better behaved than spanning dimension.

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Received August 10, 1978. Research done when the author was partially supported by NRC grant A8225.

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