PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 2, 1979

ON EXTENSION OF ROTUND NORMS II

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It is proved that if X is a Banach space, $Y \subset X$ with X/Y separable and $\|\cdot\|$ is an equivalent locally uniformly rotund norm on Y, then $\|\cdot\|$ can be extended to such a norm on X.

This generalizes [2] where it was shown that any locally uniformly rotund equivalent norm on a closed subspace of a separable Banach space X can be extended to such a norm on X.

By a subspace we mean a closed linear subspace, sp L denotes the linear hull of L and $x \to \hat{x}$ stands for the quotient map $X \to X/Y$ if Y is a subspace of X.

Let us recall that a norm $\|\cdot\|$ on a Banach space X is locally uniformly rotund (LUR) if whenever $\lim 2(\|x\|^2 + \|x_j\|^2) - \|x + x_j\|^2 = 0$, $x, x_j \in X$, then $\lim \|x - x_j\| = 0$. $\|\cdot\|$ is rotund (R) if for any $x, y \in X$, $x \neq y$, $2(\|x\|^2 + \|y\|^2) - \|x + y\|^2 > 0$.

THEOREM 1. *Let X be a Banach space, $Y \subset X$ a subspace of X. Suppose X/Y is separable and Y admits an equivalent norm $\|\cdot\|$ which is LUR (R). Then $\|\cdot\|$ can be extended to an equivalent norm $\|\cdot\|$ on X which is LUR (R).*

Proof. Let us start with the case of LUR.

First extend the given LUR norm $\|\cdot\|$ on Y to an equivalent norm $\|\cdot\|$ on X: This can easily be done as follows: Take the closed unit ball B_1^Y of Y with respect to $\|\cdot\|$ and the closed ball B of X such that $B \cap Y \subset B_1^Y$. Then, easily, the Minkowski functional of conv $(B \cup B_1^Y)$ is the desired norm on X(cf. e.g., [4], [2]).

Furthermore, let $\{\hat{a}_n\}_{n=1}^{\infty} \subset X/Y$, $\hat{a}_n \neq 0$ be a dense subset of X/Y. Let $S: X/Y \to X$ denote the Bartle-Graves continuous selection map $(S\hat{x} \in \hat{x})$ and $a_n = S\hat{a}_n$.

For $n \in N$ (N positive integers), choose $f_n \in X^*$, $f_n(a_n) = 1$, $\|f_n\| = \|\hat{a}_n\|^{-1}$, $f_n = 0$ on Y and denote by $P_n(x) = f_n(x)a_n$, $P_n' = I - P_n$ where I is the identity map on X.

Consider

$$|||x|||^2 = (1 - c)\|x\|^2 + \sum_{n=1}^{\infty} 2^{-n}(1 + \|P_n\|)^{-2} \cdot \|x - P_n x\|^2 + \|\hat{x}\|^2 ,$$

where $c = \sum_{n=1}^{\infty} (1 + \|P_n\|)^{-2} 2^{-n}$, $\|\hat{\cdot}\|$ is an equivalent LUR norm on X/Y([3]).

Then (i) $|||\cdot|||$ is an equivalent norm on X which agree with $\|\cdot\|$

on Y,

(ii) $|||\cdot|||$ is LUR.

(i) is easily seen.

To see (ii), assume there is an $\varepsilon > 0$ such that

$$\lim 2(|||x|||^2 + |||x_m|||^2) - |||x + x_m|||^2 = 0 \tag{1}$$

and

$$|||x - x_m||| > \varepsilon \tag{2}$$

and find a contradiction.

From (1),

$$\lim 2(||\hat{x}||^2 + ||\hat{x}_m||^2) - ||\hat{x} + \hat{x}_m||^2 = 0\ , \tag{3}$$

$$\lim_m 2(||P_n'x||^2 + ||P_n'x_m||^2) - ||P_n'(x + x_m)||^2 = 0\ , \quad \text{for} \quad n \in N \tag{4}$$

$$\lim 2(||x||^2 + ||x_m||^2) - ||x + x_m||^2 = 0\ , \tag{5}$$

$$K = \max(\sup ||x_n||, 1) < \infty\ . \tag{6}$$

If $x \in Y$, then $\hat{x} = 0$ and form (3), $\lim ||\hat{x}_m|| = 0$, so there is a sequence $x_m' \in Y$ with $\lim ||x_m - x_m'|| = 0$ and so, by (5), (6) $\lim 2(||x||^2 + ||x_m'||^2) - ||x + x_m'||^2 = 0$ and therefore by LUR of $||\cdot||$ on Y, $\lim ||x - x_m'|| = 0$ and thus $\lim ||x - x_m|| = 0$, a contradiction with (2).

If $x \notin Y$, write $x = y_0 + a_0$, $a_0 = S\hat{x}$, $y_0 \in Y$. From LUR of $||\cdot||$ on Y, there is $\delta \in (0, 1/2)$ such that whenever

$$y \in Y,\ ||y - y_0|| \leqq \delta,\ z \in Y, \text{ and } 2(||y||^2 + ||z||^2) - ||y + z||^2 \leqq \delta\ , \tag{7}$$

then, $||y - z|| \leqq \varepsilon/2$. By (3) and LUR of $||\hat{\cdot}||$,

$$\lim ||\hat{x}_n - \hat{x}|| = 0 \tag{8}$$

and thus,

$$\lim S\hat{x}_m = S\hat{x} = a_0\ . \tag{9}$$

Let

$$\hat{a}_n \in \{\hat{a}_n\},\ \lim \hat{a}_n = \hat{a}_0 = \hat{x} \text{ (and thus } \lim a_n = a_0)\ . \tag{10}$$

Furthermore,

$$\lim ||P_n|| = ||a_0|| \cdot ||\hat{a}_0||^{-1}\ . \tag{11}$$

Let $\delta_1 = \min\{[1 + (5(||a_0|| \cdot ||\hat{a}_0||^{-1} + 2))^2(K + 1)]^{-1}\delta,\ \varepsilon/8\}$ (δ from (7)). Choose $n_0 \in N$ so that

(a) $||P_{n_0}|| \leqq ||a_0|| \cdot ||\hat{a}_0||^{-1} + 1$

(b) $||a_n - a_0|| < \delta_1$ for each $n \geqq n_0$
(c) $||\hat{x}_m - \hat{x}|| < \delta_1$ for each $m \geqq n_0$.
Keeping this n_0 fixed, choose $n_1 \geqq n_0$ so that
(d) $2(||P'_{n_0}(x)||^2 + ||P'_{n_0}(x_m)||^2) - ||P'_{n_0}(x + x_m)||^2 < \delta_1$ for each $m \geqq n_1$.
Choose $z_{n_0} \in \hat{a}_{n_0}$ such that

$$||z_{n_0} - x|| < \delta_1 \tag{12}$$

and $x'_{n_0} \in \hat{a}_{n_0}$ such that

$$||x'_{n_0} - x_{n_1}|| < 2\delta_1 \,. \tag{13}$$

Since $x'_{n_0} = a_{n_0} + u_{n_0}$, $z_{n_0} = a_{n_0} + v_{n_0}$ for some $u_{n_0}, v_{n_0} \in Y$,

$$P'_{n_0}(x'_{n_0}) = x'_{n_0} - P_{n_0}(x'_{n_0}) = u_{n_0} \in Y \text{ and } P'_{n_0}(z_{n_0}) = v_{n_0} \in Y \,. \tag{14}$$

Furthermore, by (d), (a), (12), (13),

$$\begin{aligned} 2(||P'_{n_0}(z_{n_0})||^2 + ||P'_{n_0}(x'_{n_0})||^2) - ||P'_{n_0}(z_{n_0} + x'_{n_0})||^2 &\leqq 2(||P'_{n_0}(x)||^2 + ||P'_{n_0}(x_{n_1})||^2 \\ &\quad - P'_{n_0}||(x + x_{n_1})||^2 + 2||P'_{n_0}(z_{n_0} - x)||(||P'_{n_0}(z_{n_0})|| + ||P'_{n_0}(x)||) \\ &\quad + 2||P'_{n_0}(x'_{n_0} - x_{n_1})||(||P'_{n_0}(x'_{n_0})|| + ||P'_{n_0}(x_{n_1})||) \\ &\quad + (||P'_{n_0}(z_{n_0} - x)|| + ||P'_{n_0}(x'_{n_0} - x_{n_1})||) \\ &\quad \times(||P'_{n_0}(z_{n_0})|| + ||P'_{n_0}(x)|| + ||P'_{n_0}(x'_{n_0})|| + ||P'_{n_0}(x_{n_1})||) \\ &\leqq \delta_1(1 + (5(||\alpha_0||\cdot||\hat{a}_0||^{-1} + 2))^2(K+1)) \leqq \delta \,. \end{aligned}$$

Thus, by (7) and (14),

$$\varepsilon/2 \geqq ||P'_{n_0}(x'_{n_0}) - P'_{n_0}(z_{n_0})|| = ||x'_{n_0} - z_{n_0}|| \,.$$

So, $||x_{n_1} - x|| \leqq ||x'_{n_0} - z_{n_0}|| + ||x_{n_1} - x'_{n_0}|| + ||z_{n_0} - x|| \leqq (7/8)\varepsilon < \varepsilon$, a contradiction.

For the case of rotund norms we define the norm $|||\cdot|||$ by the same formula as above.
Again, suppose

$$2(|||x|||^2 + |||y|||^2) - |||x + y|||^2 = 0 \tag{1'}$$

and

$$||x - y|| > \varepsilon > 0 \,. \tag{2'}$$

From (1'),

$$2(||\hat{x}||^2 + ||\hat{y}||^2) - ||\hat{x} + \hat{y}||^2 = 0 \tag{3'}$$

$$2(||P'_n(x)||^2 + ||P'_n(y)|^2) - ||P'_n(x + y)||^2 = 0 \qquad \text{for} \quad n \in N \tag{4'}$$

$$2(||x||^2 + ||y||^2) - ||x + y||^2) = 0 \,. \tag{5'}$$

If $x \in Y$, $\hat{x} = 0$ and from (3′), $\hat{y} = 0$, so $y \in Y$ and from R of $\|\cdot\|$ on Y and (5′), $x = y$.

If $x \notin Y$, then by R of $|\hat{\cdot}|$ and by (3′), $\hat{x} = \hat{y}$. So, write $x = a_0 + y_0$, $y = a_0 + z_0$, $y_0, z_0 \in Y$, $a_0 = S\hat{x}$. By R of $\|\cdot\|$ on Y, there is a $(1/2) > \delta > 0$ such that whenever

$$(6') \qquad \begin{array}{l} y \in Y\,, \quad z \in Y\,, \quad \|y - y_0\| \leqq \delta\,, \quad \|z - z_0\| \leqq \delta\,, \\ 2(\|y\|^2 + \|z\|^2) - \|y + z\|^2 \leqq \delta\,, \end{array}$$

then

$$\|y - z\| \leqq \varepsilon/2\,.$$

Denote by $\delta_1 = \min\{[1 + (5(\|a_0\| \cdot \|\hat{a}_0\|^{-1} + 2))^2(K + 1)]^{-1}\delta, \varepsilon/8\}$, where $K = \max(\|x\| = \|y\|, 1)$. Let $\hat{a}_n \in \{\hat{a}_n\}$, $\lim \hat{a}_n = \hat{a}_0 = \hat{x}$, $a_n = S\hat{a}_n$. Then $\lim a_n = a_0$, $\lim \|P_n\| = \|a_0\| \cdot \|\hat{a}_0\|^{-1}$.

Thus we can choose $n_0 \in N$ so that $\|P_{n_0}\| \leqq \|a_0\| \cdot \|\hat{a}_0\|^{-1} + 1$, $\|a_{n_0} - a_0\| < \delta_1$. Choose $y_{n_0}, z_{n_0} \in \hat{a}_{n_0}$ such that $\|z_{n_0} - x\| < \delta_1$, $\|y_{n_0} - y\| < \delta_1$. Since

$$(7') \qquad \begin{array}{l} y_{n_0} = a_{n_0} + z_{n_0},\ z_{n_0} = a_{n_0} + v_{n_0},\ u_{n_0},\ v_{n_0} \in Y,\ P'_{n_0}(y_{n_0}) \\ \quad = y_{n_0} - P_{n_0}(y_{n_0}) = u_{n_0} \in Y\,. \end{array}$$

Furthermore,

$$\begin{aligned} 2(\|P'_{n_0}(z_{n_0})\|^2 + \|P'_{n_0}(y_{n_0})\|^2) - \|P'_{n_0}(y_{n_0} + z_{n_0})\|^2 &\leqq 2(\|P'_{n_0}x\|^2 \\ &+ \|P_{n_0}(y)\|^2) - \|P_{n_0}(x + y)\|^2 \\ &+ 2\cdot\|P'_{n_0}(z_{n_0} - x)\|(\|P'_{n_0}(z_{n_0})\| + \|P'_{n_0}(x)\|) \\ &+ 2\|P'_{n_0}(y_{n_0} - y)\| \cdot (\|P'_{n_0}(y_{n_0})\| + \|P'_{n_0}(y)\|) \\ &+ (\|P'_{n_0}(y_{n_0} - y)\| + \|P'_{n_0}(z_{n_0} - x)\|) \\ &\times (\|P'_{n_0}\| \cdot [\|y_{n_0}\| + \|z_{n_0}\| + \|x\| + \|y\|]) \\ &\leqq \delta_1(1 + (5(\|a_0\| \cdot \|a_0\|^{-1} + 2))^2(K + 1)) \leqq \delta\,. \end{aligned}$$

Thus, by (6′), (7′), $\varepsilon/2 \geqq \|P'_{n_0}(y_{n_0}) - P'_{n_0} - (z_{n_0})\| = \|y_{n_0} - z_{n_0}\|$. So, $\|x - y\| \leqq \|x - z_{n_0}\| + \|y_{n_0} - z_{n_0}\| + \|y_{n_0} + y\| \leqq (3/4)\varepsilon < \varepsilon$, a contradiction.

We finish the note with the following

Question. Can Theorem 1 be generalized for the case of weakly compactly generated X/Y?

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Received May 16, 1978.

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