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TAKESAKI'S DUALITY FOR REGULAR EXTENSIONS OF VON NEUMANN ALGEBRAS

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We extend Takesaki's duality to regular extensions, and hence twisted crossed products, of von Neumann algebras by locally compact groups.

Introduction. For a von Neumann algebra M, ε denotes the canonical map of the automorphism group $\mathrm{Aut}(M)$ of M to the quotient $\mathrm{Aut}(M)/\mathrm{Int}(M) = \mathrm{Out}(M)$ of $\mathrm{Aut}(M)$ by the normal subgroup of inner automorphisms. When M_* is separable, and G is a separable locally compact group (always endowed with a right Haar measure and modular function Δ), we can associate to certain Borel mappings $\alpha_{(\cdot)}: t \mapsto \alpha_t \in \mathrm{Aut}(M)$ with $t \mapsto \varepsilon(\alpha_t)$ a homomorphism, a family of extensions of M by G, known as regular extensions, or, in special cases, twisted crossed products, [7, 10, 12, 13, 15]. Indeed, since $\varepsilon(\alpha_s)\varepsilon(\alpha_t) = \varepsilon(\alpha_{st})$ there is a Borel family $(s, t) \in G \times G \mapsto u(s, t) \in M$ of unitaries such that

$$(1) \qquad \begin{cases} \alpha_s \circ \alpha_t = \mathrm{Ad}\, u(s, t) \circ \alpha_{st} \\ (\text{or } (\alpha \otimes \iota) \circ \alpha = \mathrm{Ad}\, u \circ (\iota \otimes \delta) \circ \alpha) \end{cases}$$

where δ is the isomorphism of $L^\infty(G)$ into $L^\infty(G) \otimes L^\infty(G)$ determined by $(\delta f)(s, t) \equiv f(st)$, $f \in L^\infty(G)$; $\alpha: M \to M \otimes L^\infty(G)$ is given by $(\alpha(x))(t) \equiv \alpha_t(x)$, $x \in M$ and $(u\xi)(s, t) \equiv u(s, t)\xi(s, t)$ for $\xi \in \mathscr{H} \otimes L^2(G) \otimes L^2(G)$ (where M acts on $\mathscr{H}$).

Since $t \mapsto \varepsilon(\alpha_t)$ is a homomorphism, we see

$$\alpha_r(u(s, t))u(r, st) = f_u(r, s, t)u(r, s)u(rs, t)$$

for some Borel map $f_u: G \times G \times G \to M$ with unitary values in the center of M. Also, f_u is a 3-cocycle for the natural action of G on the center of M. If f_u cobounds, we may assume, by modifying by unitaries in the center of M, that

$$(2) \qquad \alpha_r(u(s, t))u(r, st) = u(r, s)u(rs, t)$$

on $G \times G \times G$. Hence we may construct the regular extension $M \otimes_{\alpha, u} G$ of M by G, as the von Neumann algebra on $\mathscr{H} \otimes L^2(G)$ generated by the operators

$$(\alpha(x)\xi)(t) \equiv \alpha_t(x)\xi(t) \ , \quad (\lambda^u(r)\xi)(t) \equiv u(t, r)\xi(tr)$$

for $x \in M$, $r \in G$ and $\xi \in \mathscr{H} \otimes L^2(G)$. (See [13, Theorem 3.1.6] for further details on regular extensions and the significance of f_u cobounding.)

In order to formulate Takesaki's duality for a general locally compact group, we introduce the concept of a dual action of G on a von Neumann algebra N; this is an isomorphism β of N into $N \otimes R(G)$ satisfying

$$(\beta \otimes \iota)\circ\beta = (\iota \otimes \gamma)\circ\beta$$

where $R(G)$ is the von Neumann algebra generated by the right regular representation λ of G and γ is the isomorphism of $R(G)$ into $R(G) \otimes R(G)$ determined by $\gamma(\lambda(t)) = \lambda(t) \otimes \lambda(t)$, $t \in G$. The crossed dual product N by G, $N \otimes^d_\beta G$, is the von Neumann algebra generated by $\beta(N)$ and $1 \otimes L^\infty(G)$, [3, 6, 8, 9, 11, 14]. Our main result, Theorem 2 extends Takesaki's duality to regular extensions, thus answering a question raised in [13, §1].

Duality for regular extensions. Before beginning our discussion, we define unitaries U, V, V' and W on $L^2(G) \otimes L^2(G)$ by

$$(U\xi)(s, t) \equiv \xi(t, s)\ , \quad (V\xi)(s, t) \equiv \xi(st, t)\ , \quad (V'\xi)(s, t) \equiv \Delta(t)^{1/2}\xi(t^{-1}s, t)\ ,$$

and $W \equiv UVU$, so $(W\xi)(s, t) = \xi(s, ts)$. Note that $\mathrm{Ad}\,U$ is the symmetry $\sigma: x \otimes y \mapsto y \otimes x$, $\delta f = \mathrm{Ad}\,V(f \otimes 1_G)$, $f \in L^\infty(G)$, and

$$\gamma(\lambda(t)) = \mathrm{Ad}\,W^*(\lambda(t) \otimes 1_G)\ .$$

LEMMA 1. *If $\hat{\alpha}$ is defined on $M \otimes_{\alpha,u} G$ by*

$$\hat{\alpha}(y) \equiv \mathrm{Ad}\,1 \otimes W^*(y \otimes 1_G)\ ,$$

then it is a dual action of G on $M \otimes_{\alpha,u} G$.

Proof. Direct computations easily show

$$\text{(4)} \qquad \begin{cases} \mathrm{Ad}\,1 \otimes W^*(\alpha(x) \otimes 1_G) = \alpha(x) \otimes 1_G \\ \mathrm{Ad}\,1 \otimes W^*(\lambda^u(r) \otimes 1_G) = \lambda^u(r) \otimes \lambda(r)\ . \end{cases}$$

The identity $(\hat{\alpha} \otimes \iota)\circ\hat{\alpha} = (\iota \otimes \gamma)\circ\hat{\alpha}$ now follows trivially on the generators of $M \otimes_{\alpha,u} G$, and hence on all of $M \otimes_{\alpha,u} G$.

Following [6, 8], we say that actions[1] α^j of a group G on von Neumann algebras M_j, $j = 1, 2$ are equivalent if

$$(\rho \otimes \iota)\circ\alpha^1 = \alpha^2\circ\rho$$

[1] An action α of G on M means a homomorphism of G into $\mathrm{Aut}(M)$ such that $t \mapsto \alpha_t(x)$ is σ-weakly continuous for each $x \in M$.

for some isomorphism ρ of M_1 onto M_2; we denote this relation by $\{M_1, \alpha^1\} \sim \{M_2, \alpha^2\}$.

THEOREM 2. *Let* $\tilde{\alpha} \equiv \mathrm{Ad}\, 1 \otimes V' \circ (\iota \otimes \sigma) \circ \mathrm{Ad}\, u^* \circ (\alpha \otimes \iota)$, and

$$\hat{\hat{\alpha}}(x) \equiv \mathrm{Ad}\, 1 \otimes 1_G \otimes V'(x \otimes 1_G) \qquad \left(x \in \left(M \underset{\alpha\, u}{\bigotimes} G\right) \overset{d}{\underset{\hat{\alpha}}{\bigotimes}} G\right),$$

so that $\hat{\hat{\alpha}}$ *is the action*[2] *of* G *on* $(M \bigotimes_{\alpha, u} G) \bigotimes_{\hat{\alpha}}^{d} G$ *dual to* $\hat{\alpha}$. *Then* $\tilde{\alpha}$ *is an action of* G *on* $M \otimes B(L^2(G))$ *and we have*

$$\left\{\left(M \underset{\alpha, u}{\bigotimes} G\right) \overset{d}{\underset{\hat{\alpha}}{\bigotimes}} G, \hat{\hat{\alpha}}\right\} \sim \{M \otimes B(L^2(G)), \tilde{\alpha}\}\ .$$

Proof. We note first that the operators $\alpha(x)$, $x \in M$, $\lambda^u(r)$, $r \in G$ and $1 \otimes f$, $f \in L^\infty(G)$ generate $M \otimes B(L^2(G))$. Indeed, if N is the von Neumann algebra generated by the above operators, then $N' \subset B(\mathscr{H}) \otimes L^\infty(G)$. If $x \in N'$, then for all $y \in M$ we see that

$$\alpha_t(y)x(t)\xi(t) = (\alpha(y)x\xi)(t) = (x\alpha(y)\xi)(t) = x(t)\alpha_t(y)\xi(t)$$

a.e. on G, so that $x(t) \in M'$ a.e. Since also $\lambda^u(r)x = x\lambda^u(r)$ for all $r \in G$, we obtain $x(t)u(t, r) = u(t, r)x(tr)$ a.e. in t for each $r \in G$. A routine argument now shows $x \in M' \otimes 1_G$, and $N = M \otimes B(L^2(G))$. Note that in fact we have shown that $\alpha(x)$, $x \in M$ and $1 \otimes L^\infty(G)$ generate $M \otimes L^\infty(G)$.

Now define a map ρ: $M \otimes B(L^2(G)) \to M \otimes B(L^2(G)) \otimes B(L^2(G))$ by $\rho \equiv \mathrm{Ad}\, 1 \otimes V^* \circ \mathrm{Ad}\, u^* \circ (\alpha \otimes \iota)$. We have then

$$(5) \qquad \begin{cases} \rho(\alpha(x)) = \alpha(x) \otimes 1_G \\ \rho(\lambda^u(r)) = \lambda^u(r) \otimes \lambda(r) \\ \rho(1 \otimes f) = 1 \otimes 1_G \otimes f\ . \end{cases}$$

Of these, the last is trivial, the first follows from (1), and the second is checked as follows. Since, from (2),

$$\alpha_{st^{-1}}(u(t, r))u(st^{-1}, tr) = u(st^{-1}, t)u(s, r)\ ,$$

we have, for $\xi \in \mathscr{H} \otimes L^2(G) \otimes L^2(G)$,

$$\begin{aligned} &((1 \otimes V^*)u^*(\alpha \otimes \iota(\lambda^u(r)))u(1 \otimes V)\xi)(s, t) \\ &\quad = u(st^{-1}, t)^*(\alpha \otimes \iota(\lambda^u(r))u(1 \otimes V)\xi)(st^{-1}, t) \\ &\quad = u(st^{-1}, t)^*\alpha_{st^{-1}}(u(t, r))u(st^{-1}, tr)((1 \otimes V)\xi)(st^{-1}, tr) \\ &\quad = u(st^{-1}, t)^*\alpha_{st^{-1}}(u(t, r))u(st^{-1}, tr)\xi(sr, tr) \end{aligned}$$

[2] α is an action of G on M if and only if α is a normal isomorphism of M into $M \otimes L^\infty(G)$ with $(\alpha \otimes \iota) \circ \alpha = (\iota \otimes \delta) \circ \alpha$, [8, Theorem 2.1].

$$= u(s, r)\xi(sr, tr)$$
$$= ((\lambda^u(r) \otimes \lambda(r))\xi)(s, t) .$$

Since, from (4), the right hand sides of (5) generate

$$\left(M \underset{\alpha, u}{\bigotimes} G\right) \underset{\hat{\alpha}}{\overset{d}{\bigotimes}} G ,$$

ρ is an isomorphism of $M \otimes B(L^2(G))$ onto $(M \bigotimes_{\alpha,u} G) \bigotimes_{\hat{\alpha}}^d G$.

It remains to check the identity $(\rho \otimes \iota)\circ\tilde{\alpha} = \hat{\hat{\alpha}}\circ\rho$. Notice that $\tilde{\alpha} = \mathrm{Ad}\,(1 \otimes V'UV)\circ\rho$, and that

$$(V'UV\xi)(s, t) = \Delta(t)^{1/2}\xi(s, t^{-1}s) , \quad ((V'UV)^*\xi)(s, t) = \Delta(ts^{-1})^{1/2}\xi(s, st^{-1}) .$$

Thus we obtain

$$\begin{aligned}(\rho \otimes \iota)\circ\tilde{\alpha}(\alpha(x)) &= (\rho \otimes \iota)\circ\mathrm{Ad}(1 \otimes V'UV)(\alpha(x) \otimes 1_G) \\ &= (\rho \otimes \iota)(\alpha(x) \otimes 1_G) \\ &= \alpha(x) \otimes 1_G \otimes 1_G ,\end{aligned}$$

and

$$\begin{aligned}(\rho \otimes \iota)\circ\tilde{\alpha}(\lambda^u(r)) &= (\rho \otimes \iota)\circ\mathrm{Ad}(1 \otimes V'UV)(\lambda^u(r) \otimes \lambda(r)) \\ &= (\rho \otimes \iota)(\lambda^u(r) \otimes 1_G) \\ &= \lambda^u(r) \otimes \lambda(r) \otimes 1_G .\end{aligned}$$

Also

$$\begin{aligned}\tilde{\alpha}(1 \otimes f) &= \mathrm{Ad}\,(1 \otimes V')\circ(\iota \otimes \sigma)\circ\mathrm{Ad}\,u^*(1 \otimes 1_G \otimes f) \\ &= \mathrm{Ad}\,(1 \otimes V')(1 \otimes f \otimes 1_G) = 1 \otimes \kappa f ,\end{aligned}$$

where $(\kappa f)(s, t) = f(t^{-1}s)$, by direct computation.

Finally, noticing that $\mathrm{Ad}\,V'(\lambda(r) \otimes 1_G) = \lambda(r) \otimes 1_G$, and that $\mathrm{Ad}\,V'(f \otimes 1_G) = \kappa f$, we obtain also

$$\begin{aligned}\hat{\hat{\alpha}}\circ\rho(\alpha(x)) &= \alpha(x) \otimes 1_G \otimes 1_G , \\ \hat{\hat{\alpha}}\circ\rho(\lambda^u(r)) &= \hat{\hat{\alpha}}(\lambda^u(r) \otimes \lambda(r)) \\ &= \mathrm{Ad}(1 \otimes 1_G \otimes V')(\lambda^u(r) \otimes \lambda(r) \otimes 1_G) \\ &= \lambda^u(r) \otimes \lambda(r) \otimes 1_G ,\end{aligned}$$

and

$$\begin{aligned}\hat{\hat{\alpha}}\circ\rho(1 \otimes f) &= \hat{\hat{\alpha}}(1 \otimes 1_G \otimes f) \\ &= \mathrm{Ad}(1 \otimes 1_G \otimes V')(1 \otimes 1_G \otimes f \otimes 1_G) \\ &= 1 \otimes 1_G \otimes \kappa f ;\end{aligned}$$

the equality $(\rho \otimes \iota)\circ\tilde{\alpha} = \hat{\hat{\alpha}}\circ\rho$ is verified.

COROLLARY 3. *If* ${}^u\lambda(r)$ *is defined on* $\mathscr{H}\otimes L^2(G)$ *by*

$$({}^u\lambda(r)\xi)(s) \equiv \Delta(r)^{1/2}u(r, r^{-1}s)^*\xi(r^{-1}s)\ ,$$

then $\tilde{\alpha}_t = \mathrm{Ad}\,{}^u\lambda(t)\circ(\alpha_t\otimes\iota)$.

Proof. It suffices to show the indicated equality on the generators $\alpha(x)$, $\lambda^u(r)$ and $1\otimes f$ of $M\otimes B(L^2(G))$. We compute

$$\begin{aligned}
&({}^u\lambda(t)\alpha_t\otimes\iota(\alpha(x)){}^u\lambda(t)^*\xi)(s)\\
&\quad= \Delta(t)^{1/2}u(t, t^{-1}s)^*(\alpha_t\otimes\iota(\alpha(x)){}^u\lambda(t)^*\xi)(t^{-1}s)\\
&\quad= u(t, t^{-1}s)^*\alpha_t(\alpha_{t^{-1}s}(x))u(t, t^{-1}s)\xi(s)\\
&\quad= \alpha_s(x)\xi(s)
\end{aligned}$$

for $\xi\in\mathscr{H}\otimes L^2(G)$ and

$$(\tilde{\alpha}(\alpha(x))\xi)(s, t) = ((\alpha(x)\otimes 1_G)\xi)(s, t) = \alpha_s(x)\otimes 1_G\xi(s, t)\ ,$$

for $\xi\in\mathscr{H}\otimes L^2(G)\otimes L^2(G)$. Similarly, we have

$$\begin{aligned}
&(\mathrm{Ad}\,{}^u\lambda(t)\circ(\alpha_t\otimes\iota)(\lambda^u(r))\xi)(s)\\
&\quad= \Delta(t)^{1/2}u(t, t^{-1}s)^*\alpha_t(u(t^{-1}s, r))({}^u\lambda(t)^*\xi)(t^{-1}sr)\\
&\quad= u(t, t^{-1}s)^*\alpha_t(u(t^{-1}s, r))u(t, t^{-1}sr)\xi(sr)\\
&\quad= u(s, r)\xi(sr) \qquad\qquad \text{(by (2))}\\
&\quad= (\lambda^u(r)\xi)(s)\ ,
\end{aligned}$$

and

$$\begin{aligned}
(\mathrm{Ad}^u\,\lambda(t)\circ(\alpha_t\otimes\iota)(1\otimes f)\xi)(s) &= (\mathrm{Ad}^u\,\lambda(t)(1\otimes f)\xi)(s)\\
&= u(t, t^{-1}s)^*f(t^{-1}s)u(t, t^{-1}s)\xi(s)\\
&= f(t^{-1}s)\xi(s)
\end{aligned}$$

for $\xi\in\mathscr{H}\otimes L^2(G)$. Since

$$(\tilde{\alpha}(\lambda^u(r))\xi)(s, t) = ((\lambda^u(r)\otimes 1_G)\xi)(s, t)$$

and

$$\begin{aligned}
(\tilde{\alpha}(1\otimes f)\xi)(s, t) &= ((1\otimes\kappa f)\xi)(s, t)\\
&= f(t^{-1}s)\xi(s, t)
\end{aligned}$$

for $\xi\in\mathscr{H}\otimes L^2(G)\otimes L^2(G)$, the verification is complete.

This result is a partial clarification of [13, Proposition 2.1.3] asserting that the 2-cocycle $u\otimes 1_G$ cobounds with respect to $\alpha_t\otimes\iota$ in $M\otimes B(L^2(G))$. Indeed, it is trivially checked that

$$u(s, t)\otimes 1_G = (\alpha_s\otimes\iota)({}^u\lambda(t)^*){}^u\lambda(s)^*{}^u\lambda(st)$$

as required.

For a given action θ of G on a von Neumann algebra N, we write $N^\theta \equiv \{x \in N: \theta_t(x) = x, \forall t \in G\}$, the fixed point subalgebra of N.

COROLLARY 4. $M \otimes_{\alpha,u} G = (M \otimes B(L^2(G)))^{\tilde{\alpha}}$.

Proof. Since $((M \otimes_{\alpha,u} G) \otimes^d_{\hat{\alpha}} G)^{\hat{\hat{\alpha}}} = \hat{\alpha}(M \otimes_{\alpha,u} G)$ by [8, Proposition 6.4], Takesaki's duality (Theorem 2) tells us that

$$\hat{\alpha}\Big(M \underset{\alpha,u}{\otimes} G\Big) = \rho((M \otimes B(L^2(G)))^{\tilde{\alpha}}) .$$

From (4) and (5), we see that $\hat{\alpha}$ and ρ agree on $M \otimes_{\alpha,u} G$, so that $M \otimes_{\alpha,u} G = (M \otimes B(L^2(G)))^{\tilde{\alpha}}$ as claimed.

Corollary 4 gives some information on when regular extensions $M \otimes_{\alpha^1,u} G$ and $M \otimes_{\alpha^2,v} G$ of M by G, with $\varepsilon \circ \alpha^1 = \varepsilon \circ \alpha^2$, are isomorphic. For if $\tilde{\alpha}^1$ and $\tilde{\alpha}^2$ denote the actions of G on $\bar{M} \equiv M \otimes B(L^2(G))$ with fixed point algebras $M \otimes_{\alpha^1,u} G$ and $M \otimes_{\alpha^2,v} G$ respectively, then $\bar{M} \otimes_{\tilde{\alpha}^1} G$ and $\bar{M} \otimes_{\tilde{\alpha}^2} G$ will be isomorphic whenever there is a Borel map $t \in G \mapsto u_t$ with $\tilde{\alpha}^1_t = \mathrm{Ad}\, u_t \circ \tilde{\alpha}^2_t$ and $u_t \tilde{\alpha}^2_t(u_s) = u_{ts}$ for $t, s \in G$, [14]. On the other hand these crossed products are isomorphic respectively to $(M \otimes_{\alpha^1,u} G) \otimes B(L^2(G))$ and $(M \otimes_{\alpha^2,v} G) \otimes B(L^2(G))$, [8].

Also, note that $\varepsilon \circ \tilde{\alpha}^1 = \varepsilon \cdot \tilde{\alpha}^2$ whenever $\varepsilon \circ \alpha^1 = \varepsilon \circ \alpha^2$, so it is necessary only to provide conditions under which the "comparison cocycle" $\omega_{\tilde{\alpha}^1, \tilde{\alpha}^2}$ associated to $\tilde{\alpha}^1$ and $\tilde{\alpha}^2$ is trivial, [13]. The hypothesis of the next result are two situations in which this is known to happen, [1, 4].

COROLLARY 5. *Let $M \otimes_{\alpha^1,u} G$ and $M \otimes_{\alpha^2,v} G$ be regular extensions of M by G with $\varepsilon \circ \alpha^1 = \varepsilon \circ \alpha^2$. If either*

(1) *G is discrete, acts freely on the center of M, and is a locally finite extension of a solvable group*; *or*

(2) *G is a compact, abelian and connected group K, or $K \times \boldsymbol{R}$, and acts trivially on the center of M,*

then $(M \otimes_{\alpha^1,u} G) \otimes B(L^2(G))$ and $(M \otimes_{\alpha^2,v} G) \otimes B(L^2(G))$ are isomorphic.

Just as in the case of ordinary crossed products, regular extensions may be characterized by the existence of a dual action and of a distinguished family of unitaries.

THEOREM 6. *Let N be a von Neumann algebra with N_* separable and β a dual action of G on N. Then the following two conditions are equivalent*:

(i) *there is* $\{M, \alpha\}$ *with* M_* *separable such that* $\{N, \beta\} \sim \{M \otimes_{\alpha,u} G, \hat{\alpha}\}$ *for some* u; *and*

(ii) *there is a Borel map* $t = G \mapsto v(t) \in N$ *with unitary values such that* $\beta(v(t)) = v(t) \otimes \lambda(t)$, $t \in G$.

The proof goes the same way as in the proof [5, 8, 11] except the following lemma.

LEMMA 7. *Assume the condition* (ii) *in Theorem* 6. *Then,* N *is generated by* $N^\beta \equiv \{y \in N: \beta(y) = y \otimes 1_G\}$ *and* $v(t)$, $t \in G$.

Proof (Takesaki). Let $\bar{N} \equiv N \otimes F_\infty$, $\bar{\beta} \equiv (\iota \otimes \sigma)\circ(\beta \otimes \iota)$ and $\bar{v}(t) \equiv v(t) \otimes 1$, where F_∞ is a factor of type I_∞. Then $\bar{\beta}$ is a dual action of G on $\bar{N}$, $\bar{N}^{\bar{\beta}} = N^\beta \otimes F_\infty$ is properly infinite and $\bar{\beta}(\bar{v}(t)) = \bar{v}(t) \otimes \lambda(t)$ for all t. Therefore $\bar{\beta}$ is dominant[3], because $\bar{\beta}(v) = (v \otimes 1_G)(1 \otimes W)$ for a unitary v in $N \otimes L^\infty(G)$ defined by $(v\xi)(t) \equiv v(t)\xi(t)$, [2, 9]. Therefore there exists a strongly continuous unitary representation u of G in $\bar{N}$ such that $\bar{\beta}(u(t)) = u(t) \otimes \lambda(t)$ by [5, 8, 11]. In this case $\bar{N}$ is generated by $N^\beta \otimes F_\infty$ and $u(t)$, $t \in G$. If e is a projection in $\bar{N}$ of the form $1 \otimes p$ with $\dim p = 1$, then $\{N, \beta\}$ is identified with $\{\bar{N}_e, \bar{\beta}^e\}$. Since $\bar{\beta}(v(t)) = v(t) \otimes \lambda(t)$, $t \in G$, $v(t)u(t)^* \in N^\beta \otimes F_\infty$ and hence $v(t) = ew(t)u(t)e$ for some $w(t) \in N^\beta \otimes F_\infty$. Here we may assume that $w(t) = ew(t)\alpha_t(e)$. So, $w(t)$ is a partial isometry. If x is an arbitrary element in $N^\beta \otimes F_\infty$, then

$$exu(t)e = exw(t)^*w(t)u(t)e = exw(t)^*v(t)$$

and hence $e(N^\beta \otimes F_\infty)u(t)e = N^\beta v(t)$. It remains to show that $e(N^\beta \otimes F_\infty)u(t)e$, $t \in G$ generate $e\bar{N}e = N$. Since the set L of all finite linear conbinations of $xu(t)$ with $x \in N^\beta \otimes F_\infty$ and $t \in G$ is a σ-weakly dense $*$-subalgebra of $\bar{N}$, eLe is σ-weakly dense in $e\bar{N}e = N$. Consequently, $N^\beta v(t)$, $t \in G$ generate N.

Proof of Theorem 6. That (i) $\Rightarrow$ (ii) has already verified in Lemma 1.

(ii) $\Rightarrow$ (i). Let $M \equiv N^\beta$. Since $\beta(v(t)xv(t)^*) = v(t)xv(t)^* \otimes 1_G$ for $x \in M$, $v(t)$ normalizes M. Also with $u(s, t) \equiv v(s)v(t)v(st)^*$, we see $\beta(u(s, t)) = u(s, t) \otimes 1_G$, so $u(s, t) \in M$ for $s, t \in G$.

Set $\alpha_s \equiv \mathrm{Ad}\, v(s) \restriction M$. Then $\alpha_s \circ \alpha_t = \mathrm{Ad}\, u(s, t) \circ \alpha_{st}$ and $\alpha_r(u(s, t))u(r, st) = u(t, s)u(rs, t)$. Then α and u determine a regular extension $M \otimes_{\alpha,u} G$ of M by G, with generators $\alpha(M)$ and $\lambda^u(s)$, $s \in G$. Define a unitary v in $N \otimes L^\infty(G)$ by $(v\xi)(t) = v(t)\xi(t)$. Then, by di-

[3] A dual action β of G on N is said to be *dominant*, if N^β is properly infinite and $\{\bar{N}, \bar{\beta}\} \sim \{\bar{N}, \tilde{\bar{\beta}}\}$, where $\bar{N} = N \otimes B(L^2(G))$, $\bar{\beta} = (\iota \otimes \sigma)\circ(\beta \otimes \iota)$ and $\tilde{\bar{\beta}} = (\mathrm{Ad}\, 1 \otimes W)\circ\bar{\beta}$. If β is dominant, then $\{N, \beta\} \sim \{\bar{N}, \beta\} \sim \{(N \otimes^d_\beta G) \otimes_{\hat{\beta}} G, \tilde{\tilde{\beta}}\}$.

rect computation,

$$v^*\lambda^u(s)v = \beta(v(s)) \quad \text{and} \quad v^*\alpha(x)v = \beta(x)$$

for $s \in G$ and $x \in M$. Thus $v^*(M \otimes_{\alpha,u} G)v = \beta(N)$ by Lemma 7.

According to the above theorem we know the relation between [2, Theorem III. 3.1] and [5, Theorem].

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