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# CHAIN CONDITIONS IN FREE PRODUCTS OF LATTICES WITH INFINITARY OPERATIONS

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**There are many facts known about the size of subsets of certain kinds in free lattices and free products of lattices. Examples: every chain in a free lattice is at most countable; every "large" subset contains an independent set; if the free product of a set of lattices contains a "long" chain, so does the free product of a finite subset of this set of lattices. Here we investigate these problems in the setting of a variety $V$ of $\mathfrak{m}$-lattices, where $\mathfrak{m}$ is an infinite regular cardinal. An $\mathfrak{m}$-lattice $L$ is a lattice in which for any nonempty set $S$ with $|S|<\mathfrak{m}$, the meet and join exist in $L$. We obtain generalizations of many finitary results to the $\mathfrak{m}$-complete case. Our basic set-theoretic tool is the Erdös-Rado theorem.**

1. Preliminaries. Lower-case German letters denote cardinals. Lower-case Greek letters denote ordinals; cardinals are identified with initial ordinals.

A family $(S_i \mid i \in I)$ of sets is a *$\Delta$-system* with *kernel* $D$ iff $S_i \cap S_j = D$ whenever $i \neq j$ and $i, j \in I$. The cardinal $\mathfrak{n}$ is *strongly $\mathfrak{m}$-inaccessible* iff $\mathfrak{b}^{\mathfrak{a}} < \mathfrak{n}$ whenever $\mathfrak{a} < \mathfrak{m}$ and $\mathfrak{b} < \mathfrak{n}$. For example, $(2^{\underline{\mathfrak{m}}})^+$ is strongly $\mathfrak{m}$-inaccessible [2, Lemma 1.26], where $2^{\underline{\mathfrak{m}}} = \Sigma(2^{\mathfrak{a}} \mid \mathfrak{a} < \mathfrak{m})$. Note that $2^{\underline{\mathfrak{m}}} \geqq \mathfrak{m}$, and equality holds if the Generalized Continuum Hypothesis (G. C. H) is assumed. Under G. C. H., if $\mathfrak{n} > \mathfrak{m}$ is the successor of a regular cardinal, then it is strongly $\mathfrak{m}$-inaccessible.

Let $\mathfrak{n} > \mathfrak{m}$ be regular and strongly $\mathfrak{m}$-inaccessible. The *Erdös-Rado theorem* [3, Lemma 1] states that for any family $(S_\alpha \mid \alpha < \mathfrak{n})$ of sets with $|S_\alpha| < \mathfrak{m}$ whenever $\alpha < \mathfrak{n}$, there is $N \subseteqq \mathfrak{n}$ with $|N| = \mathfrak{n}$ such that $(S_\alpha \mid \alpha \in N)$ is a $\Delta$-system.

In this paper, $\mathfrak{m}$ is an infinite regular cardinal. The prefix "$\mathfrak{m}$-" is consistently used to extend concepts from the usual case of finitary joins and meets; for further details, see [6] and [7].

A *variety* $V$ of $\mathfrak{m}$-lattices or *$\mathfrak{m}$-variety* is a class of $\mathfrak{m}$-lattices that is closed under $\mathfrak{m}$-homomorphic images, $\mathfrak{m}$-sublattices and products. $V$ shall always denote a nontrivial $\mathfrak{m}$-variety.

The $V$-free $\mathfrak{m}$-product $L$ of a family $(L_i \mid i \in I)$ of $\mathfrak{m}$-lattices in $V$ is the $\mathfrak{m}$-lattice $L \in V$ (unique up to isomorphism) that contains each $L_i$ $(i \in I)$ as an $\mathfrak{m}$-sublattice and is $\mathfrak{m}$-generated by $X = \dot{\bigcup}(L_i \mid i \in I)$ (disjoint union) such that any family $\varphi_i \colon L_i \to K$ of $\mathfrak{m}$-homomorphisms into any $K \in V$ can be extended to an $\mathfrak{m}$-homomorphism of $L$ into $K$. In particular, if each $L_i$ ($i \in I$) is a one-element lattice, then $L$ is the *V-free* $\mathfrak{m}$*-lattice generated by* $X$. We omit mention of $V$ if it is the variety $\boldsymbol{L}_{\mathfrak{m}}$ of all $\mathfrak{m}$-lattices. We also omit $\mathfrak{m}$ if $\mathfrak{m} = \aleph_0$.

Let $X = \{x_\alpha \mid \alpha < \mathfrak{m}\}$ be a set of variables. The $\mathfrak{m}$*-polynomials in* $X$, defined in [6], are built up using formal joins and meets of less than $\mathfrak{m}$ elements, starting from $X$. The set $P_{\mathfrak{m}}(X)$ of all $\mathfrak{m}$-polynomials in $X$ has cardinality $2^{\underline{\mathfrak{m}}}$. Let $L$ be an $\mathfrak{m}$-lattice that is $\mathfrak{m}$-generated by a set $X$. We can express any element $a \in L$ as $a = p(\bar{a})$ where $p \in P_{\mathfrak{m}}(X)$, $Y \subset X$ is the set of variables appearing in $p$, and $\bar{a}$ is a mapping from $Y$ to $X$. By induction on the *rank* of $p$ (see [6]), it is easily shown that any $a \in L$ has such a representation with $\bar{a}$ one-to-one (that is, distinct variables are substituted by distinct elements of $X$); such a representation is called *proper*. A subset $Y$ of an $\mathfrak{m}$-lattice is $\mathfrak{m}$*-irredundant* iff the following condition and its dual hold: whenever $a \leqq \bigvee B$ with $a \in Y, B \subseteqq Y$ and $0 < |B| < \mathfrak{m}$, it follows that $a \in B$. In particular, an $\mathfrak{m}$-irredundant subset is an antichain.

2. The results. In a $V$-free lattice, every chain is countable. This result is proved in F. Galvin and B. Jónsson [4] in a much sharper form. Our first result generalizes their sharper form.

THEOREM 1. *Let $V$ be a nontrivial* $\mathfrak{m}$*-variety, and let* $\mathfrak{n}$ *be a regular cardinal that is greater than* $\mathfrak{m}$ *and strongly* $\mathfrak{m}$*-inaccessible. If a set of cardinality* $\mathfrak{n}$ *is a subset of a V-free* $\mathfrak{m}$*-lattice, then it contains an* $\mathfrak{m}$*-irredundant subset of the same cardinality.*

COROLLARY 1. *Every V-free* $\mathfrak{m}$*-lattice satisfies the* $(2^{\underline{\mathfrak{m}}})^+$*-chain condition, that is, it has no chain of cardinality* $(2^{\mathfrak{m}})^+$.

A subset $S$ of a lattice is *quasidisjoint* iff $a \wedge b = c \wedge d$ whenever $a, b, c, d \in S$ with $a \neq b$ and $c \neq d$. A lattice satisfies the $\mathfrak{n}$*-quasidisjointness condition* iff it contains no quasidisjoint set of cardinality $\mathfrak{n}$. Since no $\mathfrak{m}$-irredundant set with more than two elements can be quasidisjoint, we have

COROLLARY 2. *Every V-free* $\mathfrak{m}$*-lattice satisfies the* $(2^{\underline{\mathfrak{m}}})^+$*-quasidisjointness condition.*

A subset $Y$ of a free $\mathfrak{m}$-lattice $L$ is $\mathfrak{m}$*-independent* iff the $\mathfrak{m}$-sublattice of $L$ $\mathfrak{m}$-generated by $Y$ is (isomorphic to) the free $\mathfrak{m}$-lattice generated by $Y$. Sinde $\mathfrak{m}$-irredundancy is equivalent to $\mathfrak{m}$-independency for subsets of a free $\mathfrak{m}$-lattice [6], we obtain a result due to F. Galvin and B. Jónsson [4] in the $\mathfrak{m} = \aleph_0$ case.

COROLLARY 3. *Let* $\mathfrak{n}$ *be a regular cardinal that is greater than* $\mathfrak{m}$ *and strongly* $\mathfrak{m}$*-inaccessible. If a set of cardinality* $\mathfrak{n}$ *is a subset of a free* $\mathfrak{m}$*-lattice, then it contains an* $\mathfrak{m}$*-independent subset of the same cardinality.*

B. Jónsson [9] proved that the $\boldsymbol{V}$-free product of lattices $(L_i | i \in I)$ satisfies the $\mathfrak{m}$-chain condition ($\mathfrak{m}$ is regular and $> \aleph_0$) iff for all finite $I' \subseteq I$, the $\boldsymbol{V}$-free product of $(L_i | i \in I')$ satisfies the $\mathfrak{m}$-chain condition. Our next result generalizes this.

THEOREM 2. *Let* $\boldsymbol{V}$ *be an* $\mathfrak{m}$*-variety. Let* $\mathfrak{n}$ *be a regular cardinal that is greater than* $\mathfrak{m}$ *and strongly* $\mathfrak{m}$*-inaccessible. Let* $L$ *be the* $\boldsymbol{V}$*-free* $\mathfrak{m}$*-product of the* $\mathfrak{m}$*-lattices* $L_i \in \boldsymbol{V}, i \in I$. *If, for all* $J \subseteq I$ *with* $|J| < \mathfrak{m}$, *the free* $\mathfrak{m}$*-product of* $(L_i | i \in J)$ *satisfies the* $\mathfrak{n}$*-chain condition, then so does* $L$.

If $\mathfrak{n}$ is singular and cofinal with $\aleph_0$, then there are two lattices satisfying the $\mathfrak{n}$-chain condition whose $\boldsymbol{V}$-free product does not satisfy the $\mathfrak{n}$-chain condition. If $\mathfrak{n}$ is cofinal with $\aleph_0$, then there are countably many chains of cardinality $<\mathfrak{n}$, whose $\boldsymbol{V}$-free product does not satisfy the $\mathfrak{n}$-chain condition (B. Jónsson [9] and G. Grätzer and H. Lakser [8]). The next two results are the analogues for $\mathfrak{m}$-lattices.

$\boldsymbol{D}_{\mathfrak{m}}$ will denote the smallest nontrivial variety of $\mathfrak{m}$-lattices (generated by 2, the two-element $\mathfrak{m}$-lattice).

THEOREM 3. *Let* $\mathfrak{n}$ *be a strongly* $\mathfrak{m}$*-inaccessible singular cardinal whose cofinality is greater than* $2^{\underline{\mathfrak{m}}}$. *Then there are two Boolean* $\mathfrak{m}$*-algebras in* $\boldsymbol{D}_{\underline{\mathfrak{m}}}$ *satisfying the* $\mathfrak{n}$*-chain condition such that their* $\boldsymbol{V}$*-free* $\mathfrak{m}$*-product does not satisfy the* $\mathfrak{n}$*-chain condition.*

THEOREM 4. *If* $\mathfrak{n} > \mathfrak{m}$ *is an infinite cardinal of cofinality* $\mathfrak{m}_0$ *with* $\mathfrak{m}_0 \leqq \mathfrak{m}$, *then there are* $\mathfrak{m}_0$ *complete chains of cardinality less than* $\mathfrak{n}$ *whose* $\boldsymbol{V}$*-free* $\mathfrak{m}$*-product does not satisfy the* $\mathfrak{n}$*-chain condition.*

Some open problems are listed in § 6.

3. Proof of Theorem 1. Let $\mathfrak{n}$ be as in the statement of the theorem, let $L$ be the $\boldsymbol{V}$-free $\mathfrak{m}$-lattice generated by a set $X$, and let $Y$ be a subset of $L$ with $|Y| = \mathfrak{n}$. Since $\mathfrak{n}$ is regular, $2^{\underline{\mathfrak{m}}} < \mathfrak{n}$.

Hence, we can assume that each element of $Y$ has a proper representation $a = \boldsymbol{p}(\bar{a})$, where the *same* $\mathfrak{m}$-polynomial $\boldsymbol{p}$ is used for each element of $Y$. For notational simplicity, we further assume that, for some cardinal $\mathfrak{m}_0 < \mathfrak{m}$, $\bar{a} = \langle x^a_\alpha \mid \alpha < \mathfrak{m}_0 \rangle$ whenever $a \in Y$, where $x^a_\alpha \in X$ for all $\alpha < \mathfrak{m}_0$. (Note that $x^a_\alpha \neq x^a_\beta$ for $\alpha \neq \beta$.)

Consider the sets $S_a = \{x^a_\alpha \mid \alpha < \mathfrak{m}_0\}$ for $a \in Y$. By the Erdős-Rado theorem, there is a subset $Y' \subseteqq Y$ with $|Y'| = \mathfrak{n}$ such that $(S_a \mid a \in Y')$ is a $\Delta$-system, whose kernel we denote by $D$. For each $a \in Y'$, the inclusion $D \subseteqq S_a$ induces a map $\psi_a \colon D \to \mathfrak{m}_0$ in the obvious way. Since $|\{\psi_a \mid a \in Y'\}| \leqq \mathfrak{m}_0^{\mathfrak{m}_0} = 2^{\mathfrak{m}_0} < \mathfrak{n}$, we can assume that $\psi_a$ is the same map for all $a \in Y'$. This means that if $x^a_\alpha \in D$ ($a \in Y'$, $\alpha < \mathfrak{m}_0$), then $x^a_\alpha = x^b_\beta$ for all $b \in Y'$.

We first show that $Y'$ is an antichain in $L$. Supposing otherwise, there are $a, b \in Y'$ with $a < b$. We define an $\mathfrak{m}$-homomorphism $\varphi \colon L \to L$ as follows: $\varphi(x^a_\alpha) = x^b_\alpha$ and $\varphi(x^b_\alpha) = x^a_\alpha$ whenever $\alpha < \mathfrak{m}_0$; otherwise, if $x \in X$, $\varphi(x) = x$. Then, $\varphi(a) = b$ and $\varphi(b) = a$. Applying $\varphi$ to the inequality $a < b$, we conclude that $b \leqq a$, a contradiction.

Let $a \leqq \bigvee B$ with $a \in Y'$, $B \subseteqq Y'$ and $0 < |B| < \mathfrak{m}$. Suppose that $a \notin B$. Fix $c \in B$. We define an $\mathfrak{m}$-homomorphism $\varphi \colon L \to L$ as follows: $\psi(x^b_\alpha) = x^c_\alpha$ whenever $b \in B$ and $\alpha < \mathfrak{m}_0$; otherwise, if $x \in X$, $\varphi(x) = x$. Then $\varphi(a) = a$ and $\varphi(b) = c$ whenever $b \in B$.

Applying $\varphi$ to the inequality $a \leqq \bigvee B$, we conclude that $a < c$, contradicting that $Y'$ is an antichain. This completes the proof of the theorem.

## 4. Proof of Theorem 2.

We prepare the proof of Theorem 2 by

LEMMA 1. *Let $L$ be the V-free $\mathfrak{m}$-product of $\mathfrak{m}$-lattices $L_0, L_1, L_2$; let $L_3$ be an $\mathfrak{m}$-lattice and let $e \in L_3$; and let $\boldsymbol{p} = \boldsymbol{p}(\boldsymbol{x}, \boldsymbol{y})$ and $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{x}, \boldsymbol{y})$ be $\mathfrak{m}$-polynomials whose variables are $\boldsymbol{x} = \langle \boldsymbol{x}_\alpha \mid \alpha < \beta \rangle$ and $\boldsymbol{y} = \langle \boldsymbol{y}_\alpha \mid \alpha < \gamma \rangle$. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be $\beta$-sequences of elements of $L_0$; let $\boldsymbol{c}$ and $\boldsymbol{d}$ be $\gamma$-sequences of elements of $L_1$ and $L_2$ respectively, and let $\boldsymbol{e}$ be the $\gamma$-sequence with constant entry $e$. If*

$$\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{c}) \leqq \boldsymbol{q}(\boldsymbol{b}, \boldsymbol{d})$$

*in $L$ and*

$$\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{e}) = \boldsymbol{q}(\boldsymbol{b}, \boldsymbol{e})$$

*in the V-free product $K$ of $L_0$ and $L_3$, then*

$$\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{c}) = \boldsymbol{q}(\boldsymbol{b}, \boldsymbol{d})$$

*in* $L$.

*Proof.* Let $L^b = L \cup \{0, 1\}$, the $\mathfrak{m}$-lattice formed by adding a new zero and one to $L$. It is easily seen that $L^b \in V$. Further, let $\mathbf{0}$ and $\mathbf{1}$ be the $\gamma$-sequences with constant entry 0 and 1, respectively. We are assuming that (i) $\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{c}) \leqq \boldsymbol{q}(\boldsymbol{b}, \boldsymbol{d})$ in $L$ and (ii) $\boldsymbol{p}(\boldsymbol{a}, \boldsymbol{e}) = \boldsymbol{q}(\boldsymbol{b}, \boldsymbol{e})$ in $K$. By considering the $\mathfrak{m}$-homomorphism from $L$ to $L^b$ that maps $L_0$ identically, everything in $L_1$ to 1, and everying in $L_2$ to 0, we conclude from (i) that $\boldsymbol{p}(\boldsymbol{a}, \mathbf{1}) \leqq \boldsymbol{q}(\boldsymbol{b}, \mathbf{0})$ in $L^b$. Using (ii) and the obvious $\mathfrak{m}$-homomorphisms from $K$ to $L^b$, we also conclude that $\boldsymbol{p}(\boldsymbol{a}, \mathbf{0}) = \boldsymbol{q}(\boldsymbol{b}, \mathbf{0})$ and $\boldsymbol{p}(\boldsymbol{a}, \mathbf{1}) = \boldsymbol{q}(\boldsymbol{b}, \mathbf{1})$ in $L^b$. Thus, $\boldsymbol{q}(\boldsymbol{b}, \mathbf{1}) \leqq \boldsymbol{p}(\boldsymbol{a}, \mathbf{0})$ in $L^b$. It is easily shown by induction on the rank that $\boldsymbol{p}(\boldsymbol{a}, \mathbf{0}) \geqq \boldsymbol{p}(\boldsymbol{a}, \boldsymbol{c})$ and $\boldsymbol{q}(\boldsymbol{b}, \boldsymbol{d}) \leqq \boldsymbol{q}(\boldsymbol{b}, \mathbf{1})$ in $L^b$. Therefore, $\boldsymbol{q}(\boldsymbol{b}, \boldsymbol{d}) \geqq \boldsymbol{p}(\boldsymbol{a}, \boldsymbol{c})$ in $L$, the desired conclusion.

Let $\mathfrak{n}$ be as in the statement of Theorem 2, let $L$ be the $V$-free $\mathfrak{m}$-product of the family $(L_i \mid i \in I)$ of $\mathfrak{m}$-lattices, and let $X = \bigcup (L_i \mid i \in I)$, a subset of $L$. Suppose that $C$ is a chain in $L$ of cardinality $\mathfrak{n}$. As in the proof of Theorem 1, we can assume that there is a single $\mathfrak{m}$-polynomial $\boldsymbol{p}$ and a cardinal $\mathfrak{m}_0 < \mathfrak{m}$ such that each element $a$ of $C$ has a representation $a = \boldsymbol{p}(\langle x_\alpha^a \mid \alpha < \mathfrak{m}_0 \rangle)$, where $x_\alpha^a \in X$ for all $\alpha < \mathfrak{m}_0$. For $x \in X$, $i(x)$ denotes the element $j$ of $I$ such that $x \in L_j$. Since there are less than $\mathfrak{n}$ equivalence relations on $\mathfrak{m}_0$, we can further assume that, whenever $\alpha, \beta < \mathfrak{m}_0$, if the equality $i(x_\alpha^a) = i(x_\beta^a)$ holds for some $a \in C$, then it holds for *all* $a \in C$.

Applying the Erdös-Rado theorem to the sets $S_a = \{i(x_\alpha^a) \mid \alpha < \mathfrak{m}_0\}$ for $a \in C$, we obtain a subset $C' \subseteqq C$ with $|C'| = \mathfrak{n}$ such that $(S_a \mid a \in C')$ is a $\Delta$-system with kernel $D$. Again as in Theorem 1, we can assume that if $i(x_\alpha^a) \in D$ ($a \in C'$, $\alpha < \mathfrak{m}_0$), then $i(x_\alpha^a) = i(x_\alpha^b)$ for all $b \in C'$. We will consider only the case that $I - D \neq \varnothing$. Choose $k \in I - D$, set $J = D \cup \{k\}$, and let $K$ be a $V$-free $\mathfrak{m}$-product of $(L_i \mid i \in J)$. Further, choose $e \in L_k$. Let $\varphi: L \to K$ be the $\mathfrak{m}$-homomorphism that maps $L_i$ identically if $i \in D$, and maps everything in $L_i$ to $e$ if $i \in I - D$. If $a < b$ in $C'$, then Lemma 1 guarantees that $\varphi(a) \neq \varphi(b)$. Therefore, $\{\varphi(a) \mid a \in C'\}$ is a chain of cardinality $\mathfrak{n}$ in $K$, completing the proof.

Note that Corollary 1 of Theorem 1 also follows from Theorem 2.

**5. Proofs of Theorems 3 and 4.** In order to develop a proof of Theorem 3, we will generalize the concepts and results in § 5 of G. Grätzer and H. Lakser [8]. Let $(P_i \mid i \in I)$ be a family of posets with 0 and 1. Let $k = 0$ or 1. For each $x$ in the direct product $\Pi(P_i \mid i \in I)$, $sp_k(x) = \{i \in I \mid x_i \neq k\}$. Also, $\Pi_{\mathfrak{m}}^k(P_i \mid i \in I)$ is the set of all $x \in \Pi(P_i \mid i \in I)$ for which $|sp_k(x)| < \mathfrak{m}$. The $\mathfrak{m}$-*weak direct product* of $(P_i \mid i \in I)$ is defined as

$$\Pi_{\mathfrak{m}}(P_i \mid i \in I) = \Pi^0_{\mathfrak{m}}(P_i \mid i \in I) \cup \Pi^1_{\mathfrak{m}}(P_i \mid i \in I) .$$

LEMMA 2. *Let $\mathfrak{n}$ be a strongly $\mathfrak{m}$-inaccessible cardinal whose cofinality is greater than $2^{\underline{\mathfrak{m}}}$. If $(P_i \mid i \in I)$ is a family of posets with 0 and 1 satisfying the $\mathfrak{n}$-chain condition, then $\Pi_{\mathfrak{m}}(P_i \mid i \in I)$ satisfies the $\mathfrak{n}$-chain condition.*

*Proof.* Suppose $C$ is a chain in $\Pi_{\mathfrak{m}}(P_i \mid i \in I)$ of cardinality $\mathfrak{n}$, where each $P_i$ satisfies the $\mathfrak{n}$-chain condition. There is no loss in generality in assuming that $C \subseteqq \Pi^0_{\mathfrak{m}}(P_i \mid i \in I)$. For $x \in C$, the sets $sp_0(x)$ each have cardinality less than $\mathfrak{m}$ and form a chain under inclusion; therefore, by the Erdös-Rado theorem (a proof without appeal to this theorem is not difficult), $|\{sp_0(x) \mid x \in C\}| \leqq 2^{\underline{\mathfrak{m}}}$. Thus, there is a chain $C' \subseteqq C$ of cardinality $\mathfrak{n}$ and a set $J \subseteqq I$ of cardinality $\mathfrak{m}_0 < \mathfrak{m}$ such that $sp_0(x) = J$ whenever $x \in C'$. For $i \in J$, let $C_i = \pi_i(C')$, where $\pi_i \colon \Pi(P_i \mid i \in I) \to P_i$ is the projection map; since each $C_i$ is a chain in $P_i$, $|C_i| < \mathfrak{n}$. Choose $\mathfrak{n}_0 < \mathfrak{n}$ such that $|C| \leqq \mathfrak{n}_0$ whenever $i \in J$. Since $C'$ can be embedded in $\Pi(C_i \mid i \in J)$, we obtain $|C'| \leqq \mathfrak{n}_0^{\mathfrak{m}_0} < \mathfrak{n}$. With this contradiction, the proof is complete.

LEMMA 3. *Let $\mathfrak{n}$ be a strongly $\mathfrak{m}$-inaccessible cardinal whose cofinality is greater than $2^{\underline{\mathfrak{m}}}$. There is a Boolean $\mathfrak{m}$-algebra in $\boldsymbol{D}_{\mathfrak{m}}$ that satisfies the $\mathfrak{n}$-chain condition but contains a chain of cardinality $\mathfrak{n}'$ for every $\mathfrak{n}' < \mathfrak{n}$.*

*Proof.* Any successor ordinal, considered as a (complete) chain, is isomorphic to an $\mathfrak{m}$-sublattice of a power set. For each $\mathfrak{a} < \mathfrak{n}$, let $B_{\mathfrak{a}}$ be a Boolean $\mathfrak{m}$-algebra in $\boldsymbol{D}_{\mathfrak{m}}$ that is $\mathfrak{m}$-generated inside a Boolean $\mathfrak{m}$-algebra $A$ in $\boldsymbol{D}_{\mathfrak{m}}$ by $C \cup \{0, 1\} \cup \{c' \mid c \in C\}$, where $C$ is a successor ordinal of cardinality $\mathfrak{a}$ and $c'$ denotes the complement of $c$ in $A$. An $\mathfrak{m}$-polynomial in which $\mathfrak{m}_0 < \mathfrak{m}$ variables appear can represent at most $\mathfrak{a}^{\mathfrak{m}_0}$ elements of $B_{\mathfrak{a}}$. Since $\mathfrak{a}^{\mathfrak{m}_0} < \mathfrak{n}$ and there are $2^{\underline{\mathfrak{m}}}$ $\mathfrak{m}$-polynomials, it follows that $|B_{\mathfrak{a}}| < \mathfrak{n}$. Then $B = \Pi_{\mathfrak{m}}(B_{\mathfrak{a}} \mid \mathfrak{a} < \mathfrak{n})$ is a Boolean $\mathfrak{m}$-algebra in $D_{\mathfrak{m}}$ and, by Lemma 2, $B$ satisfies the $\mathfrak{n}$-chain condition.

Now we prove Theorem 3. Let $B_1$ be a Boolean $\mathfrak{m}$-algebra in $\boldsymbol{D}_{\mathfrak{m}}$ satisfying the condition of Lemma 3. If $\aleph_\alpha$ is the cofinality of $\mathfrak{n}$, we can write $\mathfrak{n} = \sum(\mathfrak{n}_\beta \mid \beta < \omega_\alpha)$, where $\mathfrak{n}_\beta < \mathfrak{n}$ for all $\beta < \omega_\alpha$. For each $\beta < \omega_\alpha$, let $C_\beta \subseteqq B_1$ be a chain of cardinality $\mathfrak{n}_\beta$. Let $B_2$ be a Boolean $\mathfrak{m}$-algebra that is Boolean $\mathfrak{m}$-generated by the ordinal $\omega_\alpha + 1$ inside a power set; then $|B_2| < \mathfrak{n}$. Further, let $L$ be the $V$-free $\mathfrak{m}$-product of $B_1$ and $B_2$. For $\beta < \omega_\alpha$, let $C'_\beta = \{(x \vee \beta) \wedge (\beta + 1) \mid x \in C_\beta\}$; then $C = \bigcup(C'_\beta \mid \beta < \omega_\alpha)$ is a chain in $L$. Let $\psi \colon B_2 \to 2$

be an $\mathfrak{m}$-homomorphism such that $\psi(\beta) = 0$ and $\psi(\beta + 1) = 1$. We now define the $\mathfrak{m}$-homomorphism $\varphi: L \rightarrow B_1 \cup \{0, 1\}$ by $\varphi(x)$ if $x \in B_1$, and $\varphi(x) = \psi(x)$ if $x \in B_2$. Since $\varphi((x \vee \beta) \wedge (\beta + 1)) = x$, it now follows that $|C'_\beta| = \mathfrak{n}_\beta$. Therefore, $|C| = \mathfrak{n}$, completing the proof.

Theorem 4 is easier to prove. Indeed, if $\mathfrak{n} \leqq \mathfrak{m}$, the $V$-free $\mathfrak{m}$-lattice with $\mathfrak{n}$ generators $\{x_\alpha | \alpha < \mathfrak{n}\}$ contains the chain $\{y_\alpha | \alpha < \mathfrak{n}\}$ of cardinality $\mathfrak{n}$, where $y_\alpha = \bigvee (x_\beta | \beta \leqq \alpha)$ whenever $\alpha < \mathfrak{n}$. If $\mathfrak{n} > \mathfrak{m}$, then $\mathfrak{n} = \Sigma(\mathfrak{n}_\alpha | \alpha < \mathfrak{m}_0)$, where $\mathfrak{n}_\alpha < \mathfrak{n}$ for all $\alpha < \mathfrak{m}_0$. Let $C$ and $C_\alpha$ be successor ordinals of cardinality $\mathfrak{m}_0$ and $\mathfrak{n}_\alpha$, respectively, where $\alpha < \mathfrak{m}_0$. The proof is completed similarly as in Theorem 3 by showing that each $C_\alpha$ can be embedded into the interval $(\alpha, \alpha + 1)$ in the $V$-free $\mathfrak{m}$-product of $C$ and the $C_\alpha$ $(\alpha < \mathfrak{m}_0)$.

## 6. Open problems.

*Problem* 1. Is every $V$-free $\mathfrak{m}$-lattice a union of $2^{\mathfrak{m}}$ antichains? First we show that this holds for $\mathfrak{m} = \aleph_0$.

PROPOSITION 1. *Any $V$-free lattice is a countable union of antichains.*

*Proof.* Let $L$ be the $V$-free lattice generated by a set $X$. Let $p$ be a polynomial in variables $x_1, x_2, \cdots, x_n$ and let $S$ be the set of all $a \in L$ that have a *proper* representation of the form $a = p(x_1, \cdots, x_n)$ where $x_i \in X$, $1 \leqq i \leqq n$. It is enough to show that $S$ is an antichain. Let $\sigma$ be a permutation of $\{1, 2, \cdots, n\}$. For $a = p(x_1, \cdots, x_n)$, we write $\sigma a$ for $p(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$. If $a \leqq \sigma a$, then $\sigma a \leqq \sigma^2 a, \cdots, \sigma^{n-1} a \leqq \sigma^n a = a$, from which it follows that $a = \sigma a$. (F. Galvin and B. Jónsson used similar reasoning in [4].) Now, let $a = p(x_1, \cdots, x_n)$ and $b = p(y_1, \cdots, y_n)$ be proper representations with $x_i, y_i \in X$, $1 \leqq i \leqq n$, and suppose that $a \leqq b$. Let $A = \{x_1, \cdots, x_n\}$ and $B = \{y_1, \cdots, y_n\}$. We can assume there is an integer $k$ with $0 \leqq k \leqq n$ and there are elements $z_1, \cdots, z_k \in X$ such that $A - B = \{z_1, \cdots, z_k\}$ and $A \cap B = \{y_{k+1}, \cdots, y_n\}$. Applying the obvious endomorphism of $L$ to the inequality $a \leqq b$, we obtain $p(x_1, \cdots, x_n) \leqq p(z_1, \cdots, z_k, y_{k+1}, \cdots, y_n)$; by the previous case, $a = p(z_1, \cdots, z_k, y_{k+1}, \cdots, y_n)$. Let $\varphi$ be the endomorphism of $L$ that maps $z_i$ to $y_i$, and vice-versa $(1 \leqq i \leqq k)$, and maps all other elements of $X$ identically. Applying $\varphi$ to the inequality $p(z_1, \cdots, z_k, y_{k+1}, \cdots, y_n) \leqq p(y_1, \cdots, y_n)$, we obtain $b \leqq a$, completing the proof.

The following example shows that similar reasoning will not settle the uncountable case. (For notational simplicity, we only deal with the $\mathfrak{m} = \aleph_1$ case.)

Let $V$ be a nontrivial variety of $\aleph_1$-lattices and let $L$ be a $V$-free lattice generated by an infinite set $X$. We show that, in contrast with the $\mathfrak{m} = \aleph_0$ case, permutations of $X$ can create distinct comparable elements in $L$. Let $\boldsymbol{p}$ and $\boldsymbol{q}$ be $\aleph_1$-polynomials in variables $\{\boldsymbol{x}_n \mid n < \omega\}$ such that $\boldsymbol{p} \leqq \boldsymbol{q}$ holds in $V$ (for any substitution) but $\boldsymbol{p} = \boldsymbol{q}$ does not (for example, $\boldsymbol{x}_0$ and $\boldsymbol{x}_0 \vee \boldsymbol{x}_1$). Let $x_n^i$ be distinct elements of $X$ for $i \in Z$ (the integers) and $n < \omega$. Further, let $p_i = \boldsymbol{p}(x_n^i \mid n < \omega)$ and $q_i = \boldsymbol{q}(x_n^i \mid n < \omega)$. If

$$a = \bigvee (p_i \mid i \leqq 0) \vee \bigvee (q_i \mid i > 0)$$

and

$$b = \bigvee (p_i \mid i < 0) \vee \bigvee (q_i \mid i \geqq 0)\ ,$$

then $a \leqq b$ and $b$ can be obtained from $a$ by suitably permuting the elements of $X$. If $a = b$, we obtain $p_0 = q_0$ by mapping each $x_n^i$ ($i \neq 0$, $n < \omega$) to $\wedge (x_n^0 \mid n < \omega)$. This would mean that $\boldsymbol{p} = \boldsymbol{q}$ holds in $V$, contrary to assumption. Therefore, $a < b$. In fact, a chain isomorphic to the reals $R$ can be obtained from $a$ by suitable permutations of $X$. (Let $f\colon Z \to Q$ be a bijection, and for $y \in R$, let $a_y = \bigvee (r_i \mid i \in Z)$, where $r_i = p_i$ if $f(i) < y$ and $r_i = q_i$ otherwise.)

*Problem* 2. Let $\mathfrak{n}$ be regular and $>\mathfrak{m}$. Do $V$-free $\mathfrak{m}$-products preserve the $\mathfrak{n}$-chain condition?

This problem was answered affirmatively for $\mathfrak{m} = \aleph_0$ and $V = \boldsymbol{D}$ by G. Grätzer and H. Lakser [6]. For $\mathfrak{m} = \aleph_0$ and $V = \boldsymbol{L}$, an affirmative answer was found by M. E. Adams and D. Kelly [1] by separately proving the following two statements:

(i) The free product of a family $(L_i \mid i \in I)$ of lattices is isomorphic to a subposet of the completely free lattice generated by the poset $\bigcup (L_i \mid i \in I)$.

(ii) If a poset $X$ satisfies the $\mathfrak{n}$ chain condition, then so does the completely $V$-free lattice generated by $X$.

It is shown in [6] that the statement corresponding to (i) for $\mathfrak{m}$-lattices is valid. On the other hand, the following example shows that the analogue of (ii) is false.

Let $\mathfrak{m}$ and $\mathfrak{n}$ be uncountable cardinals and consider the poset $X = \{x_n^\alpha \mid n < \omega, \alpha < \mathfrak{n}\}$ where $x_m^\alpha < x_m^\beta$ iff $m < n$ and $\alpha < \beta$. Then $X$ contains only countable chains but the completely $V$-free lattice $L$ generated by $X$ contains a chain of cardinality $\mathfrak{n}$, where $V$ is an arbitrary nontrivial variety of $\mathfrak{m}$-lattices. For $\alpha < \mathfrak{n}$ let $y_\alpha = \bigvee (x_n^\alpha \mid n < \omega)$; clearly, $\{y_\alpha \mid \alpha < \mathfrak{n}\}$ is a chain in $L$. Let $\alpha < \beta < \mathfrak{n}$. The isotone map $\varphi\colon X \to 2$ defined by $\varphi(x_n^\gamma) = 0$ if $\gamma \leqq \alpha$ and $\varphi(x) = 1$

for $x \in X$ otherwise extends to an $\mathfrak{m}$-homomorphism of $L$ onto 2 that maps $y_\alpha$ to 0 and $y_\beta$ to 1; thus, $y_\alpha \neq y_\beta$.

*Problem* 3. Is every $\mathfrak{m}$-complete chain contained in a Boolean $\mathfrak{m}$-algebra in $\boldsymbol{D}_\mathfrak{m}$?

If $\mathfrak{m} = \mathfrak{n}^+$, a Boolean $\mathfrak{m}$-algebra in $\boldsymbol{D}_\mathfrak{m}$ is called $\mathfrak{n}$-representable by R. Sikorski [10]. If, for any two distinct elements of an $\mathfrak{m}$-lattice $L$, there is an $\mathfrak{m}$-homomorphism from $L$ onto 2 separating the two elements, then $L$ is in $\boldsymbol{D}_\mathfrak{m}$. Thus, as observed in the proof of Lemma 3, any successor ordinal is an $\mathfrak{m}$-sublattice of a power set. It also follows that $\boldsymbol{D}_\mathfrak{m}$ contains every $\mathfrak{m}$-complete chain. (Replace each element of an $\mathfrak{m}$-complete chain $C$ by two elements, forming the chain $C'$; then $C'$ is an $\mathfrak{m}$-sublattice of a power set and the obvious map from $C'$ to $C$ is an $\mathfrak{m}$-homomorphism.) Since the embedding of a chain into the Boolean algebra that it $R$-generates preserves all existing joins and meets (see [5]), any $\mathfrak{m}$-complete chain is an $\mathfrak{m}$-sublattice of a Boolean $\mathfrak{m}$-algebra. However, the following example shows that $\mathfrak{m}$-congruences of maximal chains need not extend to $\mathfrak{m}$-congruences of Boolean $\mathfrak{m}$-algebras. (Contrast with the $\mathfrak{m} = \aleph_0$ case in [5].) Let $B$ be the power set of $[0, 1]$ and let $C$ be the maximal chain in $B$ consisting of all intervals of the form $[0, x)$ or $[0, x]$, where $x \in [0, 1]$. The $\mathfrak{m}$-homomorphism that only collapses $[0, x)$ and $[0, x]$, $0 \leqq x \leqq 1$, maps $C$ onto $[0, 1]$. Yet, if $\mathfrak{m} \geqq (2^{\aleph_0})^+$, any $\mathfrak{m}$-congruence of $B$ that collapses $[0, x)$ and $[0, x]$, $0 \leqq x \leqq 1$, collapses all of $B$ since $[0, 1] \subseteq \bigcup([0, x] - [0, x) \mid 0 \leqq x \leqq 1$.)

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