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POINTWISE COMPACTNESS AND MEASURABILITY

SURJIT SINGH KHURANA

Among other results it is proved that if $(X, \mathfrak{A}, \mu)$ is a probability space, E a Hausdorff locally convex space such that $(E', \sigma(E', E))$ contains an increasing sequence of absolutely convex compact sets with dense union, and $f: X \to E$ weakly measurable with $f(X) \subset K$, a weakly compact convex subset of E, then f is weakly equivalent to $g: X \to E$ with $g(X)$ contained in a separable subset of K.

In [8] and [9] some remarkable results are obtained for the pointwise compact subsets of measurable real-valued functions and some interesting applications to strongly measurable Banach space-valued functions are established. In this paper we continue those ideas a little further. We first give a somewhat different proof of ([9], Theorem 1) and then apply it to give a generalization of classical Phillip's theorem ([5]). Also some result about equicontinuous subsets of $C(X)$, the space of all continuous real-valued functions on (X, τ_ρ) (τ_p is the lifting topology, [10], p. 59; in [8] this topology is denoted by T_ρ) are obtained.

All locally convex spaces are taken over reals and notations of [6] are used. For a topological space Y, $C(Y)$ (resp. $C_b(Y)$) will denote the set of all (resp. all bounded) real-valued continuous functions of Y. N will denote the set of natural numbers.

In this paper $(X, \mathfrak{A}, \mu)$ is a complete probability measure space. Let $\mathscr{L}$ be the set of all real-valued $\mathfrak{A}$-measurable functions on X, $\mathscr{L}^\infty$, the essentially bounded elements of $\mathscr{L}$, and M^∞, the bounded elements of $\mathscr{L}$. We fix a lifting, [10], $\rho: \mathscr{L}^\infty \to M^\infty$ and on X we always take the lifting topology τ_ρ ([10], p. 59). For $f \in \mathscr{L}$, $g \in \mathscr{L}$, we write $f = g$ if $f(x) = g(x)$, $\forall x \in X$, and $f \equiv g$ if $f(x) = g(x)$, a.e. $[\mu]$. For a Hausdorff locally convex space E, a function $f: X \to E$ is said to be weakly measurable if $h \circ f$ is $\mathfrak{A}$-measurable, $\forall h \in E'$, the topological dual of E. Two weakly measurable functions $f_i: X \to E$, $i = 1, 2$, are said to be weakly equivalent if $h \circ f_1 \equiv h \circ f_2$, $\forall h \in E'$. The space $\mathscr{L}_1$ and norms $\|\cdot\|_1$ and $\|\cdot\|_\infty$ have the usual meanings. We shall call a topological space, countably compact if every sequence in it has a cluster point, and sequentially compact if every sequence has a convergent subsequence.

We start with a different proof of the following result of [9].

THEOREM 1 ([9], *Theorem* 1). *Let H be a subset of $\mathscr{L}$ such that for any $h_1 \in H$, $h_2 \in H$, $h_1 \neq h_2$ implies $h_1 \not\equiv h_2$. Then, with the pointwise topology on H, the following are equivalent*:

(i) *H is sequentially compact*;

(ii) *H is compact and metrizable.*

If H is convex, then each of (i) *and* (ii) *is also equivalent to*:

(iii) *H is compact*;

(iv) *H is countably compact.*

Proof. By ([6], Theorem 11.2, p. 187) each of (i), (ii), (iii), (iv) implies that H is relatively compact in R^X, with product topology. Thus each of these conditions implies that H is pointwise bounded. Denote by φ the homeomorphism, $[0, \infty] \to [0, 1]$, $x \to x/(1 + x)$. For any $\alpha \in I$, the directed net of all finite subsets of H, let $h_\alpha = \sup \{|h|: h \in \alpha\}$, and $p_\alpha = \rho(\varphi \circ h_\alpha)$. $\{p_\alpha\}$ is a monotone bounded net in $C_b(X)$, which is boundedly complete. Let $\sup p_\alpha = p \in C_b(X)$. This means there is an increasing sequence $\{\alpha(n)\} \subset I$ such that $p = \sup p_{\alpha(n)}$ (this follows from the fact that $\mu(p) = \sup \mu(p_\alpha)$). Since $p_\alpha \equiv \varphi \circ h_\alpha$, we get $p_\alpha^{-1}\{1\}$ is μ-null, $\forall \alpha$. From this it follows that $K = p^{-1}\{1\}$ is μ-null. Thus $q = (\varphi^{-1} \circ p)\chi_{X/K}$ is a measurable function such that $|h| \leqq q$ a.e. $[\mu]$, $\forall h \in H$.

(i) $\Leftrightarrow$ (ii) is simple ([8], Prop. 1, p. 197), the metric d of (ii) being defined by $d(f, g) = \|(f - g)/1 + q\|_1$. (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are trivial. Now we come to the proof of (iv) $\Rightarrow$ (i). Take a sequence $\{f_n'\} \subset H$. Since $1/(1 + q)H$ is relatively weakly compact in $(\mathscr{L}_1, \|\cdot\|_1)$ there exists a subsequence $\{f_n\}$ of $\{f_n'\}$ and an $f_0 \in \mathscr{L}_1$ such that $1/(1 + q)f_n \to f_0$ weakly. Thus there exists a sequence $\{g_n\}$ in the convex hull of $\{f_n: 1 \leqq n < \infty\}$ (note $\{g_n\} \subset H$) such that $1/(1 + q)g_n \to f_0$ a.e. $[\mu]$ (because a convergent sequence in $(\mathscr{L}_1, \|\cdot\|_1)$ has a subsequence converging a.e. $[\mu]$). Taking f to be a cluster point of $\{g_n\}$ in H, we get $1/(1 + q)f \equiv f_0(\mu)$. We claim $f_n \to f$ in H. If $f_n \nrightarrow f$ there exists an $x \in X$, an $\varepsilon > 0$, and a subsequence $\{f_n''\}$ of $\{f_n\}$ such that one of the two following conditions are satisfied:

(i) $f_n''(x) > f(x) + \varepsilon$, $\forall n$;

(ii) $f_n''(x) < f(x) - \varepsilon$, $\forall n$.

Since $1/(1 + q)f_n'' \to 1/(1 + q)f$ weakly, proceeding as before we get a sequence $\{g_n''\}$ in the convex hull of $\{f_n'': 1 \leqq n < \infty\}$ such that $1/(1 + q)g_n'' \to 1/(1 + q)f$ a.e. $[\mu]$. If f'' is a cluster point of $\{g_n''\}$ in H we get $f'' \equiv f(\mu)$ but because of (i) or (ii), $f''(x) \neq f(x)$, a contradiction. This proves that H is sequentially compact.

This result is also proved in [11] by a different method.

By a classical theorem of Phillips [5], if $f: X \to E$, E being a Banach space, is weakly measurable and $f(X)$ is relatively weakly compact in E, then f is weakly equivalent to a strongly measurable function ([8], Theorem 3, p. 200). What one really needs to do is to find a weakly equivalent function g such that $g(X)$ is separable. The next theorem is a generalization of Phillips' theorem.

THEOREM 2. *Let $(E, \mathscr{T})$ be a Hausdorff locally convex space such that there exists an increasing sequence $\{A_n\}$ of absolutely convex compact subsets of $(E', \sigma(E', E))$ whose union is dense in $(E', \sigma(E', E))$. Suppose $f: X \to E$ is weakly measurable and $f(X) \subset K$, for some weakly compact convex subset of E. Then there exists a weakly measurable function $g: X \to E$, $g \equiv f(w)$ and $g(X) \subset K_0$, a separable closed convex subset of K.*

Proof. Since $(E, \sigma(E, E'))$ can be considered as a subspace of $R^{E'}$, with product topology, f can be considered as $f: X \to R^{E'}$. For each $h \in E'$, define $g(h) = \rho(h \circ f)$ and let $g: X \to R^{E'}$, $(g)_h = g(h)$, $\forall h \in E'$. g is evidently continuous. If $g(x_0) \notin K$ for some $x_0 \in X$, there exists, by separation theorem ([6], p. 65), an $h \in E'$ such that $h \circ g(x_0) > \sup(K)$. This is a contradiction since $h \circ f \leqq \sup h(K)$ implies $\rho(h \circ f) \leqq \sup h(K)$. Evidently $g \equiv f(w)$. Fix $n \in N$. By Theorem 1, $B_n = \{h \circ g: h \in A_n\}$, with the topology of pointwise convergence on X, is a compact metric space. We metrize E by the seminorms p_n, $p_n(x) = \sup\{|h(x)|: h \in A_n\}$. We denote this metric topology by $\mathscr{T}_0$. For each n, $E_n = (C(B_n), \|\cdot\|)$ is a separable Banach space (here $\|\cdot\|$ is sup norm), and so $F = \prod_{n=1}^{\infty} E_n$ is a separable Frechet space. Let X_0 be the quotient space obtained from X by the equivalent relation, $x \equiv y \Leftrightarrow g(x) = g(y)$. Each $x \in X_0$ gives rise to $x \in C(B_n)$, $x(t) = t(x)$ for each $t \in B_n$, for every n. Thus X_0 can be embedded in F, $x_0 \to (x_0, x_0, \cdots) \in F$. Taking, on X_0, the topology induced by F, we easily verify that $g: X_0 \to (E, \mathscr{T}_0)$ is continuous and so $(g(X), \mathscr{T}_0)$ is separable. Let $K_0 =$ the closed convex hull, in $(E, \mathscr{T})$, of a countable dense subset of $(g(X), \mathscr{T}_0)$. If $g(X) \not\subset K_0$, by separation theorem, there exists an $h \in E'$ and $x_0 \in X$ such that $h \circ g(x_0) > \sup h(K_0)$. Since $(E, \mathscr{T}_0)' \supset \bigcup_{n=1}^{\infty} A_n$, $q \circ g(x_0) \leqq \sup q(K_0)$, $\forall q \in \bigcup_{n=1}^{\infty} A_n$. Now there exists a net $\{h_\alpha\} \subset \bigcup_{n=1}^{\infty} A_n$ such that $h_\alpha \to h$ uniformly on each compact convex subset of $(E, \sigma(E, E'))$. From this it follows $h \circ g(x_0) \leqq \sup h(K_0)$, a contradiction. This proves the result.

REMARK 3. If E is metrizable then $(E', \sigma(E', E))$ contains a sequence of compact absolutely convex sets whose union is E'. If Y is a completely regular Hausdorff space containing a σ-compact dense set and $E = C_b(Y)$ with strict topology β_0, β_1, then it is proved in ([3], Theorem 3) that $(E', \sigma(E', E))$ has an increasing sequence of absolutely convex compact sets with dense union — here E is not metrizable.

REMARK 4. The function $g: X \to (E, \sigma(E, E'))$, obtained in this theorem, is measurable in the sense of ([2], Def. 4, p. 89).

The next theorem, in some sense, is a generalization of ([9], Theorem 3).

THEOREM 5. *Let E be a Hausdorff locally convex space such that there exist, in $(E', \sigma(E', E))$, an increasing sequence $\{A_n\}$ of absolutely convex compact sets whose union is E'. Suppose $g: X \to E$ is weakly measurable such that $g \circ f \neq 0$ implies $g \circ f \not\equiv 0$, for every $f \in E'$. Then $g(X)$ is contained in a separable subspace of E.*

Proof. In the notations of Theorem 2, $B_n = \{h \circ g: h \in A_n\}$ are compact and metrizable, with the topology of pointwise convergence, and $\mathscr{T}_0$ is the metric topology, on E, of uniform convergence on A_n. Proceeding exactly as in Theorem 2, we prove that $g(X)$ is a separable subset of $(E, \mathscr{T}_0)$. Let $F = (E, \mathscr{T}_0)'$ and $E_0 =$ the closed separable subspace, in $(E, \mathscr{T})$, generated by a countable dense subset of $(g(X), \mathscr{T}_0)$. If $g(x_0) \notin E_0$ for some $x_0 \in X$ there exists, by separation theorem, an $h \in E'$ such that $h \circ g(x_0) > 0$ and $h \equiv 0$ on E_0. Since $E' = \bigcup_{n=1}^{\infty} A_n \subset F$, $h \circ g(x_0) \leqq \sup (h \circ g(X)) \leqq \sup h(E_0) = 0$, a contradiction. This proves the result.

In the next theorem we do not assume H to be uniformly bounded ([8], Theorem 4, p. 203).

THEOREM 6. *Let H be a pointwise bounded subset of $C(X)$. If H is equicontinuous then, with the topology of pointwise convergence on X, its closure in $C(X)$ is compact and metrizable. Conversely if H is sequentially compact then there is a μ-null set A such that H is equicontinuous at each point of the open set $X \backslash A$ of (X, τ_ρ).*

Proof. If H is equicontinuous then its pointwise closed convex hull H_0, in R^X, lies in $C(X)$ and is compact and convex, and so the result follows from Theorem 1.

Conversely suppose H is sequentially compact. Then, by Theorem 1, H is compact and metrizable. By the generalized Egoroff's theorem ([4], p. 198) there exists a $\mathfrak{A}$-partition of $X = \bigcup_{i=0}^{\infty} X_i$, with $\mu(X_0) = 0$ and $\mu(X_i) > 0$, $\forall i \geqq 1$ such that $H|_{X_i}$ is compact in the topology of uniform convergence on X_i, $\forall i \geqq 1$.

$Y_i = X_i \cap \rho(X_i)$, $i \geqq 1$, are nonvoid, disjoint, open subsets of (X, τ_ρ) and $\mu(A) = 0$, where $A = X \backslash \bigcup_{i=1}^{\infty} Y_i$. By the Ascoli Theorem ([1], Ch. X, §2.5), $H|_{Y_i}$ are equicontinuous for each i. The result follows now.

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THE UNIVERSITY OF IOWA
IOWA CITY, IA 52242