PACIFIC JOURNAL OF MATHEMATICS
Vol. 86, No. 2, 1980

TCHEBYCHEFF SYSTEMS AND BEST PARTIAL BASES

OVED SHISHA

This is a contribution to the *partial basis problem* and, in particular, to the case where the basis elements are cosigns or consecutive powers cosines. We contribute also to the general theory of Tchebycheff systems to which the partial basis problem is strongly related.

1. Introduction. The *partial basis problem* was formulated and studied by J. T. Lewis, D. W. Tufts and the author in 1975 in connection with their study of optimization of multichannel processing. Let X be a normed linear space, let $f, h_1, h_2, \cdots, h_N \in X$ and let n be an integer, $1 \leqq n < N$. For every sequence $\mu = \{\mu_k\}_1^n$ of integers, with $1 \leqq \mu_1 < \mu_2 \cdots < \mu_n \leqq N$, consider

$$e(\mu) = \min\left\|f - \sum_{k=1}^{n} c_k h_{\mu_k}\right\|$$

where the minimum is taken over all possible choices of the scalars $c_1, \cdots, c_n$. The problem is to minimize $e(\mu)$. It is of particular interest when X is one of the standard function spaces.

Subsequently, progress has been made both in theory and in the computational aspect. An algorithm, numerical examples and some theoretical results have been given by K. M. Levasseur and J. T. Lewis in [6]. G. G. Lorentz [5] has observed that, for $X = L^2(0, 1)$, h_k the function $x^{k-1} (k = 1, 2, \cdots, N)$ and f the function x^N, $e(\mu)$ is minimized by $\mu = \{N - n + 1, N - n + 2, \cdots, N\}$ and conjectured the same to be true for $X = C[0, 1]$. This was proved by I. Borosh, C. K. Chui and P. W. Smith [1, Theorem 1].

In Theorem 4 below we give a sufficient condition for a real function f, continuous on $[a, b] (0 < a < b < \infty)$, that $e(\mu)$ be minimized (only) by $\mu = \{N - n + 1, N - n + 2, \cdots, N\}$, where $X = L^p(a, b)$, $1 \leqq p \leqq \infty$ and h_k is the function $x^{k-1} (k = 1, 2, \cdots, N)$. For such a function f (with $a = 0, b = \pi$), Theorem 17 gives such a sufficient condition with the same X, where each h_k is a function of the form $\cos \alpha x$.

In proving Theorems 4 and 17 we use Theorem 1 which, together with Lemma 2, is due to A. Pinkus. The author is very grateful to him for that as well as for other valuable remarks. Cf. also [8, § 3].

The partial basis problem turns out to be very much interrelated with the theory of Tchebycheff systems and this paper is a contribution to both. Thus in Theorem 8 we characterize certain trigonometric sequences which are extended complete Tchebycheff systems. Lemmas 9-13 are used later: they are quite straightforward and are stated with their proof mainly for the convenience of the reader. We then state and prove Theorem 14 (used later) though much of it is known [2, Example 7, p. 42; it should read there $T = (0, \tau)$, not $T = [0, \tau)$]. A previous work in the same direction is [7], whose starting point was a conjecture made by L. Collatz in an Oberwolfach conference.

Some of the development in §4, and in particular Lemma 16, is due to R. A. Zalik. His help and interest are greatly appreciated.

Let I be a real interval and $f_1, f_2, \cdots, f_n$ real functions defined on I. The sequence $\{f_1, \cdots, f_n\}$ is called a *Tchebycheff system* or a *T-system on I* iff whenever $x_1 < x_2 < \cdots < x_n$ and all $x_k \in I$, the determinant of the $n \times n$ matrix whose kth row $(k = 1, 2, \cdots, n)$ is $f_k(x_1)\ f_k(x_2) \cdots f_k(x_n)$ is >0. The sequence is called a *complete Tchebycheff system* or a *CT-system on I* iff $\{f_1, \cdots, f_k\}$ is a T-system on I for $k = 1, 2, \cdots, n$. This is the case, e.g., if $I = (0, \infty)$ and $f_k(x) \equiv x^{\lambda_k}$ where $\lambda_1 < \lambda_2 < \cdots < \lambda_n$ [4, p. 9]. Suppose each $f_k \in C^{n-1}(I)$. Then $\{f_1, \cdots, f_n\}$ is called an *extended complete Tchebycheff system* or an *ECT-system on I* iff, for $k = 1, 2, \cdots, n$, the following property holds. If $x_1 \leqq x_2 \leqq \cdots \leqq x_k$ and if $x_j \in I$ for $j = 1, 2, \cdots, k$, then the determinant of the $k \times k$ matrix whose jth row $(j = 1, 2, \cdots, k)$ is $f_j^{(1-r_1)}(x_1)\ f_j^{(2-r_2)}(x_2) \cdots f_j^{(k-r_k)}(x_k)$ is >0. For $j = 1, 2, \cdots, k$, we denote by r_j the smallest integer r for which $x_r = x_j$.

Finally, many thanks are due to the referee for his helpful suggestions.

2. A general result concerning best partial bases.

THEOREM 1. *Let*

$$(1) \qquad f_0, f_1, \cdots, f_{N-1}, f$$

be real functions on $[a, b]$ $(-\infty < a < b < \infty)$ *and let* n *be an integer,* $1 \leqq n < N$. *Let* $\varepsilon_n, \varepsilon_{n+1}$ *be each* 1 *or* -1 *and suppose that, for* $k = n, n+1$, *every subsequence of* (1) *of length* k, *after multiplying its last element by* ε_k, *becomes a T-system on* $[a, b]$. *Let* $0 \leqq \lambda_1 < \lambda_2 \cdots < \lambda_n < N$ *be integers,* $\{\lambda_1, \lambda_2, \cdots, \lambda_n\} \neq \{N-n, N-n+1, \cdots, N-1\}$. *Let* $1 \leqq p \leqq \infty$ *and let* $f_0, f_1, \cdots, f_{N-1}, f$ *be continuous on* $[a, b]$. *Then*

$$\min_{c_k \text{ real}} \left\| f - \sum_{k=N-n}^{N-1} c_k f_k \right\|_{L^p(a,b)} < \min_{c_k \text{ real}} \left\| f - \sum_{k=1}^{n} c_k f_{\lambda_k} \right\|_{L^p(a,b)} .$$

We shall need for the proof the following

LEMMA 2. *Assume the first two sentences of Theorem 1. Let* $0 \leqq \lambda_1 < \lambda_2 \cdots < \lambda_n < N$ *and* j *be integers,* $1 \leqq j \leqq n$. *Assume that if* $j < n$, *then* $\lambda_j + 1 < \lambda_{j+1}$, *while if* $j = n$, *then* $\lambda_j + 1 < N$. *Let* $a < x_1 < x_2 \cdots < x_n < b$, $x \in [a, b] - \{x_1, \cdots, x_n\}$ *and set*

$$\omega = \begin{vmatrix} f_{\lambda_1}(x_1) \cdots\cdots f_{\lambda_1}(x_n) f_{\lambda_1}(x) \\ \vdots \qquad \vdots \qquad \vdots \\ f_{\lambda_n}(x_1) \cdots\cdots f_{\lambda_n}(x_n) f_{\lambda_n}(x) \\ f(x_1) \cdots\cdots f(x_n) \; f(x) \end{vmatrix} \Bigg/ \begin{vmatrix} f_{\lambda_1}(x_1) \cdots\cdots f_{\lambda_1}(x_n) \\ \vdots \qquad \vdots \\ f_{\lambda_n}(x_1) \cdots\cdots f_{\lambda_n}(x_n) \end{vmatrix} . \tag{2}$$

Let ω' *be obtained from* ω *by replacing* λ_j *by* $\lambda_j + 1$. *Then* $|\omega'| < |\omega|$.

Proof of Theorem 1. Let $\hat{f} = f - \sum_{k=1}^{n} c_k^* f_{\lambda_k}$, the c_k^* being real constants satisfying $||\hat{f}||_{L^p(a,b)} = \min_{c_k \text{ real}} ||f - \sum_{k=1}^{n} c_k f_{\lambda_k}||_{L^p(a,b)}$. It is known that there are $x_1 < x_2 \cdots < x_n$, all in (a, b), at which $\hat{f}$ vanishes. The right hand side of (2) is of the same form as $\hat{f}$ and vanishes at $x_1, \cdots, x_n$, hence it is $\equiv \hat{f}$. By repeated use of Lemma 2, for every $x \in [a, b] - \{x_1, \cdots, x_n\}$,

$$|\hat{f}(x)| > \left| \begin{vmatrix} f_{N-n}(x_1) \cdots\cdots f_{N-n}(x_n) f_{N-n}(x) \\ \vdots \qquad \vdots \qquad \vdots \\ f_{N-1}(x_1) \cdots\cdots f_{N-1}(x_n) f_{N-1}(x) \\ f(x_1) \cdots\cdots f(x_n) \quad f(x) \end{vmatrix} \Bigg/ \begin{vmatrix} f_{N-n}(x_1) \cdots\cdots f_{N-n}(x_n) \\ \vdots \qquad \vdots \\ f_{N-1}(x_1) \cdots\cdots f_{N-1}(x_n) \end{vmatrix} \right|$$

and so, $||\hat{f}||_{L^p(a,b)}$ is larger than the $L^p(a, b)$ norm of the last ratio, which in turn is $\geqq \min_{c_k \text{real}} ||f - \sum_{k=N-n}^{N-1} c_k f_k||_{L^p(a,b)}$.

Proof of Lemma 2. For definiteness assume $1 < j < n$. Then $(-1)^n(\omega - \omega')$ is a ratio whose numerator is

$$\begin{vmatrix} f_{\lambda_1}(x_1) \cdots\cdots f_{\lambda_1}(x_n) \\ \vdots \qquad \vdots \\ f_{\lambda_{j-1}}(x_1) \cdots\cdots f_{\lambda_{j-1}}(x_n) \\ f_{\lambda_{j+1}}(x_1) \cdots\cdots f_{\lambda_{j+1}}(x_n) \\ \vdots \qquad \vdots \\ f_{\lambda_n}(x_1) \cdots\cdots f_{\lambda_n}(x_n) \\ f_{\lambda_j+1}(x_1) \cdots\cdots f_{\lambda_j+1}(x_n) \end{vmatrix} \cdot \begin{vmatrix} f(x_1) \cdots\cdots f(x_n) \quad f(x) \\ f_{\lambda_1}(x_1) \cdots\cdots f_{\lambda_1}(x_n) \quad f_{\lambda_1}(x) \\ \vdots \qquad \vdots \qquad \vdots \\ f_{\lambda_{j-1}}(x_1) \cdots\cdots f_{\lambda_{j-1}}(x_n) \quad f_{\lambda_{j-1}}(x) \\ f_{\lambda_{j+1}}(x_1) \cdots\cdots f_{\lambda_{j+1}}(x_n) \quad f_{\lambda_{j+1}}(x) \\ \vdots \qquad \vdots \qquad \vdots \\ f_{\lambda_n}(x_1) \cdots\cdots f_{\lambda_n}(x_n) \quad f_{\lambda_n}(x) \\ f_{\lambda_j}(x_1) \cdots\cdots f_{\lambda_j}(x_n) \quad f_{\lambda_j}(x) \end{vmatrix}$$

$$
-\begin{vmatrix}
f_{\lambda_1}(x_1) & \cdots\cdots & f_{\lambda_1}(x_n) \\
\vdots & & \vdots \\
f_{\lambda_{j-1}}(x_1) & \cdots\cdots & f_{\lambda_{j-1}}(x_n) \\
f_{\lambda_{j+1}}(x_1) & \cdots\cdots & f_{\lambda_{j+1}}(x_n) \\
\vdots & & \vdots \\
f_{\lambda_n}(x_1) & \cdots\cdots & f_{\lambda_n}(x_n) \\
f_{\lambda_j}(x_1) & \cdots\cdots & f_{\lambda_j}(x_n)
\end{vmatrix}
\cdot
\begin{vmatrix}
f(x_1) & \cdots\cdots & f(x_n) & f(x) \\
f_{\lambda_1}(x_1) & \cdots\cdots & f_{\lambda_1}(x_n) & f_{\lambda_1}(x) \\
\vdots & & \vdots & \vdots \\
f_{\lambda_{j-1}}(x_1) & \cdots\cdots & f_{\lambda_{j-1}}(x_n) & f_{\lambda_{j-1}}(x) \\
f_{\lambda_{j+1}}(x_1) & \cdots\cdots & f_{\lambda_{j+1}}(x_n) & f_{\lambda_{j+1}}(x) \\
\vdots & & \vdots & \vdots \\
f_{\lambda_n}(x_1) & \cdots\cdots & f_{\lambda_n}(x_n) & f_{\lambda_n}(x) \\
f_{\lambda_{j+1}}(x_1) & \cdots\cdots & f_{\lambda_{j+1}}(x_n) & f_{\lambda_{j+1}}(x)
\end{vmatrix}
$$

and whose denominator is

$$
\begin{vmatrix}
f_{\lambda_1}(x_1) & \cdots\cdots & f_{\lambda_1}(x_n) \\
\vdots & & \vdots \\
f_{\lambda_{j-1}}(x_1) & \cdots\cdots & f_{\lambda_{j-1}}(x_n) \\
f_{\lambda_j}(x_1) & \cdots\cdots & f_{\lambda_j}(x_n) \\
f_{\lambda_{j+1}}(x_1) & \cdots\cdots & f_{\lambda_{j+1}}(x_n) \\
\vdots & & \vdots \\
f_{\lambda_n}(x_1) & \cdots\cdots & f_{\lambda_n}(x_n)
\end{vmatrix}
\cdot
\begin{vmatrix}
f_{\lambda_1}(x_1) & \cdots\cdots & f_{\lambda_1}(x_n) \\
\vdots & & \vdots \\
f_{\lambda_{j-1}}(x_1) & \cdots\cdots & f_{\lambda_{j-1}}(x_n) \\
f_{\lambda_{j+1}}(x_1) & \cdots\cdots & f_{\lambda_{j+1}}(x_n) \\
f_{\lambda_{j+1}}(x_1) & \cdots\cdots & f_{\lambda_{j+1}}(x_n) \\
\vdots & & \vdots \\
f_{\lambda_n}(x_1) & \cdots\cdots & f_{\lambda_n}(x_n)
\end{vmatrix}.
$$

By a determinant identity [3, (0.19), p. 8] the numerator equals

$$
\begin{vmatrix}
f(x_1) & \cdots\cdots & f(x_n) \\
f_{\lambda_1}(x_1) & \cdots\cdots & f_{\lambda_1}(x_n) \\
\vdots & & \vdots \\
f_{\lambda_{j-1}}(x_1) & \cdots\cdots & f_{\lambda_{j-1}}(x_n) \\
f_{\lambda_{j+1}}(x_1) & \cdots\cdots & f_{\lambda_{j+1}}(x_n) \\
\vdots & & \vdots \\
f_{\lambda_n}(x_1) & \cdots\cdots & f_{\lambda_n}(x_n)
\end{vmatrix}
\cdot
\begin{vmatrix}
f_{\lambda_1}(x_1) & \cdots\cdots & f_{\lambda_1}(x_n) & f_{\lambda_1}(x) \\
\vdots & & \vdots & \vdots \\
f_{\lambda_{j-1}}(x_1) & \cdots\cdots & f_{\lambda_{j-1}}(x_n) & f_{\lambda_{j-1}}(x) \\
f_{\lambda_{j+1}}(x_1) & \cdots\cdots & f_{\lambda_{j+1}}(x_n) & f_{\lambda_{j+1}}(x) \\
\vdots & & \vdots & \vdots \\
f_{\lambda_n}(x_1) & \cdots\cdots & f_{\lambda_n}(x_n) & f_{\lambda_n}(x) \\
f_{\lambda_{j+1}}(x_1) & \cdots\cdots & f_{\lambda_{j+1}}(x_n) & f_{\lambda_{j+1}}(x) \\
f_{\lambda_j}(x_1) & \cdots\cdots & f_{\lambda_j}(x_n) & f_{\lambda_j}(x)
\end{vmatrix}.
$$

Hence, $\omega - \omega'$ is a ratio whose denominator is as above and whose numerator is

$$
\begin{vmatrix}
f_{\lambda_1}(x_1) & \cdots\cdots & f_{\lambda_1}(x_n) \\
\vdots & & \vdots \\
f_{\lambda_{j-1}}(x_1) & \cdots\cdots & f_{\lambda_{j-1}}(x_n) \\
f_{\lambda_{j+1}}(x_1) & \cdots\cdots & f_{\lambda_{j+1}}(x_n) \\
\vdots & & \vdots \\
f_{\lambda_n}(x_1) & \cdots\cdots & f_{\lambda_n}(x_n) \\
f(x_1) & \cdots\cdots & f(x_n)
\end{vmatrix}
\cdot
\begin{vmatrix}
f_{\lambda_1}(x_1) & \cdots\cdots & f_{\lambda_1}(x_n) & f_{\lambda_1}(x) \\
\vdots & & \vdots & \vdots \\
f_{\lambda_{j-1}}(x_1) & \cdots\cdots & f_{\lambda_{j-1}}(x_n) & f_{\lambda_{j-1}}(x) \\
f_{\lambda_j}(x_1) & \cdots\cdots & f_{\lambda_j}(x_n) & f_{\lambda_j}(x) \\
f_{\lambda_{j+1}}(x_1) & \cdots\cdots & f_{\lambda_{j+1}}(x_n) & f_{\lambda_{j+1}}(x) \\
f_{\lambda_{j+1}}(x_1) & \cdots\cdots & f_{\lambda_{j+1}}(x_n) & f_{\lambda_{j+1}}(x) \\
\vdots & & \vdots & \vdots \\
f_{\lambda_n}(x_1) & \cdots\cdots & f_{\lambda_n}(x_n) & f_{\lambda_n}(x)
\end{vmatrix}.
$$

Let σ be n, 0 or r if, respectively, $x < x_1$, $x > x_n$ or $x_{n-r} < x < x_{n+1-r} (1 \leqq r < n)$. Then $\operatorname{sgn}(\omega - \omega') = \varepsilon_n \varepsilon_{n+1}(-1)^\sigma = \operatorname{sgn} \omega = \operatorname{sgn} \omega'$. Hence $|\omega'| < |\omega|$.

3. On best partial power bases. Our main result here is Theorem 4 which will follow immediately from

THEOREM 3. *Let $0 < a < b < \infty$ and let N, n be integers, $1 \leqq n < N$. Let f be a real function, continuous in $[a, b]$ and assume that, for $k = 0, 1, \cdots, n$, $(x^{k-N}f)^{(k)}$ exists and is $\geqq 0$ in (a, b), with strict inequality there for $k = n$. Let $0 \leqq \lambda_1 < \lambda_2 \cdots < \lambda_n < N$ be integers. Then $\{x^{\lambda_1}, x^{\lambda_2}, \cdots, x^{\lambda_n}, f\}$ is a T-system on $[a, b]$.*

THEOREM 4. *Assume the hypotheses of Theorem 3 and also that $(x^{n-1-N}f)^{(n-1)} > 0$ on (a, b), and that $\{\lambda_1, \lambda_2, \cdots, \lambda_n\} \neq \{N - n, N - n + 1, \cdots, N - 1\}$. Let $1 \leqq p \leqq \infty$. Then*

$$\min_{c_k \text{ real}} \left\| f(x) - \sum_{k=N-n}^{N-1} c_k x^k \right\|_{L^p(a,b)} < \min_{c_k \text{ real}} \left\| f(x) - \sum_{k=1}^{n} c_k x^{\lambda_k} \right\|_{L^p(a,b)} .$$

Proof of Theorem 4. Set $f_k(x) \equiv x^k$, $k = 0, 1, \cdots, N - 1$, and observe that every subsequence of $f_0, f_1, \cdots, f_{N-1}$ is a T-system on $[a, b]$. If $n > 1$, then the first two sentences of Theorem 1 hold, with $\varepsilon_n = \varepsilon_{n+1} = 1$. Hence, by that theorem, the result. Examining the proofs of Theorem 1 and Lemma 2, we see that if $n = 1$, we do not need for the conclusion of Theorem 4 the hypothesis $f > 0$ on $[a, b]$ but merely our hypothesis $f > 0$ on (a, b).

To prove Theorem 3 we need

LEMMA 5. *Let f be a real function, r a real number and s an integer $\geqq 0$. Suppose, at some $x > 0$, $f^{(s)}$ exists. Then, at that x,*

$$(x^r f)^{(s)} = x \sum_{k=0}^{s} k! \binom{s}{k} (x^{r-1-k} f)^{(s-k)} .$$

Proof of Theorem 3. Let $a \leqq t_1 < t_2 < \cdots < t_{n+1} \leqq b$. Suppose we have proved that

$$\Delta(f) = \begin{vmatrix} t_1^{\lambda_1} & \cdots & t_{n+1}^{\lambda_1} \\ \vdots & & \vdots \\ t_1^{\lambda_n} & \cdots & t_{n+1}^{\lambda_n} \\ f(t_1) & \cdots & f(t_{n+1}) \end{vmatrix} \neq 0 .$$

For every $t \in [0, 1]$, $tf(x) + (1 - t)x^N$ satisfies the hypotheses made on f. Hence $\Delta(tf(x) + (1 - t)x^N)$ is either >0 for all $t \in [0, 1]$ or <0

there. Since $\Delta(x^N) > 0$, also $\Delta(f) > 0$. Let $c_1, \cdots, c_{n+1}$ be reals not all 0. Suppose $(\sum_{k=1}^{n} c_k x^{\lambda_k}) + c_{n+1} f(x)$ vanished at $n+1$ points of $[a, b]$. We shall reach a contradiction which will prove the theorem. For $j = 1, 2, \cdots, n$, let

$$g_j(x) \equiv (x^{1+\lambda_{j-1}-\lambda_j}(x^{1+\lambda_{j-2}-\lambda_{j-1}} \cdots (x^{1+\lambda_1-\lambda_2}(x^{-\lambda_1}f(x)))')' \cdots)')'$$

(meaning $(x^{-\lambda_1}f(x))'$ if $j = 1$). Using induction on j, one readily shows by Rolle's theorem that, for $j = 1, 2, \cdots, n$,

$$\left[\sum_{k=j+1}^{n} c_k \left\{\prod_{r=1}^{j} (\lambda_k - \lambda_r)\right\} x^{\lambda_k - \lambda_j - 1}\right] + c_{n+1} g_j(x) ,$$

where the Σ means 0 if $j = n$, vanishes at some $n + 1 - j$ points of (a, b). In particular, since $c_{n+1} \neq 0$, g_n must vanish somewhere in (a, b). We shall reach the desired contradiction by showing that $g_n > 0$ throughout (a, b). This, in turn, follows from the fact that, for $k = 1, 2, \cdots, n$, (*) throughout (a, b),

$$g_k(x) = x^{N-n-\lambda_k+k-1} \sum_{j=0}^{k} c_{k,j} (x^{n-N-j} f)^{(k-j)}$$

where $c_{k,j}$ are constants $\geqq 0$ and $c_{k,0} = 1$.

Now (*) holds for $k = 1$, since on (a, b),

$$(x^{-\lambda_1} f)' = (x^{N-n-\lambda_1}(x^{n-N} f))' = x^{N-n-\lambda_1}[(x^{n-N} f)' + (N - n - \lambda_1) x^{n-N-1} f] .$$

Suppose it holds for some k, $1 \leqq k < n$. Then throughout (a, b), by Lemma 5,

$$g_{k+1} = (x^{1+\lambda_k-\lambda_{k+1}} g_k)' = x^{N-n-\lambda_{k+1}+k}\left[\sum_{j=0}^{k} c_{k,j}(x^{n-N-j} f)^{(k+1-j)}\right.$$
$$\left. +(N-n-\lambda_{k+1}+k)\sum_{j=0}^{k} c_{k,j} \sum_{p=j+1}^{k+1} (p-1-j)! \binom{k-j}{p-1-j} (x^{n-N-p} f)^{(k+1-p)}\right] ,$$

which establishes (*) for $k + 1$ and completes the proof of the theorem.

Proof of Lemma 5. We may assume $s \geqq 1$. For $0 \leqq n \leqq s - 1$,

$$(**) \quad (x^r f)^{(s)} = \left[x \sum_{k=0}^{n} k! \binom{s}{k} (x^{r-1-k} f)^{(s-k)} \right] + (n+1)! \binom{s}{n+1} (x^{r-1-n} f)^{(s-n-1)} .$$

Indeed, for $n = 0$, (**) reduces to

$$(***) \qquad (x^r f)^{(s)} = x(x^{r-1} f)^{(s)} + s(x^{r-1} f)^{(s-1)} ,$$

which is true. Assuming (**) holds for some n, $0 \leqq n < s-1$, (***) yields

$$(x^r f)^{(s)} = \left[x \sum_{k=0}^{n} k! \binom{s}{k} (x^{r-1-k} f)^{(s-k)} \right]$$
$$+ (n+1)! \binom{s}{n+1} [x(x^{r-s-n} f)^{(s-n-1)} + (s-n-1)(x^{r-2-n} f)^{(s-n-2)}]$$
$$= \left[x \sum_{k=0}^{n+1} k! \binom{s}{k} (x^{r-1-k} f)^{(s-k)} \right] + (n+2)! \binom{s}{n+2} (x^{r-2-n} f)^{(s-n-2)} .$$

Take now, in (**), $n = s-1$.

REMARKS 6. Theorem 1 continues to hold if $[a, b]$ is replaced by (a, b), $1 \leqq p \leqq \infty$ by $1 \leqq p < \infty$, and if $f_0, \cdots, f_{N-1}, f$ belong to $L^p(a, b)$. Similarly, Theorem 4 continues to hold if $0 < a < b < \infty$ is replaced by $0 \leqq a < b < \infty$, $1 \leqq p \leqq \infty$ by $1 \leqq p < \infty$, and "continuous in $[a, b]$" by "in $L^p(a, b)$." As to the case $a = 0$, $p = \infty$, we have the following result: Let $0 < b < \infty$ and let N, n be integers, $1 \leqq n < N$. Let f be a real function, continuous in $[0, b]$, with $f(0) = 0$ and assume that, for $k = 0, 1, \cdots, n$, $(x^{k-N} f)^{(k)}$ exists and is $\geqq 0$ in $(0, b)$ with strict inequality there for $k = n-1, n$. Let $0 \leqq \lambda_1 < \lambda_2 \cdots < \lambda_n < N$ be integers, $\{\lambda_1, \lambda_2, \cdots, \lambda_n\} \neq \{N-n, N-n+1, \cdots, N-1\}$. Then

$$\min_{c_k \text{ real}} \max_{0 \leqq x \leqq b} \left| f(x) - \sum_{k=N-n}^{N-1} c_k x^k \right| < \min_{c_k \text{ real}} \max_{0 \leqq x \leqq b} \left| f(x) - \sum_{k=1}^{n} c_k x^{\lambda_k} \right| .$$

Finally, Theorem 3 continues to hold if $a = 0$, in case $\lambda_1 = 0$.

4. Trigonometric Tchebycheff systems and partial bases.

THEOREM 7 [4, p. 376]. *Let $-\infty < a < b < \infty$ and let $u_0, u_1, \cdots, u_n$ be real functions in $C^n[a, b]$. Then $\{u_k\}_0^n$ is an ECT-system on $[a, b]$ iff, for each $k = 0, 1, \cdots, n$ and each $x \in [a, b]$,*

$$W(u_0, \cdots, u_k)(x) = \begin{vmatrix} u_0(x) & u_0'(x) & \cdots\cdots & u_0^{(k)}(x) \\ u_1(x) & u_1'(x) & \cdots\cdots & u_1^{(k)}(x) \\ \vdots & \vdots & & \vdots \\ u_k(x) & u_k'(x) & \cdots\cdots & u_k^{(k)}(x) \end{vmatrix} > 0 .$$

In what follows we shall use the following fact. Let $a_0, \cdots, a_n$ $(n \geqq 1)$ be reals and consider the matrix

$$\begin{pmatrix} 1 & a_0 & \cdots\cdots\cdots\cdots & a_0^n \\ 1 & a_1 & \cdots\cdots\cdots\cdots & a_1^n \\ \vdots & \vdots & & \vdots \\ 1 & a_n & \cdots\cdots\cdots\cdots & a_n^n \end{pmatrix} . \tag{3}$$

By adding to the last column a suitable linear combination of the previous ones, we can obtain the matrix

$$(4)\qquad \begin{pmatrix} 1 & a_0 & \cdots\cdots\cdots\cdots & a_0^{n-1} & 0 \\ \vdots & \vdots & & \vdots & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & a_{n-1} & \cdots\cdots\cdots & a_{n-1}^{n-1} & 0 \\ 1 & a_n & \cdots\cdots\cdots\cdots & a_n^{n-1} & \prod_{j=0}^{n-1}(a_n - a_j) \end{pmatrix} .$$

THEOREM 8. *Let $0 < \alpha_0 < \alpha_1 \cdots < \alpha_n (n \geqq 0)$. A necessary and sufficient condition that*

$$\cos \alpha_0 x,\ \sin \alpha_0 x,\ -\cos \alpha_1 x,\ -\sin \alpha_1 x,\ \cdots,\ (-1)^n \cos \alpha_n x,\ (-1)^n \sin \alpha_n x$$

be an ECT-system on $[0, \pi]$ is $\alpha_n < 1/2$.

Proof. Necessity. If $\alpha_0 \geqq 1/2$, then $\pi/(2\alpha_0) \in (0, \pi]$, $\cos \alpha_0(\pi/(2\alpha_0)) = 0$, contradicting our hypothesis. Thus we can assume $n > 0$. For $k = 1, 2, \cdots, n$, consider the differential equation $[\prod_{j=0}^{k-1} D^2 + \alpha_j^2]y = 0$ having the linearly independent solutions $\cos \alpha_0 x$, $\sin \alpha_0 x$, $\cdots$, $(-1)^{k-1}\cos \alpha_{k-1}x$, $(-1)^{k-1}\sin \alpha_{k-1}x$ and the (never vanishing in $(-\infty, \infty)$) Wronskian

$$W_{k-1}(x) \equiv \begin{vmatrix} \cos \alpha_0 x & -\alpha_0 \sin \alpha_0 x \cdots\cdots\cdots\cdots & (-1)^k \alpha_0^{2k-1} \sin \alpha_0 x \\ \sin \alpha_0 x & \alpha_0 \cos \alpha_0 x \cdots\cdots\cdots & (-1)^{k-1}\alpha_0^{2k-1}\cos \alpha_0 x \\ -\cos \alpha_1 x & \alpha_1 \sin \alpha_1 x \cdots\cdots\cdots & (-1)^{k-1}\alpha_1^{2k-1}\sin \alpha_1 x \\ \vdots & \vdots & \vdots \\ (-1)^{k-1}\cos \alpha_{k-1}x & (-1)^k \alpha_{k-1} \sin \alpha_{k-1} x \cdots & -\alpha_{k-1}^{2k-1} \sin \alpha_{k-1} x \\ (-1)^{k-1}\sin \alpha_{k-1}x & (-1)^{k-1}\alpha_{k-1} \cos \alpha_{k-1}x \cdots & \alpha_{k-1}^{2k-1} \cos \alpha_{k-1} x \end{vmatrix} .$$

By Theorem 7, $U_k(x) > 0$ on $[0, \pi]$ where $U_k(x)$ is the determinent of the $(2k+1)\times(2k+1)$ matrix whose $(2j+1)$th row $(j = 0, 1, \cdots, k)$ is $(-1)^j$ times the row $\cos \alpha_j x \ \ -\alpha_j \sin \alpha_j x \ \ -\alpha_j^2 \cos \alpha_j x \cdots (-\alpha_j^2)^k \cos \alpha_j x$ and whose $(2j+2)$th row $(j = 0, 1, \cdots, k-1)$ is $(-1)^j$ times the row $\sin \alpha_j x \ \ \alpha_j \cos \alpha_j x \ \ -\alpha_j^2 \sin \alpha_j x \cdots (-\alpha_j^2)^k \sin \alpha_j x$.

By performing on the odd (1st, 3rd, $\cdots$) columns of the last matrix, operations similar to those transforming (3) into (4), we obtain that

$$(5)\quad U_k(x) \equiv \begin{vmatrix} \cos\alpha_0 x \cdots (-1)^k\alpha_0^{2k-1}\sin\alpha_0 x & 0 \\ \sin\alpha_0 x \cdots (-1)^{k-1}\alpha_0^{2k-1}\cos\alpha_0 x & 0 \\ -\cos\alpha_1 x \cdots (-1)^{k-1}\alpha_1^{2k-1}\sin\alpha_1 x & 0 \\ -\sin\alpha_1 x \cdots (-1)^k\alpha_1^{2k-1}\cos\alpha_1 x & 0 \\ \vdots \qquad \vdots & \vdots \\ (-1)^{k-1}\cos\alpha_{k-1}x \cdots -\alpha_{k-1}^{2k-1}\sin\alpha_{k-1}x & 0 \\ (-1)^{k-1}\sin\alpha_{k-1}x \cdots \alpha_{k-1}^{2k-1}\cos\alpha_{k-1}x & 0 \\ (-1)^k\cos\alpha_k x \cdots\cdots \alpha_k^{2k-1}\sin\alpha_k x & \alpha \end{vmatrix} \equiv W_{k-1}(x)(\cos\alpha_k x)\prod_{j=0}^{k-1}(\alpha_k^2-\alpha_j^2)$$

where $\alpha = (-1)^k(\cos\alpha_k x)\prod_{j=0}^{k-1}(\alpha_j^2 - \alpha_k^2)$.

Thus, $\cos\alpha_k x$ has to be $\neq 0$ on $[0, \pi]$ and hence $\alpha_k < 1/2$, and in particular, $\alpha_n < 1/2$.

Sufficiency. For $1 \leqq k \leqq n$ consider again

$$W_k(x) \equiv \begin{vmatrix} \cos\alpha_0 x & -\alpha_0\sin\alpha_0 x \cdots (-\alpha_0^2)^k(-\alpha_0\sin\alpha_0 x) \\ \sin\alpha_0 x & \alpha_0\cos\alpha_0 x \cdots (-\alpha_0^2)^k(\alpha_0\cos\alpha_0 x) \\ \vdots & \vdots \qquad \vdots \\ (-1)^k\cos\alpha_k x & (-1)^{k-1}\alpha_k\sin\alpha_k x \cdots (-\alpha_k^2)^k((-1)^{k-1}\alpha_k\sin\alpha_k x) \\ (-1)^k\sin\alpha_k x & (-1)^k\alpha_k\cos\alpha_k x \cdots (-\alpha_k^2)^k((-1)^k\alpha_k\cos\alpha_k x) \end{vmatrix}.$$

By performing on the even (2nd, 4th, $\cdots$) columns of the last matrix operations similar to those transforming (3) into (4), we obtain that

$$W_k(x) \equiv \begin{vmatrix} \cos\alpha_0 x \cdots\cdots (-1)^k\alpha_0^{2k}\cos\alpha_0 x & 0 \\ \sin\alpha_0 x \cdots\cdots (-1)^k\alpha_0^{2k}\sin\alpha_0 x & 0 \\ \vdots \qquad \vdots & \vdots \\ \vdots \qquad \vdots & 0 \\ \vdots \qquad \vdots & 0 \\ (-1)^k\cos\alpha_k x \cdots\cdots \alpha_k^{2k}\cos\alpha_k x & (-1)^{k-1}\alpha_k(\sin\alpha_k x)\prod_{j=0}^{k-1}(\alpha_j^2-\alpha_k^2) \\ (-1)^k\sin\alpha_k x \cdots\cdots \alpha_k^{2k}\sin\alpha_k x & (-1)^k\alpha_k(\cos\alpha_k x)\prod_{j=0}^{k-1}(\alpha_j^2-\alpha_k^2) \end{vmatrix}.$$

Therefore $W_k(0) = U_k(0)\alpha_k\prod_{j=0}^{k-1}(\alpha_k^2 - \alpha_j^2)$ and by (5),

$$\operatorname{sgn} W_k(0) = \operatorname{sgn} W_{k-1}(0)\ .$$

As $W_0(x) \equiv \alpha_0 > 0$, we have, using (5), that $W_k(x)$ and $U_k(x)$ are >0 on $[0, \pi]$. By Theorem 7 the desired conclusion follows.

LEMMA 9. *Let $-\infty < a < b < \infty$ and let $y_1, y_2, \cdots, y_n$ be real* *functions defined on (a, b) and Lebesgue integrable on each (a, x), $a < x < b$. Suppose each $y_k \in C^{n-1}(a, b)$ and that $\{y_k\}_1^n$ is an ECT-system on (a, b). Then so is $\{z_k\}_1^n$, where*

$$z_k(x) = \int_a^x y_k(t)dt;\ k = 1, 2, \cdots, n;\ a < x < b\ . \tag{6}$$

Proof. Follows from Theorem 7, as for such $k \geqq 2$ and for $x \in (a, b)$,

$$W(z_1, \cdots, z_k)(x) = \int_a^x \begin{vmatrix} y_1(t) & y_1(x) & \cdots & y_1^{(k-2)}(x) \\ \vdots & \vdots & & \vdots \\ y_k(t) & y_k(x) & \cdots & y_k^{(k-2)}(x) \end{vmatrix} dt > 0\ .$$

LEMMA 10. *Assume the first two sentences of Lemma 9 and let*

$$z_k(x) = c_k + \int_a^x y_k(t)dt;\ c_k \text{ real constants};\ k = 1, 2, \cdots, n,\ a \leqq x < b\ . \tag{7}$$

Then $\{1, z_1(x), \cdots, z_n(x)\}$ is an ECT-system on (a, b).

Proof. For $k = 1, 2, \cdots, n$ and $a < x < b$,

$$W(1, z_1, \cdots, z_k)(x) = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ z_1(x) & \cdots & \cdots & z_1^{(k)}(x) \\ \vdots & & & \vdots \\ z_k(x) & \cdots & \cdots & z_k^{(k)}(x) \end{vmatrix}$$

$$= \begin{vmatrix} y_1(x) & \cdots & y_1^{(k-1)}(x) \\ \vdots & \cdots & \vdots \\ \vdots & & \vdots \\ y_k(x) & \cdots & y_k^{(k-1)}(x) \end{vmatrix} > 0\ .$$

LEMMA 11. *Assume the first sentence of Lemma 9 and suppose $\{y_k\}_1^n$ is a T-system on (a, b). Then with (6), so is $\{z_k\}_1^n$.*

Proof. We may assume $n > 1$. Let $a < t_1 < t_2 \cdots < t_n < b$. Then

$$\begin{vmatrix} z_1(t_1) & z_1(t_2) & \cdots & z_1(t_n) \\ z_2(t_1) & z_2(t_2) & \cdots & z_2(t_n) \\ \vdots & \vdots & & \vdots \\ z_n(t_1) & z_n(t_2) & \cdots & z_n(t_n) \end{vmatrix}$$

$$= \begin{vmatrix} \int_a^{t_1} y_1(s_1)ds_1 & \int_{t_1}^{t_2} y_1(s_2)ds_2 & \cdots & \int_{t_{n-1}}^{t_n} y_1(s_n)ds_n \\ \vdots & \vdots & & \vdots \\ \int_a^{t_1} y_n(s_1)ds_1 & \int_{t_1}^{t_2} y_n(s_2)ds_2 & \cdots & \int_{t_{n-1}}^{t_n} y_n(s_n)ds_n \end{vmatrix}$$

$$= \int_a^{t_1} \begin{vmatrix} y_1(s_1) & \int_{t_1}^{t_2} y_1(s_2)ds_2 & \cdots & \int_{t_{n-1}}^{t_n} y_1(s_n)ds_n \\ \vdots & \vdots & & \vdots \\ y_n(s_1) & \int_{t_1}^{t_2} y_n(s_2)ds_2 & \cdots & \int_{t_{n-1}}^{t_n} y_n(s_n)ds_n \end{vmatrix} ds_1 = \cdots$$

$$= \int_a^{t_1}\int_{t_1}^{t_2} \cdots \int_{t_{n-1}}^{t_n} \begin{vmatrix} y_1(s_1) & \cdots & y_1(s_n) \\ \vdots & & \vdots \\ y_n(s_1) & \cdots & y_n(s_n) \end{vmatrix} ds_n \cdots ds_1 > 0 .$$

LEMMA 12. *Assume the first sentence of Lemma* 11. *Then, with* (7), $\{1, z_1, \cdots, z_n\}$ *is a T-system on* $[a, b)$.

Proof. Let $a \leqq t_1 < t_2 < \cdots < t_{n+1} < b$. Then

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1(t_1) & z_1(t_2) & \cdots & z_1(t_{n+1}) \\ \vdots & \vdots & & \vdots \\ z_n(t_1) & z_n(t_2) & \cdots & z_n(t_{n+1}) \end{vmatrix} = \begin{vmatrix} 1 & 0 & \cdots & 0 \\ z_1(t_1) & z_1(t_2)-z_1(t_1) & \cdots & z_1(t_{n+1})-z_1(t_n) \\ \vdots & \vdots & & \vdots \\ z_n(t_1) & z_n(t_2)-z_n(t_1) & \cdots & z_n(t_{n+1})-z_n(t_n) \end{vmatrix}$$

$$= \begin{vmatrix} \int_{t_1}^{t_2} y_1(x_1)dx_1 & \int_{t_2}^{t_3} y_1(x_2)dx_2 & \cdots & \int_{t_n}^{t_{n+1}} y_1(x_n)dx_n \\ \vdots & \vdots & & \vdots \\ \int_{t_1}^{t_2} y_n(x_1)dx_1 & \int_{t_2}^{t_3} y_n(x_2)dx_2 & \cdots & \int_{t_n}^{t_{n+1}} y_n(x_n)dx_n \end{vmatrix}$$

$$= \int_{t_1}^{t_2} \cdots \int_{t_n}^{t_{n+1}} \begin{vmatrix} y_1(x_1) & \cdots & y_1(x_n) \\ \vdots & & \vdots \\ y_n(x_1) & \cdots & y_n(x_n) \end{vmatrix} dx_n \cdots dx_1 > 0 .$$

LEMMA 13. *Let* $-\infty < a < b < \infty$ *and let* $u, u_0, \cdots, u_n (n \geqq 0)$ *be real functions in* $C^n[a, b]$ *such that* $\{u_k\}_0^n$ *is an ECT-system on* $[a, b]$ *and* $u > 0$ *there. Then* $\{uu_k\}_0^n$ *is an ECT-system on* $[a, b]$.

Proof. By an identity for Wronskians and by Theorem 7, as on $[a, b]$, for $k=0, 1, \cdots, n$, $W(uu_0, \cdots, uu_k)=u^{k+1}W(u_0, u_1, \cdots, u_k)>0$, the result follows.

THEOREM 14. *Let* $0 \leqq \alpha_n < \cdots < \alpha_0$. *The following statements are equivalent*: (a) $\alpha_0 \leqq 1/2$. (b) $\{\cos \alpha_{m_k}x\}_1^s$ *is a T-system on* $[0, \pi)$ *for every subsequence* $\{m_1, \cdots, m_s\}$ *of* $\{0, \cdots, n\}$. (c) $\{\cos \alpha_k x\}_0^n$ *is an ECT-system on* $(0, \pi)$ *and a CT-system on* $[0, \pi)$.

Proof. (a) $\Rightarrow$ (c). True for $n = 0$. Suppose true for some $n - 1 \geqq 0$. So $\{\cos \alpha_k x\}_1^n$ is an *ECT*-system on $(0, \pi)$ and a *CT*-system on $[0, \pi)$. Set

$$(8) \qquad y_k(x) \equiv (\alpha_0^2 - \alpha_k^2)\cos \alpha_0 x \cos \alpha_k x\ , \quad k = 1, 2, \cdots, n\ .$$

Then $\{y_k\}_1^n$ is a *CT*-system on $[0, \pi)$ and an *ECT*-system on $(0, \pi)$. For $k = 1, 2, \cdots, n$, let

$$(9) \qquad z_k(x) \equiv \alpha_0 \sin \alpha_0 x \cos \alpha_k x - \alpha_k \cos \alpha_0 x \sin \alpha_k x$$

so that

$$(10) \quad z_k'(x) \equiv y_k(x);\ z_k(x) = \int_0^x y_k(t)dt = \cos^2\alpha_0 x[(\cos \alpha_k x)/\cos \alpha_0 x]'\ , \quad 0 < x < \pi\ .$$

By Lemma 9, $\{z_k\}_1^n$ is an *ECT*-system on $(0, \pi)$ and, in particular, it is a *CT*-system there. By (10) and Lemma 13, $\{[(\cos \alpha_k x)/\cos \alpha_0 x]'\}_{k=1}^n$ is an *ECT*-system on $(0, \pi)$. By Lemmas 10 and 12, $\{1, (\cos \alpha_1 x)/\cos \alpha_0 x, \cdots, (\cos \alpha_n x)/\cos \alpha_0 x\}$ is an *ECT*-system on $(0, \pi)$ and a *CT*-system on $[0, \pi)$. Hence $\{\cos \alpha_k x\}_0^n$ is an *ECT*-system on $(0, \pi)$ and a *CT*-system on $[0, \pi)$.

Clearly, now, (a) $\Rightarrow$ (b). Trivially (b) and likewise (c), implies (a) for if $\{\cos \alpha_0 x\}$ is a *T*-system on $[0, \pi)$, α_0 must be $\leqq 1/2$.

LEMMA 15. *Let* $0 \leqq \alpha_n < \alpha_{n-1} \cdots < \alpha_0 \leqq 1/2$. *Let* y *be a real function with* $y^{(2n)}$ *continuous in* $[0, \pi)$; $y^{(2k-1)}(0) = 0, k = 1, 2, \cdots, n$ (*if* $n > 0$), *and suppose* $(\cos \alpha_n x)^{-1}[\prod_{k=0}^{n-1} D^2 + \alpha_k^2]y$ (*meaning* $(\cos \alpha_0 x)^{-1}y$ *if* $n = 0$) *is strictly increasing on* $(0, \pi)$ (*hence on* $[0, \pi)$). *Then* $\{\cos \alpha_0 x, \cdots, \cos \alpha_n x, y\}$ *is a T-system on* $[0, \pi)$.

Proof. True for $n = 0$. Suppose true for some $n - 1 \geqq 0$. Applying it to $\alpha_1, \cdots, \alpha_n$ and to $y'' + \alpha_0^2 y$, we obtain that $\{\cos \alpha_1 x, \cdots, \cos \alpha_n x, y'' + \alpha_0^2 y\}$ is a *T*-system on $[0, \pi)$. Hence, with (8), so is $\{y_1, \cdots, y_n, (y'' + \alpha_0^2 y)\cos \alpha_0 x\}$. Let, on $[0, \pi)$, $z_{n+1}(x) = \alpha_0 y \sin \alpha_0 x + y' \cos \alpha_0 x$. We have there, $z_{n+1}'(x) = (y'' + \alpha_0^2 y)\cos \alpha_0 x$, $z_{n+1}(x) = \int_0^x (y'' + \alpha_0^2 y)\cos \alpha_0 x dx$. Hence, with (9), we have by (10) and Lemma 11 that $\{z_k\}_1^{n+1}$ is a T-system on $(0, \pi)$. By (10) and the fact that $[y/\cos\alpha_0 x]' = (\cos^{-2}\alpha_0 x)z_{n+1}(x)$, $0 < x < \pi$, so is $\{[(\cos\alpha_1 x)/\cos\alpha_0 x]', \cdots, [(\cos\alpha_n x)/\cos\alpha_0 x]', [y/\cos\alpha_0 x]'\}$. By Lemma 12, $\{1, (\cos\alpha_1 x)/\cos\alpha_0 x, \cdots, (\cos\alpha_n x)/\cos\alpha_0 x, y/\cos\alpha_0 x\}$ is a T-system on $[0, \pi)$. Hence so is $\{\cos\alpha_0 x, \cos\alpha_1 x, \cdots, \cos\alpha_n x, y\}$.

LEMMA 16. *Let* $0 \leqq \alpha_n < \alpha_{n-1} \cdots < \alpha_0 \leqq 1/2 (n \geqq 0)$. *Let* y *be a real function with* $y^{(2k+1)}(0) = 0$, $k = 0, 1, \cdots, n$; $y^{(2n+1)}$ *continuous at* 0 *from the right and* $y^{(2k)} > 0$ *on* $(0, \pi)$, *for* $k = 0, 1, \cdots, n+1$. *Then, for every subsequence* $\{m_1, m_2, \cdots, m_s\}$ *of* $\{0, 1, \cdots, n\}$, $\{\cos\alpha_{m_1}x, \cdots, \cos\alpha_{m_s}x, y\}$ *is a* T-*system on* $[0, \pi)$.

Proof. Consider $[\prod_{k=1}^{s-1} D^2 + \alpha_{m_k}^2]y$ (meaning y if $s = 1$) $\equiv \sum_{k=0}^{s-1} a_k D^{2k} y$; all a_k are $\geqq 0$, $a_{s-1} = 1$. By Lemma 15, it is enough to show that, for $k = 0, 1, \cdots, s-1$, $z_k(x) \equiv (\cos\alpha_{m_s}x)^{-1}y^{(2k)}$ is strictly increasing on $(0, \pi)$. But there,

$$z_k'(x) = \cos^{-2}\alpha_{m_s}x[y^{(2k+1)}(x)\cos\alpha_{m_s}x + \alpha_{m_s}y^{(2k)}(x)\sin\alpha_{m_s}x] > 0 .$$

THEOREM 17. *Let* $0 \leqq \alpha_{N-1} < \alpha_{N-2} \cdots < \alpha_0 < 1/2$ *and let* $1 \leqq n < N$, n *an integer. Let* f *be a real function with* $f^{(2k+1)}(0) = 0$, $k = 0, 1, \cdots, N-1$, *and* $f^{(2k)}(x) > 0$ *on* $(0, \pi]$ *for* $k = 0, 1, \cdots, N$. *Assume* $f^{(2N-1)}(x)$ *is continuous from the right at* 0. *Let* $0 \leqq m_1 < m_2 \cdots < m_n < N$ *be integers*, $\{m_1, m_2, \cdots, m_n\} \neq \{N-n, N-n+1, \cdots, N-1\}$. *Let* $1 \leqq p \leqq \infty$. *Then*

$$\min_{c_k \text{ real}} \left\| f(x) - \sum_{k=N-n}^{N-1} c_k \cos\alpha_k x \right\|_{L^p(0,\pi)} < \min_{c_k \text{ real}} \left\| f(x) - \sum_{k=1}^{n} c_k \cos\alpha_{m_k}x \right\|_{L^p(0,\pi)} . \tag{11}$$

Proof. Let $\{m_1, \cdots, m_s\}$ be a subsequence of $\{0, \cdots, N-1\}$. Then, by Theorem 14, $\{\cos(2\alpha_0)^{-1}\alpha_{m_k}x\}_1^s$ is a T-system on $[0, \pi)$, and therefore $\{\cos\alpha_{m_k}x\}_1^s$ is a T-system on $[0, (2\alpha_0)^{-1}\pi)$ and hence on $[0, \pi]$. Redefine f, for $x > \pi$, as $\sum_{j=0}^{2N} f^{(j)}(\pi)(x-\pi)^j/j!$ and observe that now $f^{(2k)} > 0$ on $(0, \infty)$ for $k = 0, 1, \cdots, N$. By Lemma 16, $\{\cos(2\alpha_0)^{-1}\alpha_{m_1}x, \cdots, \cos(2\alpha_0)^{-1}\alpha_{m_s}x, f((2\alpha_0)^{-1}x)\}$ is a T-system on $[0, \pi)$, and therefore $\{\cos\alpha_{m_1}x, \cdots, \cos\alpha_{m_s}x, f(x)\}$ is a T-system on $[0, (2\alpha_0)^{-1}\pi)$ and hence on $[0, \pi]$. We can use now Theorem 1 to obtain (11), observing (as in the proof of Theorem 4) that if $n = 1$, our positivity hypothesis on f suffices for our purpose.

EXAMPLE. Let $0 \leqq \alpha_{N-1} < \alpha_{N-2} \cdots < \alpha_0 < 1/2$, $1 \leqq n < N$, n and $M(\geqq 2N)$ integers. If $0 \leqq m_1 < \cdots < m_n < N$ are integers,

$\{m_1, m_2, \cdots, m_n\} \neq \{N-n, N-n+1, \cdots, N-1\}$ and $1 \leqq p \leqq \infty$, then

$$\min_{c_k \text{ real}} \left\| x^M - \sum_{k=N-n}^{N-1} c_k \cos \alpha_k x \right\|_{L^p(0,\pi)} < \min_{c_k \text{ real}} \left\| x^M - \sum_{k=1}^{n} c_k \cos \alpha_{m_k} x \right\|_{L^p(0,\pi)} .$$

REFERENCES

1. I. Borosh, C. K. Chui and P. W. Smith, *Best uniform approximation from a collection of subspaces*, Math. Z., **156** (1977), 13-18.
2. D. Braess, *Chebyshev approximation by γ-polynomials*, J. Approximation Theory, **9** (1973), 20-43.
3. S. Karlin, *Total Positivity*, Vol I, Stanford University Press, Stanford, 1968.
4. S. Karlin and W. J. Studden, *Tchebycheff Systems: With Applications in Analysis and Statistics*, Interscience, New York, 1966.
5. G. G. Lorentz, *Approximation by incomplete polynomials (problems and results)*, in "Padé and Rational Approximation: Theory and Applications", edited by E. B. Saff and R. S. Varga, Academic Press, New York, (1977), 289-302.
6. K. M. Levasseur and J. T. Lewis, *Antenna design as a partial basis problem*, Technical Report No. 66, Department of Mathematics, University of Rhode Island, June 1977.
7. G. Meinardus, *Über ein Problem von L. Collatz*, Computing, **8** (1971), 250-254.
8. P. W. Smith, *An improvement theorem for Descartes systems*, Proc. Amer. Math. Soc., **70** (1978), 26-30.

Received April 11, 1978 and in revised form March 21, 1979. Research sponsored by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Grant No. AFOSR 77-3174. The United States Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright notation hereon.

UNIVERSITY OF RHODE ISLAND
KINGSTON, RI 02881