DIFFERENTIAL SYSTEMS WITH IMPULSIVE PERTURBATIONS

S. G. PANDIT

Properties of solutions of measure differential equations are investigated with emphasis on the impulses. A variation of parameters formula, expressing solutions of nonlinear measure differential system in terms of the solution of linear measure differential system and the strength of the impulses, is developed and a result on the asymptotic stability is established.

1. Introduction. Measure differential equations have been investigated by Das and Sharma [3], Leela [4, 5], Raghavendra and Rao [6] and Schmaedeke [7], among others. These equations provide good models for many a physical and biological system. The fact that their solutions are discontinuous renders the conventional methods of ordinary differential equations unapplicable, and thus their study becomes interesting. In [3-6], the equation

$$(1.1) \qquad Dx = F(t, x) + G(t, x)Du$$

is studied as an impulsively perturbed system of the ordinary differential equation

$$x' = F(t, x) \qquad \left(' = \frac{d}{dt} \right).$$

In [7], it is investigated from the view point of optimal control theory, that is, G is assumed to be independent of x. In this paper, we are concerned with the system

$$(1.2) \qquad Dx = f(t, x) + AxDu + g(t, x)Du ,$$

which is treated as a perturbed system of the linear system

$$(1.3) \qquad Dx = AxDu .$$

This gives a more clear picture of the effect of impulses on the behavior of solutions. Deviations from the conventional theory, which are obviously expected, are noted in particular.

Theorem 2.1 indicates the possible abrupt behavior of solutions of (1.3) at the points of discontinuity of u (see also Remark 2.3 and Example 3.1). Theorem 3.1 is a "variation of parameters formula" for the system (1.2). Using Theorem 3.1 and an auxilliary result (Lemma 2.1), we obtain in Theorem 3.2, asymptotic stability

of the null solution of (1.2), employing the techniques of Brauer [1] and Strauss and Yorke [8]. When the impulses cease to act, that is when u is an absolutely continuous function, our results reduce to the corresponding ones known for ordinary differential equations [2].

2. **Preliminaries and basic results.** Let $J = [t_0, \infty)$, $t_0 \geqq 0$ and R^n denote the n-Euclidean space with any convinient norm $|\cdot|$. The same symbol will be used to denote the norm of an n by n matrix. Consider (1.2) where $x \in R^n$, A is an n by n matrix, $u: J \to R$ is a right-continuous function of bounded variation on every compact sub-interval of J, $f: J \times R^n \to R^n$ is Lebesgue integrable, $g: J \times R^n \to R^n$ is integrable with respect to the Lebesgue-Stieltjes measure du and Dx, Du denote the distributional derivatives of x and u respectively.

A function $x(t) = x(t, t_0, x_0)$ is a solution of (1.2) on J if and only if it satisfies the integral equation

$$(2.1) \quad x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds + \int_{t_0}^{t} [Ax(s) + g(s, x(s))]du(s) .$$

For the proof of this and for the definition of solution of (1.2), along with other relevent details, see [3].

REMARK 2.1. In equation (1.2), $f(t, x) + AxDu + g(t, x)Du$ is identified with the derivative (in the sense of distributions) of

$$\int_{t_0}^{t} f(s, x(s))ds + \int_{t_0}^{t} [Ax(s) + g(s, x(s))]du(s) .$$

When u is an absolutely continuous function, it has the identification $f(t, x) + [Ax + g(t, x)]u'$, where u' is the ordinary derivative (which exists a.e. on J) of u. In particular, if $u' \equiv 1$, (1.2) reduces to the conventional system $x' = f(t, x) + Ax + g(t, x)$.

Let $t_1 < t_2 < \cdots$ denote the discontinuities of u such that $t_1 > t_0$ and $t_k \to \infty$ as $k \to \infty$. Suppose further that these discontinuities are isolated. Throughout this paper, except in Lemma 2.1 (in which u may be any function of bounded variation), we assume that u has the form

$$(2.2) \quad u(t) = t + \sum_{k=1}^{\infty} a_k H_k(t); \quad H_k(t) = \begin{cases} 0, & \text{for } t < t_k \\ 1, & \text{for } t \geqq t_k \end{cases}$$

where a_k are real numbers. Generally, a right-continuous function of bounded variation contains an absolutely continuous part and a singular part. The latter usually resembles $\sum_{k=1}^{\infty} a_k H_k(t)$ when the

discontinuities (which are obviously countable) are isolated. The above assumption is thus reasonable. Moreover, in this case, the predominent effect of the impulses can be visualized. It follows from (2.2) that

$$Du = 1 + \sum_{k=1}^{\infty} a_k \delta(t_k)$$

where $\delta(t_k)$ is the Dirac measure concentrated at t_k. Note that $u' \equiv 1$ a.e. on J. For any $t \in J$, it is clear that there is a unique integer $k \geq 1$ such that $t \in [t_{k-1}, t_k)$.

Let $B_k = E - a_k A$, $k = 1, 2, \cdots$ where E is the identity n by n matrix. From the assumptions on u, it is easy to establish the following.

THEOREM 2.1. *Let B_k be nonsingular for each $k = 1, 2 \cdots$. Then, for $t \in [t_{k-1}, t_k)$ and any $x_0 \in R^n$, the (unique) solution $x(t) = x(t, t_0, x_0)$ of (1.3) is given by*

$$(2.3) \qquad x(t) = \left(\prod_{i=1}^{k-1} B_{k-i}^{-1} \right) e^{(t-t_0)A} x_0 \, .$$

(*Here the product $\prod_{i=1}^{k-1}$ is to be understood as E if $k = 1$.*)

REMARK 2.2. If $a_k = 0$ for all k, then $B_k(=E)$ is clearly invertible. In this case, (2.3) reduces to $x(t) = e^{(t-t_0)A} x_0$, which obviously solves $x' = Ax$. On the other hand, if $a_k \neq 0$ for some k, then a sufficient condition for B_k to be invertible is that a_k^{-1} is not an eigenvalue of A.

REMARK 2.3. Suppose a_k^{-1} is an eigenvalue of A for some k. Then, in general, the solution $x(t)$ of (1.3) does not exist at $t = t_k$. If $x_0 = 0$, then $x(t)$ is arbitrarily determined at $t = t_k$.

We need the following lemma which is similar to Lemma 2 in [1] or Lemma 3.6 in [8], when u is an absolutely continuous function.

LEMMA 2.1. *Let u be a scalar function of bounded variation on $[t_0, T]$ and let v denote the total variation function of u. Suppose that r and p are non-negative, scalar functions such that r is integrable and p is dv-integrable on $[t_0, T]$. Then, for any positive constants c and M, the inequality*

$$(2.4) \qquad r(t) \leq c + \int_{t_0}^{t} Mr(s)ds + \int_{t_0}^{t} p(s)dv(s), \quad t \in [t_0, T]$$

implies

$$(2.5) \qquad r(t) \leqq ce^{M(t-t_0)} + \int_{t_0}^{t} e^{M(t-s)} p(s)dv(s), \quad t \in [t_0, T] .$$

Proof. Clearly, $r(t) \leqq y(t)$, where $y(t)$ is the maximal solution of the integral equation

$$(2.6) \qquad y(t) = c + \int_{t_0}^{t} My(s)ds + \int_{t_0}^{t} p(s)dv(s), \quad t \in [t_0, T] .$$

Therefore, it is enough to show that any solution of (2.6) satisfies the inequality

$$(2.7) \qquad y(t) < (c + \delta)e^{M(t-t_0)} + \int_{t_0}^{t} e^{M(t-s)} p(s)dv(s) = z(t)$$

for $t \in [t_0, T]$ and for every $\delta > 0$. This will obviously follow if we show that $z(t)$ in (2.7) is a solution of the equation

$$(2.8) \qquad z(t) = (c + \delta) + \int_{t_0}^{t} Mz(s)ds + \int_{t_0}^{t} p(s)dv(s) .$$

Here, for the right hand side of (2.8), we obtain

$$(2.9) \quad \begin{aligned} &(c + \delta) + \int_{t_0}^{t} Mz(s)ds + \int_{t_0}^{t} p(s)dv(s) \\ &= (c + \delta) + \int_{t_0}^{t} M(c + \delta)e^{M(s-t_0)}ds + \int_{t_0}^{t} Me^{Ms} \left\{ \int_{t_0}^{s} e^{-M\tau} p(\tau)dv(\tau) \right\} ds \\ &\quad + \int_{t_0}^{t} p(s)dv(s) . \end{aligned}$$

Denote the first two integrals on the right hand side of (2.9) by I' and I'' respectively. Then

$$(2.10) \qquad I' = (c + \delta)e^{M(t-t_0)} - (c + \delta) ,$$

and, integration by parts yields

$$(2.11) \quad \begin{aligned} I'' &= \int_{t_0}^{t} \left\{ \int_{t_0}^{s} e^{-M\tau} p(\tau)dv(\tau) d(e^{Ms}) \right. \\ &= \int_{t_0}^{t} e^{M(t-s)} p(s)dv(s) - \int_{t_0}^{t} e^{Ms} d \left(\int_{t_0}^{s} e^{-M\tau} p(\tau)dv(\tau) \right) \\ &= \int_{t_0}^{t} e^{M(t-s)} p(s)dv(s) - \int_{t_0}^{t} p(s)dv(s) . \end{aligned}$$

From (2.9)-(2.11), it follows that the equation (2.8) reduces to identity, which of course, was our objective. This completes the proof of the lemma.

3. Variation of parameters formula and a stability result.
It is well known that an important technique in obtaining the

qualitative properties of solutions of linear and nonlinear ordinary differential equations under perturbations, is through the use of the variation of parameters formula. The theorem that follows gives an analytic expression for solutions of (1.2) in terms of solution of (1.3) and the strength of the impulses a_k. In the absence of the impulses, the result reduces to the well known formula for ordinary differential equations [2].

THEOREM 3.1. *Let the conditions of Theorem 2.1 hold. Then, for $t \in [t_{k-1}, t_k)$, any solution $y(t) = y(t, t_0, x_0)$ of (1.2) is given by*

$$(3.1) \quad \begin{aligned} y(t) = x(t) &+ \int_{t_0}^{t} e^{(t-s)A} f(s, y(s)) ds + \int_{t_0}^{t} e^{(t-s)A} g(s, y(s)) du(s) \\ &+ e^{tA} \sum_{i=1}^{k-1} a_i \Big(\prod_{j=1}^{k-i} B_{k-j}^{-1} \Big) A(I_i + J_i) \end{aligned}$$

where $x(t)$ is given by (2.3) and

$$I_i = \int_{t_0}^{t_i} e^{-sA} f(s, y(s)) ds; \quad J_i = \int_{t_0}^{t_i} e^{-sA} g(s, y(s)) du(s), \quad 1 \leq i \leq k - 1 .$$

Proof. Since $u(t) = t$ for $t \in [t_0, t_1)$, we have

$$(3.2) \quad \begin{aligned} y(t) = e^{(t-t_0)A} x_0 &+ \int_{t_0}^{t} e^{(t-s)A} f(s, y(s)) ds \\ &+ \int_{t_0}^{t} e^{(t-s)A} g(s, y(s)) du(s), \quad t \in [t_0, t_1) . \end{aligned}$$

At $t = t_1$, (2.1) gives

$$(3.3) \quad \begin{aligned} y(t_1) = y(t_1 - h) &+ \int_{t_1-h}^{t_1} f(s, y(s)) ds \\ &+ \int_{t_1-h}^{t_1} [Ay(s) + g(s, y(s))] du(s) \end{aligned}$$

where $h > 0$. Letting $h \to 0_+$ and using the fact that

$$\lim_{h \to 0_+} \int_{t_1-h}^{t_1} f(s, y(s)) ds = 0 ,$$

we obtain from (3.2) and (3.3),

$$(3.4) \quad \begin{aligned} y(t_1) = e^{t_1 A} \Big[e^{-t_0 A} x_0 + I_1 &+ \int_{t_0}^{t_1^-} e^{-sA} g(s, y(s)) du(s) \Big] \\ &+ a_1 Ay(t_1) + a_1 g(t_1, y(t_1)) . \end{aligned}$$

Now,

$$\int_{t_0}^{t_1^-} e^{(t_1-s)A} g(s, y(s)) du(s) + a_1 g(t_1, y(t_1)) = e^{t_1 A} J_1 .$$

Therefore, in view of the facts that B_1 is inverticle, and B_1^{-1} and $e^{t_1 A}$ commute with each other, (3.4) yields

$$
\begin{aligned}
(3.5) \quad y(t_1) &= B_1^{-1} e^{t_1 A}[e^{-t_0 A} x_0 + I_1 + J_1] \\
&= x(t_1) + e^{t_1 A}[I_1 + J_1 + a_1 B_1^{-1} A(I_1 + J_1)] .
\end{aligned}
$$

For $t \in [t_1, t_2)$, we know that

$$
\begin{aligned}
y(t) = e^{(t-t_1) A} y(t_1) &+ \int_{t_1}^{t} e^{(t-s) A} f(s, y(s)) ds \\
&+ \int_{t_1}^{t} e^{(t-s) A} g(s, y(s)) du(s) ,
\end{aligned}
$$

where $y(t_1)$ is determined by (3.5). Thus

$$
\begin{aligned}
y(t) = x(t) &+ \int_{t_0}^{t} e^{(t-s) A} f(s, y(s)) ds + \int_{t_0}^{t} e^{(t-s) A} g(s, y(s)) du(s) \\
&+ e^{t A} a_1 B_1^{-1} A(I_1 + J_1), \quad t \in [t_1, t_2) .
\end{aligned}
$$

As above, it can be shown that

$$
y(t_2) = e^{t_2 A}\left[I_2 + J_2 + \sum_{i=1}^{2} a_i \left(\prod_{j=1}^{3-i} B_{3-j}^{-1} \right) A(I_i + J_i) \right] .
$$

In general, for $t \in [t_{k-1}, t_k)$, (3.1) follows by induction, completing the proof of the theorem.

Assume the following hypothesis:

(H$_1$) given any $\varepsilon > 0$, there exist $\delta(\varepsilon) > 0$ and $T(\varepsilon) > t_0$ such that

$$
|f(t, x)| \leqq \varepsilon |x|, \text{ for } |x| \leqq \delta(\varepsilon) \text{ and } t \geqq T(\varepsilon) ;
$$

(H$_2$) g satisfies

$$
|g(t, x)| \leqq p(t), \text{ for } t \geqq t_0 \text{ and } |x| \leqq r, \ r > 0
$$

where p is a dv-integrable ($v(t)$ is the total variation function of $u(t)$ on $[t_0, t]$, $t \in J$) function such that

$$
(3.6) \qquad\qquad \int_{t_0}^{\infty} p(s) dv(s) < \infty ;
$$

(H$_3$) there exist constants P and Q such that

$$
\left(\prod_{i=1}^{k} |B_i^{-1}| \right) \text{ and } \sum_{i=1}^{k} |a_i A| \left(\sum_{j=i}^{k} |B_j^{-1}| \right)
$$

are bounded by P and Q respectively, as $k \to \infty$.

Note that, for each $c > 0$, (3.6) implies

$$
(3.7) \qquad\qquad \lim_{t \to \infty} e^{-ct} \int_{t_0}^{t} e^{cs} p(s) dv(s) = 0 .
$$

THEOREM 3.2. *Let* (H_1)-(H_2) *hold. Suppose that all the characteristic roots of A have negative real parts. Then, under the conditions of Theorem* 3.1, *there exist T_0 and $\delta > 0$ such that for every $t_0 \geqq T_0$ and x_0 with $|x_0| < \delta$, any solution $y(t) = y(t, t_0, x_0)$ of* (1.2) *satisfies $|y(t)| \to 0$ as $t \to \infty$. In particular, if* (1.2) *possesses the null solution, then it is asymptotically stable.*

Proof. Let $t \in J$ be arbitrary. Then there is an index k such that $t \in [t_{k-1}, t_k)$. Since all the characteristic roots of A have negative real parts, there are positive constants K and α such that $|e^{tA}| \leqq Ke^{-\alpha t}$, for all $t > 0$. Let $0 < \varepsilon < \min(\alpha M^{-1}, r)$, where $M = K(Q+1)$. By (H_1), choose $T(\varepsilon)$ and $\delta(\varepsilon)$ so that $T(\varepsilon) > t_0$ and $\delta(\varepsilon) \leqq \varepsilon$. Select $T_0 \geqq T(\varepsilon)$ so large that (by (3.7)),

$$\int_{t_0}^{t} e^{-(\alpha - M\varepsilon)(t-s)} p(s) dv(s) < (2M)^{-1} \delta(\varepsilon), \text{ for } t \geqq T_0 .$$

Let $\delta = (2PK)^{-1}\delta(\varepsilon)$, and consider any $t_0 \geqq T_0$ and x_0 satisfying $|x_0| < \delta$. From (2.3), (3.1) and the conditions of the theorem, we then have

$$|y(t)| \leqq PK\delta e^{-\alpha(t-t_0)} + \int_{t_0}^{t} M\varepsilon e^{-\alpha(t-s)} |y(s)| ds$$
$$+ \int_{t_0}^{t} Me^{-\alpha(t-s)} p(s) dv(s) ,$$

as long as $|y(t)| < \delta(\varepsilon)$. By Lemma 2.1, this gives

$$|y(t)| \leqq PK\delta e^{-(\alpha - M\varepsilon)(t-t_0)} + M \int_{t_0}^{t} e^{-(\alpha - M\varepsilon)(t-s)} p(s) dv(s) ,$$

from which the conclusion of the theorem follows in the usual way (see [1, 8]).

Theorem 3.2 remains valid if the condition (3.6) in (H_2) is replaced by a more general condition

$$\int_{t}^{t+1} p(s) dv(s) \longrightarrow 0 \text{ as } t \longrightarrow \infty .$$

EXAMPLE 3.1. Let $J = [0, \infty)$. Consider the system (1.3) where

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } A = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix} .$$

Let $t_k = k$ and $a_k = k^{-1}$ for $k = 1, 2, \cdots$. Then a_2^{-1} is an eigenvalue of A but a_1^{-1} is not. Choose $x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. It can be verified that the solution of (1.3) through $(0, x_0)$ on the interval $[0, 2)$ is given by

$$x_1(t) = \begin{cases} 5te^{2t}, & 0 \leqq t < 1 \\ 5(1-t)e^{2t}, & 1 \leqq t < 2 \end{cases}, \quad x_2(t) = \begin{cases} e^{2t}, & 0 \leqq t < 1 \\ -e^{2t}, & 1 \leqq t < 2 \end{cases}.$$

However, $x(t)$ does not exist at $t = 2$. If we choose $x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $x(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for $0 \leqq t < 2$, whereas $x(2) = \begin{bmatrix} c \\ 0 \end{bmatrix}$, c an arbitrary constant.

EXAMPLE 3.2. Let $J = [1, \infty)$. Choose $A = -1$, $t_k = k$ and $a_k = 2(k^3 - 1)^{-1}$ for $k = 2, 3, \cdots$. Then the hypothesis (H_3) is satisfied. Indeed, we have

$$\prod_{k=2}^{\infty} |B_k^{-1}| = \prod_{k=2}^{\infty} (1 - 2(k^3 + 1)^{-1}) = 2/3$$

and

$$\lim_{k \to \infty} \sum_{i=2}^{k} |a_i A| \left(\prod_{j=1}^{k} \left| B_j^{-1} \right| \right) \sum_{k=2}^{\infty} 2k^{-2} < \infty .$$

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UNIVERSITY OF BOMBAY
PANAJI-GOA, 403001, INDIA