PACIFIC JOURNAL OF MATHEMATICS
Vol. 86, No. 2, 1980

DIRECT LIMIT GROUPS AND THE KEESLING-MARDESIC SHAPE FIBRATION

K. R. GOODEARL AND T. B. RUSHING

We show that the Keesling-Mardešić shape fibration has an uncountable number of fibers of different shape type. This is done by showing that an uncountable number of nonisomorphic groups can arise as direct limits of direct limit sequences having all groups $Z \oplus Z$ and all bonding homomorphisms given by one of the two matrices $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$.

A. Introduction. The notion of shape fibration has been developed by Mardešić and Rushing in [4, 5, 6]. In [4] the following question was raised: Let $p: E \to B$ be a shape fibration and let $x, y \in B$ be points belonging to the same component. Do the fibers $p^{-1}(x)$ and $p^{-1}(y)$ have the same shape? In that paper, this was shown to be the case if x and y belong to the same path component, and in [5, Corollary 1], this was shown to be the case if x and y belong to a subcontinuum of B of trivial shape. In [3], Keesling and Mardešić gave an ingenious example of a shape fibration with connected base space and showed that it has two fibers of different shape. We show that their example, in fact, has an uncountable number of fibers of different shape.

Let an inverse limit sequence of spaces be given where each space is $T^2 = S^1 \times S^1$ and each bonding map is given by one of the two matrices $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$. Here the given matrices each induce a continuous homomorphism on R^2 which in turn defines a continuous homomorphism on T^2 via the covering map $e \times e: R^2 \to T^2$ where $e(t) = e^{2\pi i t}$. Let X be the inverse limit space of such a sequence. Then the discussion in [3, §§2, 3] and [3, Lemma 1] implies that the Keesling-Mardešić fibration has a fiber homeomorphic to X.

Consider the direct limit sequence of groups where each group is $Z \oplus Z$ and each bonding map is given by the transpose of the corresponding matrix in the inverse sequence above. Then by [3, §5], the first Čech cohomology group $\check{H}^1(X, Z)$ of X is isomorphic to the direct limit of this sequence. Since Čech cohomology is an invariant of shape type, it suffices to show that there are an uncountable number of isomorphism classes of such direct limit groups.

Thus the groups we are interested in arise as direct limits of sequences

$$G_0 \xrightarrow{f_0} G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \cdots$$

of abelian groups and group homomorphisms, where each $G_n = \boldsymbol{Z} \oplus \boldsymbol{Z}$ and each f_n is given by one of two matrices $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$. (We are viewing the elements of $\boldsymbol{Z} \oplus \boldsymbol{Z}$ as column vectors, so that f_n acts as left multiplication by the chosen matrix.) Our task is to show that among the groups G appearing as direct limits of the above form, there are uncountably many isomorphism classes. This result is folklore among abelian group theorists. An effort is made here to give a presentation that is as accessible as possible.

For convenience in labelling these groups and comparing them, we shall use a few computations within the field of 2-adic numbers. For the reader's convenience, we review the basics of this field and its construction in the following section. Details may be found, for example, in [1, Chapter I] or [2, Chapter V, §5]. We would like to thank Richard S. Pierce for bringing these 2-adic methods to our attention.

B. **2-Adic numbers.** For any rational number x, the *2-adic valuation of* x, denoted $\upsilon_2(x)$, is defined as follows. If $x = 0$, then $\upsilon_2(x) = \infty$. If $x \neq 0$, then x may be uniquely written in the form $x = 2^n a/b$, where n, a, b are integers, a and b are odd, and $b \neq 0$; in this case, $\upsilon_2(x) = n$. Clearly $\upsilon_2(xy) = \upsilon_2(x) + \upsilon_2(y)$, and it may be checked that $\upsilon_2(x+y) \geqq \min\{\upsilon_2(x), \upsilon_2(y)\}$.

The 2-adic valuation $\upsilon_2(-)$ is used to define the *2-adic absolute value* $|\cdot|_2$ on $\boldsymbol{Q}$, by the formula

$$|x|_2 = 2^{-\upsilon_2(x)}\,.$$

(Some authors refer to $|\cdot|_2$, rather than $\upsilon_2(-)$, as the 2-adic valuation on $\boldsymbol{Q}$.) The basic properties of $\upsilon_2(-)$ translate into the following basic properties of $|\cdot|_2$:

(i) $|x|_2$ is a nonnegative real number, and $|x|_2 = 0$ if and only if $x = 0$.

(ii) $|xy|_2 = |x|_2 |y|_2$.

(iii) $|x+y|_2 \leqq \max\{|x|_2, |y|_2\} \leqq |x|_2 + |y|_2$.

It is clear from these properties that the rule $\delta_2(x, y) = |x - y|_2$ defines a metric on $\boldsymbol{Q}$, called the *2-adic metric*.

We shall use $\boldsymbol{Q}_2^*$ to denote the completion of $\boldsymbol{Q}$ with respect to the metric δ_2. Since addition and multiplication in $\boldsymbol{Q}$ are uniformly continuous with respect to δ_2, they induce addition and multiplication operations in $\boldsymbol{Q}_2^*$. Then $\boldsymbol{Q}_2^*$ becomes a field, known as the *field of 2-adic numbers*. The completion of $\boldsymbol{Z}$ with respect to δ_2 is then a subring $\boldsymbol{Z}_2^*$ of $\boldsymbol{Q}_2^*$, known as the *ring of 2-adic integers*, and it may be checked that $\boldsymbol{Q} \cap \boldsymbol{Z}_2^* = \boldsymbol{Z}$.

The elements of Z_2^* may be explicitly represented as (limits of) series. Given any sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \cdots$ of zeros and ones, the series $\sum_{k=0}^{\infty} \varepsilon_k 2^k$ converges in Z_2^*; conversely, every element of Z_2^* may be uniquely represented as such a series. In particular, it follows that Z_2^* is uncountable.

C. Change of perspective. Given a group G obtained as a direct limit as in §A, we wish to present G in a form more suitable for computation. Each of the maps f_k is given by a matrix $\begin{pmatrix} 1 & \varepsilon_k \\ 0 & 2 \end{pmatrix}$, where $\varepsilon_k = 0$ or 1. Set $\alpha_n = \sum_{k=0}^{n} \varepsilon_k 2^k$ for all $n = 0, 1, 2, \cdots$, and let α be the 2-adic integer $\sum_{k=0}^{\infty} \varepsilon_k 2^k$.

Let V be a 2-dimensional vector space over $\boldsymbol{Q}$, with basis $\{x_1, x_2\}$. Set $z_0 = x_2$, set

$$z_n = (x_2 - \alpha_{n-1} x_1)/2^n \tag{1}$$

for $n = 1, 2, \cdots$, and for all n let $A_{\alpha,n}$ be the sub*group* of V generated by x_1 and z_n. Observing that

$$z_n = \varepsilon_n x_1 + 2 z_{n+1}\ , \tag{2}$$

we see that $A_{\alpha,n} \subseteqq A_{\alpha,n+1}$. Thus $A_\alpha = \bigcup_{n=0}^{\infty} A_{\alpha,n}$ is a subgroup of V, and we claim that $A_\alpha \cong G$.

Since $A_{\alpha,n}$ is a free abelian group with basis $\{x_1, z_n\}$, there is a group isomorphism $g_n \colon G_n \to A_{\alpha,n}$ given by the rule $g_n \begin{pmatrix} a \\ b \end{pmatrix} = a x_1 + b z_n$. Note, using (2), that the following diagram commutes:

$$\begin{array}{ccccccc}
G_0 & \xrightarrow{f_0} & G_1 & \xrightarrow{f_1} & G_2 & \xrightarrow{f_2} & \cdots \\
\downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \\
A_{\alpha,0} & \xrightarrow{\subseteqq} & A_{\alpha,1} & \xrightarrow{\subseteqq} & A_{\alpha,2} & \xrightarrow{\subseteqq} & .
\end{array}$$

As the direct limits of the top and bottom rows are G and A_α, we obtain $G \cong A_\alpha$, as claimed.

There is a projection $V \to \boldsymbol{Q}$ mapping $x_1 \mapsto 0$ and $x_2 \mapsto 1$. The image of A_α under this projection is the group of all rational numbers of the form $a/2^n$ (where $a, n \in \boldsymbol{Z}$ and $n \geqq 0$), which is not finitely generated. Therefore A_α is not finitely generated.

D. Isomorphic groups. The question now is, for what values of α are the groups A_α isomorphic? Consider another group A_β, where $\beta = \sum_{k=0}^{\infty} \delta_k 2^k$ in Z_2^*, and each $\delta_k = 0$ or 1. Set $\beta_n = \sum_{k=0}^{n} \delta_k 2^k$ for all n. Set $w_0 = x_2$, set

$$w_n = (x_2 - \beta_{n-1} x_1)/2^n \tag{3}$$

for $n = 1, 2, \cdots$, and for all n let $A_{\beta,n}$ be the subgroup of V generated by x_1 and w_n. Then

$$w_n = \delta_n x_1 + 2w_{n+1} \tag{4}$$

for all n, so that $A_{\beta,n} \subseteqq A_{\beta,n+1}$, and we set $A_\beta = \bigcup_{n=0}^{\infty} A_{\beta,n}$.

We claim that if $A_\alpha \cong A_\beta$, then

$$(\alpha r_{12} - r_{22})\beta = r_{21} - \alpha r_{11} \tag{5}$$

for some integers r_{ij}.

Thus assume that there is a group isomorphism $A_\alpha \to A_\beta$. Observing that we may identify $A_\alpha \otimes_Z \boldsymbol{Q}$ and $A_\beta \otimes_Z \boldsymbol{Q}$ with V, we see that the group isomorphism $A_\alpha \to A_\beta$ induces a vector space automorphism $g: V \to V$ such that $g(A_\alpha) = A_\beta$. There exist integers $r_{11}, r_{12}, r_{21}, r_{22}, s$ (with $s \neq 0$) such that

$$g(x_i) = (r_{i1}/s)x_1 + (r_{i2}/s)x_2\ , \tag{6}$$

and we shall prove (5) using these r_{ij}. Note, from combining (6) and (1), that $g(z_0) = (r_{21}/s)x_1 + (r_{22}/s)x_2$ and

$$g(z_n) = [(r_{21} - \alpha_{n-1}r_{11})/2^n s]x_1 + [(r_{22} - \alpha_{n-1}r_{12})/2^n s]x_2 \tag{7}$$

for $n = 1, 2, \cdots$.

E. **The computation.** As $g(A_\alpha) = A_\beta$, there is a nonnegative integer t such that $g(x_1) \in A_{\beta,t}$. For all $n = 0, 1, 2, \cdots$, let $k(n)$ be the least integer such that $k(n) \geqq t$ and $g(z_n) \in A_{\beta,k(n)}$. In view of (2), we see that $g(z_n)$ lies in $A_{\beta,k(n+1)}$, whence $k(n) \leqq k(n+1)$. Thus $k(0) \leqq k(1) \leqq \cdots$ is an increasing sequence of nonnegative integers, and we claim this sequence is unbounded.

If there is an integer k with all $k(n) \leqq k$, then $g(x_1) \in A_{\beta,k}$ and all $g(z_n) \in A_{\beta,k}$, whence $g(A_\alpha) \subseteqq A_{\beta,k}$. Since $A_{\beta,k}$ is finitely generated, this would imply that A_α is finitely generated, which is false. Thus the $k(n)$ are unbounded, as claimed.

In particular, there exists a positive integer N such that $k(n) > t$ for all $n \geqq N$.

As $g(z_n) \in A_{\beta,k(n)}$, we must have

$$g(z_n) = a_n x_1 + b_n w_{k(n)} \tag{8}$$

for some $a_n, b_n \in \boldsymbol{Z}$. If $k(n) > t$ and $b_n = 2c$ for some $c \in \boldsymbol{Z}$, then we see using (4) that

$$g(z_n) = a_n x_1 + c(w_{k(n)-1} - \delta_{k(n)-1}x_1) \tag{9}$$

and so $g(z_n) \in A_{\beta,k(n)-1}$, which contradicts the minimality of $k(n)$. Thus

b_n must be odd whenever $k(n) > t$. In particular, b_n is odd for all $n \geqq N$.

Comparing (7) and (8), with the help of (3), we obtain

$$(2^{k(n)}a_n - \beta_{k(n)-1}b_n)/2^{k(n)} = (r_{21} - \alpha_{n-1}r_{11})/2^n s \tag{10}$$

$$b_n/2^{k(n)} = (r_{22} - \alpha_{n-1}r_{12})/2^n s \tag{11}$$

for all $n \geqq 1$. Cross-multiplying (10) and (11) yields

$$(r_{21} - \alpha_{n-1}r_{11})b_n = (r_{22} - \alpha_{n-2}r_{12})(2^{k(n)}a_n - \beta_{k(n)-1}b_n) . \tag{12}$$

When $n \geqq N$, we have b_n odd and so can divide by it, yielding

$$r_{21} - \alpha_{n-1}r_{11} = (r_{22} - \alpha_{n-1}r_{12})[2^{k(n)}(a_n/b_n) - \beta_{k(n)-1}] . \tag{13}$$

We intend to take the limit of (13) in Q_2^* as $n \to \infty$.

As b_n is odd, $\upsilon_2(b_n) = 0$ and so $|b_n|_2 = 1$. As a_n is an integer, $\upsilon_2(a_n) \geqq 0$ and so $|a_n|_2 \leqq 1$. Consequently,

$$|2^{k(n)}(a_n/b_n)|_2 \leqq |2^{k(n)}|_2 = 2^{-k(n)} . \tag{14}$$

Since the $k(n)$ form an unbounded increasing sequence, $2^{-k(n)} \to 0$, hence $2^{k(n)}(a_n/b_n) \to 0$ in Q_2^*.

Thus we can now compute the limit of (13) as $n \to \infty$, obtaining

$$r_{21} - \alpha r_{11} = (r_{22} - \alpha r_{12})(-\beta) , \tag{14}$$

which is equivalent to (5).

F. Uncountably many groups. Put an equivalence relation $\sim$ on Z_2^*, defined by $\alpha \sim \beta$ if and only if $A_\alpha \cong A_\beta$. If $\alpha \notin Z$ and $\alpha \sim \beta$, then (5) shows that

$$\beta = (r_{21} - \alpha r_{11})/(r_{22} - \alpha r_{12}) \tag{15}$$

for some integers r_{ij}, hence there are only countably many possibilities for β. Thus if $\alpha \notin Z$, then the equivalence class of α is countable. For $\alpha \in Z$, either the equivalence class of α is contained in Z, or it coincides with the equivalence class of some $\beta \notin Z$; in either case, the equivalence class of α is again countable.

Thus all the equivalence classes with respect to $\sim$ are countable. As Z_2^* is uncountable, there must be uncountably many equivalence classes. Therefore there are uncountably many isomorphism classes of the groups A_α.

REFERENCES

1. N. Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta-Functions*, New York, Springer-Verlag, 1977. (Graduate Texts in Math., 58.)

2. N. Jacobson, *Lectures in Abstract Algebra*, Vol. III, Princeton, Van Nostrand, 1964.
3. James Keesling and Sibe Mardešić, *A shape fibration with fibers of different shape*, Pacific J. Math., (to appear).
4. Sibe Mardešić and T. B. Rushing, *Shape fibrations I*, Gen. Top. and its Appl., **9** (1978), 193–215.
5. ———, *Shape fibrations II*, Rocky Mountain J. Math., **9** (2), (1979), 283–298.
6. ———, *n-shape fibrations*, Topology Proceedings, **3** (1978), 429–459.

Received March 30, 1979. Both authors partially supported by National Science Foundation grants.

UNIVERSITY OF UTAH
SALT LAKE CITY, UT 84112