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HERMITIAN LIFTINGS IN ORLICZ SEQUENCE SPACES

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Let M and N be complimentary Orlicz functions satisfying the Δ_2-condition, and let l_M and $l_{(M)}$ be the Orlicz sequence spaces associated with M with the two usual norms. We show that if 2 is not in the associated interval for M, then every essentially Hermitian operator on l_M or $l_{(M)}$ is a compact perturbation of a real diagonal operator.

1. Introduction. If B is a unital Banach algebra, let $S=\{f \in B^*: f(e) = 1 = ||f||\}$ be the state space and for each element $x \in B$, and set $W(x) = \{f(x): f \in S\}$. Let X be a complex Banach space, $B(X)$ the space of bounded linear operators on X, and $C(X)$ the space of compact linear operators on X. The quotient algebra $A(X) = B(X)/C(X)$ is called the Calkin algebra and both $B(X)$ and $A(X)$ are unital Banach algebras. If $T \in B(X)$, the set $W(T)$ is called the *numerical range* of T, and the set $W_e(T) = \bigcap_{K \in C(X)} W(T+K)$ is called *essential numerical range of* T. An operator $T \in B(X)$ is called *Hermitian* if $W(T) \subseteq R$, the real line, and *essentially Hermitian* if $W_e(T) \subseteq R$.

Clearly any compact perturbation of a Hermitian operator $T \in B(X)$ is essentially Hermitian, but the converse is by no means obvious. The converse is easy if X is a Hilbert space, and has been shown to be true if $X = l_p$, $1 \leqq p < \infty$, (cf. [1] and [4]). In this paper, we show the converse is true for those Orlicz sequence spaces X for which 2 is not in the so called associated interval. This term is defined below.

2. Orlicz sequence spaces. We refer the reader to [3] and [6] for references on Orlicz spaces. In [3], Orlicz function spaces are considered, and many of the results translate directly into the sequence space setting.

In this paper, assume that M is a continuous, strictly increasing, convex function defined on $[0, \infty)$, with $M(0) = 0$, and $\lim_{t\to\infty} M(t) = \infty$. Any function M satisfying these properties is called an Orlicz function. The complementary function will be denoted by N. We assume M and N *both* satisfy the Δ_2-condition; that is, there exists $K_0 > 0$ such that $M(2t) \leqq K_0 M(t)$ and $N(2t) \leqq K_0 N(t)$ for all t. By [5, Prop. 2.9], this means there exists $K_1 \geqq 1$ such that

$$(1) \qquad 1 \leqq \frac{tM'(t)}{M(t)} \leqq K_1 \quad \text{and} \quad 1 \leqq \frac{tN'(t)}{N(t)} \leqq K_1$$

for all t.

Since we are assuming the Δ_2-condition, we may further assume that $p \equiv M'$ and $q \equiv N'$ are continuous and strictly increasing (cf. [5], Prop. 2.15). Recall also that p and q are inverse functions of each other.

The following are equivalent norms on the Orlicz sequence spaces:

$$\|\bar{a}\|_M = \|\{a_n\}\|_M = \inf\left\{k: \sum_{n=1}^{\infty} M\left(\frac{|a_n|}{k}\right) \leqq 1\right\}.$$

$$\|\bar{a}\|_{(M)} = \|\{a_n\}\|_{(M)} = \sup\left\{\left|\sum^{\infty} a_n b_n\right| : \sum_{n=1}^{\infty} N(|b_n|) \leqq 1\right\}.$$

Note that $\|\bar{a}\|_M = 1$ if and only if $\sum_{n=1}^{\infty} M(|a_n|) = 1$. Denote by l_M and $l_{(M)}$ the Orlicz sequence spaces endowed with the $\|\cdot\|_M$ and $\|\cdot\|_{(M)}$ norms, respectively. The dual space l_M^* is isometrically isomorphic to $l_{(N)}$ (cf. [6], Prop. 4.b.1), and the dual space $l_{(M)}^*$ is isometrically isomorphic to l_N (cf. [3], p. 135). Because both M and N are assumed to satisfy the Δ_2-condition, l_M (and l_N) are uniformly convex [7, Thm. 1] and thus reflexive (condition (iv) in Theorem 11 of [7] is extraneous in the case of sequence spaces as has been noted in [2, Theorem. 3]).

For each Orlicz function define the following two numbers:

$$\alpha_M = \sup\left\{p: \sup_{0<\lambda, t \leqq 1} \frac{M(\lambda t)}{M(\lambda)t^p} < \infty\right\} \tag{2}$$

$$\beta_M = \inf\left\{p: \inf_{0<\lambda, t \leqq 1} \frac{M(\lambda t)}{M(\lambda)t^p} > 0\right\}. \tag{3}$$

It is easy to see that $1 \leqq \alpha_M \leqq \beta_M \leqq \infty$, and that $\beta_M < \infty$ if and only if M satisfies the Δ_2-condition near 0 (cf. [6, Theorem 4.a.9]). Let α_N and β_N be the values defined as above for the complementary function N. Then it is known that $\alpha_M^{-1} + \beta_N^{-1} = 1$ and $\alpha_N^{-1} + \beta_M^{-1} = 1$ (cf. [6, Theorem 4.b.3]). Hence if M and N satisfy the Δ_2-condition, we have $1 < \alpha_M \leqq \beta_M < \infty$ and $1 < \alpha_N \leqq \beta_N < \infty$. The interval $[\alpha_M, \beta_M]$ is called the *associated interval* for M.

If $2 < \alpha_M \leqq \beta_M < \infty$, r and s can be chosen so that $2 < r < \alpha_M \leqq \beta_M < s < \infty$. Then from (2) there is a constant $K_4 < \infty$ such that

$$\sup_{0<\lambda, t \leqq 1} \frac{M(\lambda t)}{M(\lambda)t^r} = K_4. \tag{4}$$

Using (1), (2) and the fact that $M(\lambda) = \int_0^\lambda p(t)dt \leqq \lambda p(\lambda)$ we have

$$(5)\qquad \sup_{0<\lambda,t\leq 1}\frac{p(\lambda t)}{p(\lambda)t^{r-1}} \leqq \sup_{0<\lambda,t\leq 1}\frac{K_1M(\lambda t)}{\lambda t\lambda^{-1}M(\lambda)t^{r-1}} = K_1K_4 = Q_1 < \infty .$$

Similarly, using (3) and (1), it follows that

$$(6)\qquad \inf_{0<\lambda,t\leq 1}\frac{p(\lambda t)}{p(\lambda)t^{s-1}} = Q_2 > 0 .$$

These inequalities will be used later.

3. Vector states on $B(l_M)$ and $B(l_{(M)})$.

THEOREM 3.1. *If $\bar{a} = \{a_n\}$ is a unit vector in l_M, let $\bar{a}' = \{a'_n\}$, where $a'_n = kp(|a_n|)\operatorname{sgn} a_n$ and $k = ||\{p(|a_n|)\}||^{-1}_{(N)}$. Then the mapping $A \to \langle A\bar{a}, \bar{a}'\rangle$ defines a state on $B(l_M)$. Furthermore, there is a $K_2 > 0$ such that $K_2 \leqq k \leqq 1$ for all unit vectors $\bar{a} \in l_M$.*

Proof. $\bar{a}'$ is a unit vector in $l_{(N)}$ by the definition of k. Now $||\bar{a}||_M = 1$ implies $\sum_{n=1}^{\infty} M(|a_n|) = 1$, and this is the same as $\sum_{n=1}^{\infty} M(q(p(|a_n|))) = 1$. By [3, Theorem 10.4],

$$\langle \bar{a}, \bar{a}'\rangle = \sum_{n=1}^{\infty} a_n kp(|a_n|)\operatorname{sgn}\bar{a}_n = k\sum_{n=1}^{\infty}|a_n|p(|a_n|)$$
$$= k\sum_{n=1}^{\infty}|p(|a_n|)|q(p(|a_n|)) = k||\{p(|a_n|)\}||_{(N)} = 1 .$$

Hence $A \to \langle A\bar{a}, \bar{a}'\rangle$ defines a vector state on $B(l_M)$ for each unit vector $\bar{a} \in l_M$.

Since $||\{p(|a_n|)\}||_{(N)} \geqq 1$, it follows that $k \leqq 1$. Using (1) and the equality above, $\sum|a_n|p(|a_n|) = ||p(|a_n|)||_{(N)}$, it follows that $||\{p(|a_n|)\}||_{(N)} \leqq K_1$. Thus $K_1^{-1} \leqq k \leqq 1$. Take $K_2 = K_1^{-1}$ and the proof is complete.

THEOREM 3.2. *If $\bar{a} = \{a_n\}$ is a unit vector in $l_{(M)}$, let $\bar{a}'' = \{a''_n\}$, where $a''_n = p(k|a_n|)\operatorname{sgn} a_n$ and $k>0$ is chosen so that $\sum N(p(k|a_n|)) = 1$. Then the mapping $A \to \langle A\bar{a}, \bar{a}''\rangle$ defines a state on $B(l_{(M)})$. Furthermore, there is a $K_3 \geqq 1$ such that $1 \leqq k \leqq K_3$ for all unit vectors $\bar{a} \in l_{(M)}$.*

Proof. The proof is similar to that of Theorem 3.1. In this case, note that

$$||\{a''_n\}||_N = 1 = \left|\left|\left\{\frac{1}{k}q(|a''_n|)\right\}\right|\right|_{(M)} = ||\{|a_n|\}||_{(M)} .$$

It follows that $\langle \bar{a}, \bar{a}''\rangle = 1/k||\{q(|a''_n|)\}||_{(M)} = 1$. So $A \to \langle A\bar{a}, \bar{a}''\rangle$ defines a vector state on $B(l_{(M)})$ for each unit vector $\bar{a} \in l_{(M)}$. Also $K_1^{-1} \leqq k^{-1} \leqq 1$, so take $K_3 = K_1$ and the proof is complete.

4. Essentially Hermitian operators on l_M or $l_{(M)}$. Let A be an operator on l_M or $l_{(M)}$ and define

$$r_i(A) = \max\{|\operatorname{Im} z| : z \in W(A)\} .$$

Let $\mathscr{P}$ be the set of projections onto the span of a subset of the canonical basis vectors for l_M or $l_{(M)}$. If $P \in \mathscr{P}$, define $P^\perp = I - P$, where I is the identity operator.

Our first result in this section is trivially true in the l_p spaces $p \neq 2$, $1 < p < \infty$, and is also true for the Orlicz spaces under consideration here. But due to the state structure in l_M the result must be proved. Recall that throughout this paper M and N satisfy the Δ_2-condition and hence that l_M is reflexive and uniformly convex.

LEMMA 4.1. *There is a constant $c > 0$ so that $r_i(PAP) < cr_i(A)$ for all $P \in \mathscr{P}$ and $A \in B(l_M)$.*

Proof. Suppose for a given $A \in B(l_M)$ and $P \in \mathscr{P}$ with $P^\perp$ infinite dimensional that there exists a vector $\sigma = \{\sigma_n\}$ in l_M for which $r_i(PAP) \equiv \delta = \operatorname{Im}\langle PAP\sigma, \sigma'\rangle$. From Theorem 3.1, it follows that $\sigma' = \{kp(|\sigma_n|)\operatorname{sgn}\sigma_n\}$ where $k = \|\{p(|\sigma_n|)\}\|_{(N)}^{-1}$ and that

$$r_i(PAP) = k \operatorname{Im}\langle A\hat{\sigma}, \{p(|\hat{\sigma}_n|)\operatorname{sgn}\hat{\sigma}_n\}\rangle$$

where $\hat{\sigma} = \{\hat{\sigma}_n\}$ satisfies $P\hat{\sigma} = \sigma$ and $P^\perp\hat{\sigma} = 0$. Clearly $\|\hat{\sigma}\| \leqq 1$. We wish to perturb $\hat{\sigma}$ into a unit vector γ for which $\operatorname{Im}\langle A\gamma, \gamma'\rangle \geqq c\delta$ for some $c > 0$, c independent of σ, P and A. Since l_M is reflexive the basis $\{e_i\}$ is shrinking [6]. Furthermore the sequences $\{e_i\}$ and $\{Ae_i\}$ converge weakly to zero. From this it follows that for given $\varepsilon > 0$, there exists an N so that

$$|\langle A(\hat{\sigma}+re_N), (\hat{\sigma}+re_N)'\rangle - k'\langle A\hat{\sigma}, \{p(|\hat{\sigma}_n|)\operatorname{sgn}\hat{\sigma}_n\}\rangle - k'\langle Are_N, p(r)e'_N\rangle| < \varepsilon$$

where $0 \leqq r < 1$ is chosen so that $\|\hat{\sigma} + re_N\| = 1$ and $k' = \|\{p(\hat{\sigma}_n), p(r)\}\|_{(N)}^{-1}$. From Theorem 3.1, $K_2 \leqq k'/k$. Hence it follows that

$$\begin{aligned} &\operatorname{Im}\langle A(\hat{\sigma} + re_N), (\hat{\sigma} + re_N)'\rangle \\ &\qquad \geqq \operatorname{Im}[k'\langle A\hat{\sigma}, \{p(|\hat{\sigma}_n|)\operatorname{sgn}\hat{\sigma}_n\}\rangle + k'\langle Are_N, p(r)e'_N\rangle] - \varepsilon . \end{aligned}$$

So

$$r_i(A) \geqq \frac{k'}{k}[k \operatorname{Im}\langle A\hat{\sigma}, \{p(|\hat{\sigma}_n|)\operatorname{sgn}\hat{\sigma}_n\}\rangle + k \operatorname{Im}\langle Are_N, p(r)e'_N\rangle] - \varepsilon .$$

Now if $|\operatorname{Im}\langle Ae_N, e'_N\rangle| \geqq K_2\delta/2$, the lemma is proved with $c = K_2/2$. So assume $|\operatorname{Im}\langle Ae_N, e'_N\rangle| < K_2\delta/2$ (K_2 as in Theorem 3.1). In this case, note that the quantities r and $kp(r)K_2$ are less than or equal to 1 since $p(r)K_2k < p(r)k' = p(r)/\|\{p(\hat{\sigma}), p(r)\}\|_{(N)}$ and it follows that

$$\begin{aligned} r_i(A) &\geqq \frac{k'}{k}[\delta - krp(r)K_2\delta/2] - \varepsilon \\ &\geqq \frac{k'}{k}[\delta/2] - \varepsilon \geqq K_2\delta/2 - \varepsilon \end{aligned}$$

and the lemma still holds with $c = K_2/2$.

Consider next the case $P \in \mathscr{P}$ with $P^\perp$ finite dimensional. Then P eventually "looks like" the identity. Suppose for such P, $r_i(PAP) > cr_i(A)$ with c as above. Then there exists a unit vector σ such that

$$\operatorname{Im}\langle PAP\sigma, \sigma'\rangle > cr_i(A)$$

and due to the continuity of the inner product assume σ has finite support. The projection P can now be altered to a projection P' for which $P'^\perp$ is infinite dimensional and $\operatorname{Im}\langle P'AP'\sigma, \sigma'\rangle > cr_i(A)$. But this is impossible and so the lemma is valid for all projections.

LEMMA 4.2. *If $2 < \alpha_M$, then there is a constant c_M such that $\sup_{P\in\mathscr{P}}\|PAP^\perp\| \leqq c_M r_i(A)$ for all $A \in B(l_M)$.*

Proof. Let $A \in B(l_M)$ be fixed, and let $\sup_{P\in\mathscr{P}}\|PAP^\perp\| = \alpha$. Assume, without loss of generality, that the supremums of the above expression are attained; that is, there exists some $P \in \mathscr{P}$ and fixed unit vectors $\bar{a} \in l_M$ and $\bar{b}' \in l_{(N)}$ satisfying $\alpha = \langle PAP^\perp\bar{a}, \bar{b}'\rangle$. Letting $\bar{b}$ be associated with $\bar{b}'$ as above (i.e., $\langle\bar{b}, \bar{b}'\rangle = 1$, $\|\bar{b}\| = 1$) assume $P^\perp\bar{a} = \bar{a}$, $P\bar{b} = \bar{b}$. So $\bar{a}$ and $\bar{b}$ have disjoint supports. Let $\hat{\sigma} = c\bar{a} + d\bar{b}$, where c and d are chosen so that $\|\hat{\sigma}\|_M = 1$ and $c \operatorname{sgn}\bar{d} = i|c|$. Since $\sum_{n=1}^\infty M(|c||a_n| + |d||b_n|) = 1$ and M is convex, we must have $|d| \geqq 1 - |c| \geqq 0$.

Now it follow that

$$\begin{aligned} r_i(A) &\geqq |\operatorname{Im}\langle A\bar{\sigma}, \bar{\sigma}'\rangle| \\ &= |\operatorname{Im}\{\langle PAP^\perp\bar{\sigma}, \bar{\sigma}'\rangle + \langle P^\perp AP^\perp\bar{\sigma}, \bar{\sigma}'\rangle + \langle P^\perp AP\bar{\sigma}, \bar{\sigma}'\rangle \\ &\qquad + \langle PAP\bar{\sigma}, \bar{\sigma}'\rangle\}| \\ &\geqq |\operatorname{Im}\{\langle PAP^\perp\bar{\sigma}, \bar{\sigma}'\rangle + \langle P^\perp AP\bar{\sigma}, \bar{\sigma}'\rangle\}| - 2cr_i(A)\,, \end{aligned}$$

where the last inequality follows from Lemma 4.1. Hence letting $c' = 2c + 1$ we have

$$\begin{aligned} c'r_i(A) &\geqq |\operatorname{Im}\{\langle PAP^\perp\bar{\sigma}, \bar{\sigma}'\rangle + \langle P^\perp AP\bar{\sigma}, \bar{\sigma}'\rangle\}| \\ &= \Bigg|\operatorname{Im}\Bigg\{\sum_{n=1}^\infty (PAP^\perp\bar{a})_n ck_1 p(|db_n|)\operatorname{sgn}\overline{db_n} \end{aligned}$$

$$
\begin{aligned}
(7) \qquad & + \sum_{n=1}^{\infty} (P^{\perp}AP\bar{b})_n dk_1 p(|ca_n|) \operatorname{sgn} \overline{ca_n}\Big\} \Big| \\
&= \Big|\mathrm{Im}\Big\{\sum_{n=1}^{\infty} (PAP^{\perp}\bar{a})_n k_2 p(|b_n|) \operatorname{sgn} \bar{b}_n \cdot c \operatorname{sgn} \bar{d}\frac{k_1}{k_2}\frac{p(|db_n|)}{p(|b_n|)} \\
&\quad + \sum_{n=1}^{\infty} (P^{\perp}APb)_n k_3 p(|a_n|) \operatorname{sgn} (\overline{P^{\perp}APb})_n \\
&\quad \times d \operatorname{sgn} \bar{c}\frac{k_1}{k_3}\frac{p(|ca_n|)}{p(|a_n|)}\frac{\operatorname{sgn} \bar{a}_n}{\operatorname{sgn} (\overline{P^{\perp}APb})_n}\Big\} \Big|
\end{aligned}
$$

where k_1, k_2 and k_3 are the positive weights associated with $\bar{\sigma}'$, $\bar{b}'$, $\bar{a}'$ as in Theorem 3.1.

From (5) and (6), the inequality (7) continues as

$$
\begin{aligned}
& c' r_i(A) \geqq \Big|\mathrm{Im}\Big\{\sum_{n=1}^{\infty} (PAP^{\perp}\bar{a})_n k_2 p(|b_n|) \operatorname{sgn} \bar{b}_n \cdot c \operatorname{sgn} \bar{d} \cdot \frac{k_1}{k_2} Q_2 |d|^{s-1}\Big\} \Big| \\
(8) \qquad & \qquad - \sum_{n=1}^{\infty} (P^{\perp}AP\bar{b})_n k_3 p(|a_n|) \operatorname{sgn} (\overline{P^{\perp}AP\bar{b}})_n \cdot |d| \cdot \frac{k_1}{k_3} Q_1 |c|^{r-1}
\end{aligned}
$$

where each term in the second series is nonnegative. Since $c \operatorname{sgn} \bar{d} = |c| i$ it follows from (5) that

$$
\begin{aligned}
& c' r_i(A) \geqq \langle PAP^{\perp}\bar{a}, \bar{b}'\rangle R_2' |c||d|^{s-1} - \langle P^{\perp}AP\bar{b}, \bar{a}''\rangle R_1' |d||c|^{r-1} \\
(9) \qquad & \qquad \geqq \{R_2' |c||d|^{s-1} - R_1' |d||c|^{r-1}\}\alpha \\
& \qquad \geqq \{R_2 |c||d|^{s-1} - R_1 |d||c|^{r-1}\}\alpha
\end{aligned}
$$

where

$$
R_2' = \frac{k_1}{k_2} Q_2 , \qquad R_1' = \frac{k_1}{k_3} Q_1 , \qquad R_2 = K_2 Q_2 , \qquad R_1 = K_2^{-1} Q_1
$$

and $\bar{a}'' = \{k_3 p(|a_n|) \operatorname{sgn} (\overline{P^{\perp}APb})_n\}$. Notice that the constants R_2 and R_1 are independent of the vectors $\bar{\sigma}$, $\bar{a}$ and $\bar{b}$. Now choose $|c|$ so small that

$$
\frac{(1 - |c_0|)^{s-2}}{|c_0|^{r-2}} > 2\frac{R_1}{R_2} .
$$

Then $R_2(1 - |c_0|)^{s-2} > 2R_1 |c_0|^{r-2}$, so $R_2(1 - |c_0|)^{s-2} - R_1 |c_0|^{r-2} > R_1 |c_0|^{r-2}$. Finally, choose c such that $|c| = |c_0|$. Recalling that $|d| \geqq 1 - |c_0|$, it follows that

$$
\begin{aligned}
& R_2 |c||d|^{s-1} - R_1 |d||c|^{r-1} = |c_0||d|(R_2 |d|^{s-2} - R_1 |c_0|^{r-2}) \\
(10) \qquad & \qquad \geqq |c_0||d|(R_2(1 - |c_0|)^{s-2} - R_1 |c_0|^{r-2}) \\
& \qquad \geqq |c_0||d| \, |R_1 |c_0|^{r-2} \\
& \qquad \geqq R_1 |c_0|^{r-1}(1 - |c_0|) .
\end{aligned}
$$

Hence by (9) and (10), we may take $c_M = c'[R_1 |c_0|^{r-1}(1 - |c_0|)]^{-1}$ and the lemma is proved.

LEMMA 4.3. *If* $2 < \alpha_M$, *then there exists a constant* c_M *such that* $\sup_{P \in \mathscr{P}} \|PAP^{\perp}\| \leqq c_M r_i(A)$ *for all* $A \in B(l_{(M)})$.

Proof. The proof is almost identical with the proof of Lemma 4.2, with $\bar{b}'$ replaced with b'' (of Theorem 3.2).

THEOREM 4.4. *If* $2 \notin [\alpha_M, \beta_M]$, *then there exists a constant* c_M *such that* $\sup_{P \in \mathscr{P}} \|PAP^{\perp}\| \leqq c_M r_i(A)$ *for all* $A \in B(l_M)$ *or* $B(l_{(M)})$.

Proof. If $2 < \alpha_M$, the conclusion follows from Lemmas 4.2 and 4.3. If $1 < \alpha_M \leqq \beta_M < 2$, then consider the transpose operator $A^t \in B(l_{(N)})$ or $B(l_N)$. From the above relations between α_M, β_N and β_M, α_N, and since $2 < \alpha_N \leqq \beta_N < \infty$, the conclusion follows from Lemmas 4.2 and 4.3.

REMARK. Theorem 4.4 implies that Hermitian elements in $B(l_M)$ or $B(l_{(M)})$, $2 \notin [\alpha_M, \beta_M]$, must be diagonal with respect to the canonical basis. Results of this type were first obtained by Tam (see [8]).

THEOREM 4.5. *If* $A \in B(l_M)$ *or* $B(l_{(M)})$, *then* $\|A - \operatorname{diam} A\| \leqq 8 \sup_{P \in \mathscr{P}} \|PAP^{\perp}\|$.

The proof of this result requires nothing special about the function M. Indeed, below, we sketch the proof which in detail can be found in [1], Lemmas 3, 4, 5 and 6. Since l_M is reflexive, the canonical basis $\{e_i\}$ is unconditionally monotone and shrinking. From those facts it can be verified that there are diagonal operators $u_k \in B(l_M)$ for which $\bar{u}_k u_k = 1$ and for which the

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{n} (\bar{u}_k A u_k) = \operatorname{diag} A ,$$

with the limit being taken in the w^* topology of $B(l_M)$. With this and the w^*-lower-semicontinuity of the norm it follows that

$$\begin{aligned} \|\operatorname{diag} A - A\| &\leqq \limsup_{n \to \infty} \left\| \sum_{k=1}^{n} \bar{u}_k A u_k - A \right\| \\ &\leqq \limsup_{n \to \infty} \max_{1 \leqq k \leqq n} \|A u_k - u_k A\| \\ &\leqq \sup \{\|SA - AS\| \colon S \text{ is a diagonal operator in } B(l_M),\ \|S\| = 1\} . \end{aligned}$$

Finally, by a result of Arveson [1, Lemma 6], this quantity is shown to be $\leqq 8 \sup_{P \in \mathscr{P}} \|PAP^{\perp}\|$. This completes a sketch of the proof of the theorem.

THEOREM 4.6. *Let $2 \notin [\alpha_M, \beta_M]$. If A is an essentially Hermitian operator in $B(l_M)$ or $B(l_{(M)})$, then there is a real diagonal operator D and a compact operator K such that $A = D + K$.*

Proof. We show that $A - \operatorname{Re}\operatorname{diag} A$ is compact. Suppose that $\operatorname{diag} A = \operatorname{Re}\operatorname{diag} A$, since $\operatorname{Im}\operatorname{diag} A$ must be compact for essentially Hermitian operators. Recall that $P_n^\perp$ is the projection onto span $\{e_{n+1}, e_{n+2}, \cdots\}$. If $r_i((A - \operatorname{re}\operatorname{diag} A)P_n^\perp)$ is not convergent to zero as $n \to \infty$, it is simple to construct a sequence of mutually disjoint norm one vectors v_n for which $\inf_n |\operatorname{Im}\langle (A - \operatorname{Re}\operatorname{diag} A)v_n, v_n'\rangle| = k > 0$. If glim denotes Banach limit, then $\phi(\cdot) \equiv \operatorname{glim}\langle \cdot v_n, v_n'\rangle$ is a state on the Calkin algebra for which $\operatorname{Im}\phi(A) = k > 0$. This contradicts the hypothesis that A is essentially Hermitian. Hence by Theorems 4.4 and 4.5 it follows that $\|(A - \operatorname{Re}\operatorname{diag} A)P_n^\perp\| \to 0$ as $n \to \infty$. This means that, in the uniform norm,

$$\lim_{n \to \infty} (A - \operatorname{Re}\operatorname{diag} A)P_n = A - \operatorname{Re}\operatorname{diag} A .$$

Since each P_n is compact, the theorem is proved.

5. Concluding remarks. It is conjectured that if $2 \in [\alpha_M, \beta_M]$ the main result does not hold in general. The reason is this: if $2 \in [\alpha_M, \beta_M]$ then l_M contains a subspace isomorphic to l_2, and indeed the subspace can even be complemented. However even with the assumption that l_M contains a complemented subspace isomorphic to l_2 we have been unable to establish the conjecture. The existence of the isomorphism is simply not enough; in fact there is a modular Orlicz sequence space, isomorphic to l_2, which contains only diagonal Hermitian operators.

The analogous result to Theorem 4.5 in Orlicz function spaces, even in L_p $1 \leqq p < \infty$, is another matter altogether and it is posed as an open problem.

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