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A MAXIMUM PRINCIPLE ON CLIFFORD TORUS AND NON-EXISTENCE OF PROPER HOLOMORPHIC MAP FROM THE BALL TO POLYDISC

B. WONG

A maximum principle which estimates the gradients on the clifford torus of plurisubharmonic functions defined in the polydisc is derived. With this result we give a new proof that there exists no proper holomorphic map from the ball to polydisc in C^2.

1. Introduction. It is an old theorem of H. Poincaré that the bidisc $\Delta_2 = \{(z_1, z_2): |z_1| < 1, |z_2| < 1\}$ and the ball $B_2 = \{(z_1, z_2): |z_1|^2 + |z_2|^2 < 1\}$ are holomorphically distinct. Around 1935, V. Rothstein proved the remarkable fact that there exists no proper holomorphic map from Δ_2 to B_2 ([6]). Much later, G. H. Henkin generalised this result of Poincaré to the care of analytic polyhedron and strictly pseudo-convex domain ([3]). His proof was based on the comparison of Carathéodory metrics on these two different domains. He also indicated that it is possible to yield a generalization of Rothstein's theorem to polyhedron and strictly pseudo-convex domain by using the techniques employed in his paper (i.e. there exists no proper holomorphic map from polyhedron to strictly pseudo-convex domain). But his method can only be applied to the case of proper holomorphic map from polyhedron to strictly pseudo-convex domain. Despite all these old and new developments, a proof the following statement is unknown to the author.

"There exists no proper holomorphic map from B_2 to Δ_2".

The proofs of Poincaré and Rothstein are rather easy; they used the groups of biholomorphisms, Carathéodory metrics as well as the continuity principle to gain contradiction. In order to obtain a proof of the above statement a deeper investigation is in demand. In this paper we give a proof of this fact along the line of the theory of intrinsic measures (Carathéodory measures, Eisenman-Kobayashi measures) ([4], [5], [7]). The proof involves a boundary estimate of Eisenman-Kobayashi measure in $\Delta_2 - W$, where W is a complex analytic variety.

For simplicity of notations we shall restrict all our statements in C^2, but our proof can equally be applied to C^n without difficulty. One can also generalize our result to strictly pseudo-convex domain by passing through a localization process. It is also important to point out that, throughout this paper, the intrinsic measures are defined with respect to polydisc ([5]). They are different, in an essential way, from those measures defined with respect to the ball ([7]). For the basic theory of intrinsic measures one should consult [4] and [5]. They will be used frequently in the sequel of this paper.

After the completion of our work we were informed by professors Yum-Tong Siu and Karl Stein that our result had already been proved by H. Rischel (Math. Scand. 15 (1964), 49-63). Our proof is entirely different from his, and the method of invariant measures can be used to prove further interesting results on proper holomorphic maps between strongly pseudo-convex domains in $\boldsymbol{C}^n$. Although there are difficulties for us to understand the proof due to Rischel, it should be stressed that his paper has been available for fifteen years and our proof was obtained independently.

1. Maximum principle on clifford torus. The distinguished boundary of Δ_2, namely

$$T = \{(z_1, z_2) \,|\, |z_1| = 1 \text{ and } |z_2| = 1\}\,,$$

can be regarded as a regularly embedded torus in

$$S^3 = \{(z_1, z_2) \,|\, |z_1|^2 + |z_2|^2 = 2\}\,,$$

which is a three dimensional sphere. In the terms of classical differential geometry, T is called a clifford torus. The clifford torus plays an important role in the complex function theory of polydisc. On the other hand, it is an interesting submanifold of S^3 from a geometric viewpoint.

Let x be a point of $T \subset S^3$. We shall use the following notations throughout;

(1) n_x is the line pointing from the origin of S^3 to x (i.e., radial direction).

(2) If $z \in n_x$ $(x \in T)$ we denote by $d(z, T)$ the euclidean distance from z to T.

(3) $N_\varepsilon = \{z \in \Delta_2 \,|\, z \in n_t,\, t \in T,\, d(z, T) < \varepsilon\}$, were ε is a sufficiently small positive number.

We are going to derive the following maximum principle in this paragraph.

THEOREM 1.1. *Let U be a funciton in Δ_2 which is continuous up to the boundary of Δ_2. We assume that*

(1) *$U \equiv 0$ on $\partial\Delta_2$ and $U < 0$ in Δ_2.*

(2) *U is plurisubharmonic in Δ_2.*

Then there exists a small $\varepsilon > 0$ and a positive constant C such *that for all* $z \in N_\varepsilon$, $-U(z)/d(z, T) \geqq C > 0$.

Proof. Let us write

$$g(z) = e^{-ar^2} - e^{-2a} ,$$

where $z = (z_1, z_2) \in \boldsymbol{C}^2$, a is a constant to be determined, $r = |z| = \sqrt{|z_1|^2 + |z_2|^2}$.

A direct computation would show that g satisfies the following conditions

(1) $\forall z \in \Delta_2$, $g(z) > 0$

(2) $\forall z \in T$, $g(z) \equiv 0$

(3) There exists a positive constant K such that $dg/dn_t(t) = -2\sqrt{2}a\ e^{-2a} < -k$ for all $t \in T$.

Let r_1 be a positive number which is slightly less that $\sqrt{2}$. Then $\partial B_2^{r_1} = \{(z_1, z_2) \,|\, |z_1|^2 + |x_2|^2 = r_1^2\}$. We also denote by D_1 the region between $\partial B_2^{\sqrt{2}}$ and $\partial B_2^{r_1}$ and $D = D_1 \cap \Delta_2$. Let $A = D \cap (\bigcup_{x \in T} \bar{n}_x)$, where $\bar{n}_x = \{\lambda x \,|\, \lambda$ is a complex number $|\lambda| < 1$, $x \in T\}$.

The following is a result subject to only straightforward computation.

(4) If a is a sufficiently large positive constant, then $\forall z \in A$, $\Delta g(z) > 0$, where Δ is the Laplacian operator on $\bar{n}_x$ which contains z with $x \in T$.

Now we can proceed with our proof.

Let $V = U + s \cdot g$, where s is a positive constant such that $V(z) < 0$ for all z belonging to $\partial A \cap \partial B_2^{r_1}$. It should be noticed that the sets T and $\partial A \cap \partial B_2^{r_1}$ constitute the boundary of A constructed above.

Claim. If we regard V as a function defined on A, then it is impossible for V to attain its maximum in the interior of A.

Suppose $w \in A$ such that $V(w)$ is the maximum. Let $\bar{n}_x$ be the complex disc as above which contains w (where $x \in T$). However, U is plurisubharmonic in Δ_2 (therefore subharmonic on $\bar{n}_x$) and g is subharmonic on $\bar{n}_x$, one can easily see that $V = U + s \cdot g$ is then subharmonic on $\bar{n}_x$. Thus one obtains a contradiction to the assumption that V attains a maximum at w. Furthermore $V(z) < 0$ for all $z \in \partial A \cap \partial B_2^{r_1}$, hence it is led to the conclusion that V attains its maximum on T; here we have used the fact that $g = 0$, $U = 0$ on T and that both g and U are continuous up to the boundary.

By the compactness of T and the fact that the vector field n_t on T is smooth we can choose a sufficiently small $\varepsilon > 0$ such that

$$\frac{U(t) - U(z)}{d(z, \partial T)} + s \cdot \frac{dg}{dn_x}(t) \geqq 0$$

for all $t \in T$, $z \in N_\varepsilon$.

Since $U(t) \equiv 0 \forall t \in T$, together with condition (3) we obtain

$$-\frac{U(z)}{d(z, T)} \geqq -s \cdot \frac{dg}{dn_t} \geqq s \cdot K > 0$$

for all $t \in T$, $z \in N_\varepsilon$.

Letting $C = s \cdot K$ we complete the proof.

2. A theorem on the proper holomorphic map from B_2 to Δ_2. Suppose $f\colon B_2 \to \Delta_2$ is a proper holomorphic map. By a theorem of Remmert (B_2, f, Δ_2) is a complex analytic branched covering. The branching locus is given by $s = \{z \in B_2 \mid \det(df(z)) = 0\}$ and $f\colon B_2 - s \to \Delta_2 - f(s)$ is a unramified finite complex analytic covering.

We define a complex analytic function $L\colon \Delta_2 - f(s) \to C$ as follows,

$$L(z) = \prod_{i=1}^{m} \det(df(x_i)), \qquad \text{where} \quad x \in \Delta_2 - f(s),$$

$f^{-1}(z) = \{x_1, x_2, \cdots, x_m\}$, $df(x_i) =$ Jacobian of f at x_i. We make an observation here that L is locally bounded around the complex subvariety $f(s) \subset \Delta_2$, hence L extends holomorphically across $f(s)$. The following theorem will be proved in this section.

THEOREM 2.1. *L is a bounded holomorphic function in Δ_2.*

Proof. Let r be a positive number slightly less than 1. It is well-known fact that $L \mid \Delta_2^r$ assumes its maximum on the distinguished boundary

$$T(\Delta_2^r) = \{(z_1, z_2) \mid |z_1| = r, |z_2| = r\}$$

of Δ_2^r.

With some considerations it suffies to prove that $\forall z \in N_\varepsilon$, $|\det(df(x))| \leqq Q$, where $x \in \{f^{-1}(x)\}$, Q is a constant and ε is a sufficiently small positive number.

The following notations will be used in the rest of this paper:

$E_{B_2} =$ Eisenman-Kobayashi measure on B_2

$E_{B_2^{\sqrt{2}}} =$ Eisenman-Kobayashi measure on $B_2^{\sqrt{2}} = \{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 < 2\}$

$E_{\Delta_2} =$ Eisenman-Kobayashi measure on Δ_2

(For the definition of Eisenman-Kobayashi measure, consult [4]). The following fact are immediate

(a) Since $f\colon B_2 \to \Delta_2 \subset B_2^{\sqrt{2}}$ is holomorphic map, by the measure-decreasing property we have

$$|E_{B_2}(x)| \geqq |\det(df(x))| \cdot |E_{B_2^{\sqrt{2}}}(z)|$$

where

$$x \in \{f^{-1}(z)\}\ , \qquad z \in \Delta_2 \subset B_2^{\sqrt{2}}\ .$$

(b) From the explicit formulas of E_{B_2} and $E_{B_2^{\sqrt{2}}}$, we have

$$\frac{C_1}{((y, \partial B_2))^3} \leqq |E_{B_2}(y)| \leqq \frac{C_2}{((y, \partial B_2))^3}$$

$$\frac{K_1}{(d(w, \partial B_2^{\sqrt{2}}))^3} \leqq |E_{B_2^{\sqrt{2}}}(w)| \leqq \frac{K_2}{(d(w, \partial B_2^{\sqrt{2}}))^3}$$

where $y \in B_2$, $w \in B_2^{\sqrt{2}}$, sufficiently close to the boundary, C_1, C_2, K_1 and K_2 are constants.

(c) From the explicit formulas of Kobayashi metrics in B_2 and Δ_2, and also the distance-decreasing property, we have

$$d(z, \partial\Delta_2) \geqq l \cdot d(x, \partial B_2)\ ,$$

where $x \in \{f^{-1}(z)\}$ for all $z \in \Delta_2$ sufficiently close to $\partial\Delta_2$, l is a positive constant.

With the above facts in mind we can prove the following consequences.

(1)
$$\left(\frac{C_2}{K_2}\right) \cdot \left(\frac{d(z, \partial B_2^{\sqrt{2}})}{d(x, \partial B_2)}\right)^3 \geqq |\det (df(x))|$$

with $x \in \{f^{-1}(z)\}$, $z \in N_\varepsilon$, ε is a sufficiently small number.

(2) Suppose that $U: \Delta_2 \to R$ is defined as follows;

$$U(z) = \max \{-d(x_1, \partial B_2), -d(x_2, \partial B_2), \cdots, \\ -d(x_m, \partial B_2) \mid f^{-1}(z) = \{x_1, x_2, \cdots, x_m\}, z \in \Delta_2\}\ .$$

Obviously U is a bounded plurisubharmonic function on $\Delta_2 - \{w \mid L(w) = 0\}$, it extends across $\{w \mid L(w) = 0\}$ by a well-known lemma of Grauert. By (c) it is easy to see that U is continuous up to the boundary of Δ_2.

Finally we apply Theorem 1.1 to U defined above. We therefore obtain

$$\frac{1}{C} \geqq \frac{d(z, \partial B_2^{\sqrt{2}})}{d(x, \partial B_2)} \qquad (C \text{ is the constant in Theorem 1.1})\ ,$$

where $x \in \{f^{-1}(z)\}$, $z \in N_\varepsilon$ for a well-chosen small positive constant ε (Note: if $z \in N_\varepsilon$, $d(z, \partial B_2^{\sqrt{2}}) = d(z, T)$.)

Combining all above inequalities, we have

$$\left(\frac{1}{C}\right)^3 \left(\frac{C_2}{K_1}\right) \geqq |\det (f(x))|\ .$$

The proof is thereby completed.

3. Boundary behavior of the complex analytic variety $\{z \in \Delta_2 | L(z) = 0\}$ in Δ_2. Let P be a point belonging to $\partial\Delta_1$. We denote by Γ_P^θ the cone extending θ with vertex at P in Δ_1 (i.e., $\Gamma_P^\theta = \{z | z = (x, y),\ z \in \Delta_1,\ \tan^{-1}(y/x) < \theta,\ 0 \leqq \theta < \pi/2\}$), where we choose P to be the origin of our coordinates, x − axis = normal of $\partial\Delta_1$ at P, y − axis = tangent of $\partial\Delta_1$ at P. Furthermore, $P_1: \Delta_1 \times \Delta_1 \to \Delta_1$, $P_2: \Delta_1 \times \Delta_1 \to \Delta_1$ are first and second projections in a natural way.

LEMMA 3.1. *With the same notations as before, there exists a point $P \in T(\Delta_2)$ such that it is not an accumulation point of $\{z | L(z) = 0\} \cap \Lambda_P^\theta$ for all $0 \leqq \theta < \pi/2$, where $\Lambda_P^\theta = \Gamma_P^\theta \times \Gamma_P^\theta \subset \Delta_2$.*

Proof. By Theorem 2.1. L is a bounded holomorphic function, the radial limits L^* of L approaching $T(\Delta_2)$ exist almost everywhere. It is well-known fact that $L = \bar{P}[L^*]$ (see our remark below), where $\bar{P}$ is the poisson kernel of Δ_2 (for instance, see Rudin: Function theory in polydiscs P. 31 exercise). Furthermore one can model from the proof of Fatou theorem to give the following assertion: For almost every point $y \in T(\Delta_2)$, the non-tangential limit

$$\lim_{\substack{x \to y \\ x \in \Lambda_y^\theta}} L(x)$$

exists, i.e.,

$$L^*(y) = \lim_{\substack{x \to y \\ x \in \Lambda_y^\theta}} L(x)$$

for $y \in T(\Delta_2)$ a.e. However, if every point $y \in T(\Delta_2)$ is an accumulation of

$$\{z | L(z) = 0\} \cap \Lambda_P^\theta \qquad \text{for some} \quad 0 \leqq \theta < \frac{\pi}{2},$$

it would imply immediately that $L^*(y) = 0$ a.e., hence $L \equiv 0$ by poisson formula. It is a contradiction.

REMARK. To be rigorus we should write $L = L_1 + iL_2$, where L_1 and L_2 are n-harmonic (see Rudin, p. 16), correspondingly $L^* = L_1^* + iL_2^*$, and notation $L = \bar{P}[L^*]$ means $L_1 = \bar{P}[L_1]$ and $L_2 = \bar{P}[L_2^*]$.

The proof of our claim concerning the boundary values of bounded holomorphic functions on $T(\Delta_2)$ is rather long; it is a reproduction of a theorem of Fatou which is a folklore in the area of boundary values of holomorphic functions, we therefore take the liberty to skip the proof here. Proofs and references of relevant results can be found in "A. Koranyi—E. M. Stein: Fatou's theorem for generalized half planes". (Estratto dagli Annali della, Scuola

Normale Superiore di Pisa classe di Science, Vol. XXII, Fasc. I, (1968).)

4. An estimate of Eisenman-Kobayashi measure. We start with the definitions of Carathéodory and Eisenman-Kobayashi measures. It is easy to check the measure U on Δ_2 given below is invariant under biholomorphisms;

$$U = \prod_{j=1}^{2} \frac{4\sqrt{-1}}{(1-|z^j|^2)^2} dz^j \Lambda d\bar{z}^j .$$

(Note: We use the notation $|U|$ to stand for $\prod_{j=1}^{2} (4/|(1-|z^j|^2)^2|)$.)

Let D be a bounded domain in C^2. The Carathéodory measure C_D is defined as follows:

$$C_D(x) = \sup_f (f^* U)_x ,$$

where the supremun is taken over all holomorphic maps $f: D \to \Delta_2$.

The Eisenman-Kobayashi E_D is defined as follows:

$$E_D(x) = \inf_f ((f^{-1})^* U)_0 .$$

Where the infimun is taken over all holomorphic maps $f: \Delta_2 \to D$ which maps the origin "O" of Δ_2 to x and is nondegenerate at "O".

In this section we shall derive an estimate of Eisenman-Kobayashi measure (in the case of metric, such an estimate was first obtained by R. L. Royden).

DEFINITION. $z, w \in D$, then $d_D(z, w) = \inf\{P(a, b) \mid f \in \mathrm{Hol}(\Delta_2, D)$ s.t. $f(a) = z$, $f(b) = w$, P is the Kobayashi metric in $\Delta_2\}$.

Let D_1 be another domain in C^2 such that $D_1 \cap D$ is nonempty.

DEFINITION. For $z \in D \cap D_1$

$$d_{D-D_1}(z) = \inf\{d_D(z, w) \mid w \text{ belongs to } D \text{ but not } D_1\} .$$

THEOREM 4.1. *Let $\bar{D} = D \cap D_1$. Then for all $z \in \bar{D}$, we have*

$$|E_{\bar{D}}(z)| \leqq |\cot h d_{D-D_1}(z)|^4 \cdot |E_D(z)| .$$

Proof. First of all let us fix a constant r as follows, $r = \sup\{t \mid \text{there exists } f \in \mathrm{Hol}(\Delta_2^t, \bar{D}),\ f(0) = z,\ |\det(df(0))| = 16\}$.

Then we choose a number R slightly larger than r. From our choice of r it is obvious that there is a $f \in \mathrm{Hol}(\Delta_2^R, D)$ s.t. $f(0) = z$, $|\det(df(0))| = 16$, and it maps a boundary point of Δ_2^r to a point belonging to $D - \bar{D}$. One can see, if w is this point belonging to

$D-\bar{D}$, then $d_{D-D_1}(z) \leqq d_D(z, w)$.

From the definition of $d_D(z, w)$ we observe that

$$d_D(z, w) \leqq \frac{1}{2} \ln\left(\frac{1+\frac{r}{R}}{1-\frac{r}{R}}\right).$$

Then it implies

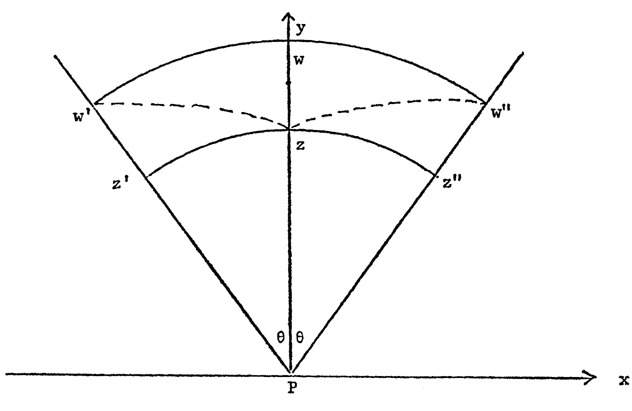
$$\left(\frac{1}{r}\right) \leqq (\cot h d_{D-D_1}(z)) \cdot \left(\frac{1}{R}\right)$$

$$\left(\frac{1}{r}\right)^4 \leqq (\cot h d_{D-D_1}(z))^4 \cdot \left(\frac{1}{R}\right)^4 .$$

It is easy to see that the above inequality together with the definition of Eisenman-Kobayashi measure imply our theorem immediately.

5. Some estimates of Kobayashi metric inside the cones.

Let D be a domain in C^n, we denote by d_D^k the Kobayashi metric on D. If D is the upper-half plane $H = \{x + iy \mid y > 0\}$, d_H^k is then induced by $(2\sqrt{dx^2 + dy^2})/y$. Suppose P is the origin of z-plane and Γ_P^θ is a cone in H with vertex at P and extended angle θ, where $0 \leqq \theta < \pi/2$



In the following theorem we assume that $z = (0, y)$ is a point on the y-axis.

THEOREM 5.1. *$d_H^k(z, \partial\Gamma_P^\theta) = 2\ln(\tan\theta + \sec\theta)$ where $d_H^k(z, \partial\Gamma_P^\theta) =$ the distance from z to the boundary of Γ_P^θ* (i.e., $\partial\Gamma_P^\theta$) *with respect to d_H^k.*

Proof. It is well-known that the geodesic of the metric

$(2\sqrt{dx^2+dy^2})/y$ passing through z is a great circle (with radius $|y|$ and center at P). Let z' and z'' be the points of intersection of this circle with $\partial\Gamma_P^\theta$. Then the length of the arc zz' is equal to

$$2\int_0^\theta \frac{yd\phi}{(y\cos\phi)} = 2\int_0^\theta \frac{d\phi}{\cos\phi} = 2\ln(\tan\theta + \sec\theta)\ .$$

We need the following known result in Riemannian geometry for our proof.

THEOREM. *Let M be a simply connected complete Riemannian manifold of negative sectional curvature, then for an two points in M there exists one and only one minimizing geodesic joining them.*

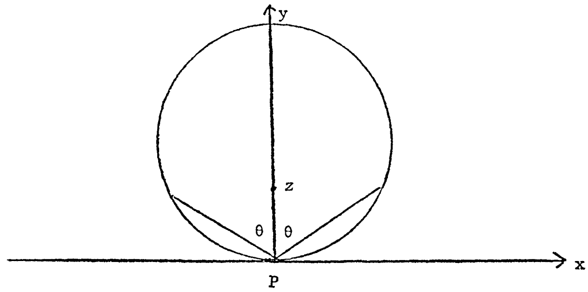
Now suppose that w' is a point on $\partial\Gamma_P^\theta$ such that the length of $zw' = d_H^k(z,\partial\Gamma_P^\theta)$. By the symmetric properties of Γ_P^θ and $(2\sqrt{dx^2+dy^2})/y$ one easily observes that there exists a point w'' on the other side of Γ_P^θ s.t. $Pw'' = Pw'$ (with respect to euclidean length). Let w be the point of intersection of y-axis and the great circle of radius Pw' and with center at P. Since the arc ww' is a minimizing geodesic joining w and w', hence $d_H^k(w,w') \leqq d_H^k(z,w')$. $d_H^k(w,w') = d_H^k(z,z') = 2\ln(\tan\theta + \sec\theta)$ from the previous computation. Thus, one obtains

$$d_H^k(z,\ z') \leqq d_H^k(z,\ w') = d_H^k(z,\ \partial\Gamma_P^\theta)\ .$$

It is now easy to conclude that

$$d_H^k(z,\ \partial_P^\theta\Gamma) = 2\ln(\tan\theta + \sec\theta)\ .$$

THEOREM 5.2. *Let P be a boundary point of the unit disc Δ_1. Suppose that Γ_P^θ is a cone in Δ_1 with vertex at P and extended angle θ, $0 \leqq \theta < \pi/2$, z is a point of the normal of $\partial\Delta_1$ at P. (See our figure.) Then we have $d_{\Delta_1}^k(z,\ \partial\Gamma_P^\theta) \geqq 2\ln(\tan\theta + \sec\theta)$.*



Proof. Since $H \supset \Delta_1$, by the distance-decreasing property of Kobayashi metrics we have

$$d^k_{\Delta_1}(z, \partial\Gamma^\theta_P) \geqq d^k_H(z, \partial\Gamma^\theta_P) .$$

Our theorem follows immediately from Theorem 5.1.

THEOREM 5.3. *Let P be a point on $T(\Delta_2)$ and Λ^θ_P a product cone as before,* $0 \leqq \theta < \pi/2$. *Then we have*

$$d^k_{\Delta_2}(z, \partial\Lambda^\theta_P) \geqq 2 \ln (\tan\theta + \sec\theta) .$$

Proof. It is clear from Theorem 5.2 and the remark below.

REMARK. It is a well-known fact in the theory of intrinsic metrics that

$$d^k_{\Delta_2}(z_1, z_2) = \sup \{d^k_{\Delta_1}(P_1(z_1), P_1(z_2)), d^k_{\Delta_1}(P_2(z_1), P_2(z_2))\} .$$

It is now easy to observe the following.

THEOREM 5.4. *With the assumptions in Theorem* 5.3, *we have*
(1) $\mathrm{Lim}_{\theta\to\pi/2}\, d^k_{\Delta_2}(z, \partial\Lambda^\theta_P) = \infty$
(2) $\mathrm{Lim}_{\theta\to\pi/2} \cot h d^k_{\Delta_2}(z, \partial\Lambda^\theta_P) = 1$.

Proof. Elementary.

6. Main part of the proof. We break our proof into several steps:

(1) From Lemma 3.1 there exists a point, namely $p \in T(\Delta_2)$, which is not an accumulation point of $f(s) \cap \Lambda^\theta_P$ in Λ^θ_P for all $0 \leqq \theta < \pi/2$. We choose a sequence $\{z_i\} \to P$, where z_i belongs to $\Delta_2 - f(s)$ and lies on the line R perpendicular to S^3 at P. Let $\{x_i\}$ be another sequence in B_2, where $x_i \in \{f^{-1}(z_i)\}$. One easily verifies that $\{x_i\} \to q$ (passing through a subsequence if necessary), where q is a boundary point B_2.

(2) $f\colon B_2 \to \Delta_2$ is a proper holomorphic map. Then

$$s = \{z \in B_2 \,|\, \det(df(z)) = 0\}$$

and $f(s)$ are complex analytic varieties in B_2 and Δ_2 respectively. Furthermore, $f\colon B_2 - s \to \Delta_2 - f(s)$ is a finite complex analytic covering. From the standard facts in theory of intrinsic measures ([4], [5]) we have

(I) $C_{\Delta_2 - f(s)} = C_{\Delta_2}$, $C_{B_2 - s} = C_{B_2}$
(II) $E_{\Delta_2 - f(s)} \geqq E_{\Delta_2}$, $E_{B_2 - s} \geqq E_{B_2}$
(III) $C_{B_2} \geqq f^*(C_{\Delta_2})$
$E_{B_2} \geqq f^*(E_{\Delta_2})$.
(Volume-decreasing property under holomorphic maps.)

(IV) $E_D \geqq C_D$, where D is any bounded domain

(V) $E_{B_2-s} = f^*(E_{\Delta_2-f(s)})$

(note: $(B_2 - s, f, \Delta_2 - f(s))$ is a covering).

(3) Making use of the above inequalities and equalities we obtain

$$1 \geqq \frac{|C_{B_2}(x_i)|}{|E_{B_2}(x_i)|} \geqq \frac{|f^*(C_{\Delta_2}(z_i))|}{|E_{B_2-s}(x_i)|} = \frac{|C_{\Delta_2}(z_i)|}{|E_{\Delta_2-f(s)}(z_i)|} = \frac{|E_{\Delta_2}(z_i)|}{|E_{\Delta_2-f(s)}(z_i)|},$$

for all i.

(4) The following lemma was derived in ([7]).

LEMMA. *Let D be a complete hyperbolic bounded domain in C^n. Suppose that E_D and C_D are defined with respect to polydisc Δ_n. If there exists $x \in D$ such that*

$$\frac{|E_D(x)|}{|C_D(x)|} = 1,$$

then D is biholomorphic to Δ_n.

Since B_2 is homogeneous we have

$$\frac{|C_{B_2}(x_i)|}{|E_{B_2}(x_i)|} = C, \qquad \text{for all } i,$$

where C is a constant which is not equal to 1. It is clear from (3) that we would obtain a contradiction if the following identity holds

$$\lim_{z_i \to P} \frac{|E_{\Delta_2}(z_i)|}{|E_{\Delta_2-s}(z_i)|} = 1.$$

(5) First of all we know that

$$E_{\Delta_2-f(s)} \geqq E_{\Delta_2} \qquad \text{from} \quad (2)\ (\text{II}).$$

Secondly, in the view of Theorem 4.1 we have the following inequality

$$|E_{\Delta_2-f(s)}(z_i)| \leqq |\cot hd_{D-D_1}(z_i)|^4 \cdot |E_{\Delta_2}(z_i)|$$

where $D = \Delta_2$, $D_1 = \Delta_2 - f(s)$. Our proof would be completed if one could show

$$\operatorname*{Lim}_{z_i \to P} \cot hd_{D-D_1}(z_i) = 1.$$

We note that $d_{D-D_1}(z_i) = d_{\Delta_2}(z_i, f(s))$, where d_{Δ_2} was defined in §4. It is trivial to observe $d_{\Delta_2} = d^k_{\Delta_2}$ from our definition. Thus it is enough for us to prove

$$\lim_{z_i \to P} \cot h d^k_{\Delta_2}(z_i, f(s)) = 1$$

(i.e., $\lim_{z_i \to P} d^k_{\Delta_2}(z_i, f(s)) = \infty$).

From our assumption P is not an accumulation point of $f(s) \cap \Lambda^\theta_P$ in Λ^θ_P for all $0 \leqq \theta < \pi/2$, for a fixed θ between 0 and $\pi/2$ we have

$$d^k_{\Delta_2}(z_i, f(s)) \geqq d^k_{\Delta_2}(z, \partial\Lambda^\theta_P)$$

if i is sufficiently large. However, it follows from Theorem 5.4 that

$$d^k_{\Delta_2}(z_i, \partial\Lambda^\theta_P) \geqq 2 \ln (\tan\theta + \sec\theta) .$$

Now we can conclude

$$\lim_{i \to \infty} d^k_{\Delta_2}(z_i, f(s)) \geqq 2 \ln (\tan\theta + \sec\theta)$$

for a fixed θ.

Letting $\theta \to \pi/2$, we thereby complete our proof (Theorem 5.4).

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THE CHINESE UNIVERSITY OF HONG KONG
SHATIN, N. T., HONG KONG