CONTINUOUS SELECTIONS AND
FINITE-DIMENSIONAL SETS

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Some known selection theorems are strengthened by weakening the hypotheses on a finite-dimensional subset of the domain.

1. **Introduction.** The following selection theorem was recently obtained in [7] by C. Pixley and the author.

THEOREM 1.1. *Let X be paracompact, Y a Banach space, $Z \subset X$ with $\dim_X Z \leqq 0$, and $\phi: X \to \mathscr{F}(Y)$ l.s.c. with $\phi(x)$ convex for all $x \in X - Z$. Then ϕ has a selection.*

Let us quickly define our terms: We write $2^Y = \{S \subset Y: S \neq \varnothing\}$ and $\mathscr{F}(Y) = \{S \in 2^Y: S \text{ closed in } Y\}$. A map $\phi: X \to 2^Y$ is *lower semicontinuous*, or l.s.c., if $\{x \in X: \phi(x) \cap V \neq \varnothing\}$ is open in X for every open V in Y. A *selection* for a map $\phi: X \to 2^Y$ is a continuous $f: X \to Y$ such that $f(x) \in \phi(x)$ for every $x \in X$. Finally, if $Z \subset X$, then $\dim_X Z \leqq n$ means that $\dim S \leqq n$ for every subset S of Z which is closed in X.[1]

The purpose of this paper is first to generalize Theorem 1.1 to the case where $\dim_X Z \leqq n$, and then to prove two other theorems of this type in which different assumptions are made on $\phi(x)$ when $x \in X - Z$. Just like Theorem 1.1, all of these "hybrid" selection theorems reduce to known results when $Z = \varnothing$ or $Z = X$. Theorem 1.1, along with the special cases of Theorems 1.3 and 1.4 where $\dim Z = 0$, will be applied in [6].

In contrast to Theorem 1.1, the theorems to be proved in this paper all have both a local and a global version. To state these results succinctly, we introduce some more terminology: A map $\phi: X \to 2^Y$ has the *selection extension property*, or SEP, if, whenever $A \subset X$ is closed, every selection g for $\phi | A$ extends to a selection for ϕ. If g only extends to a selection for $\phi | U$ for some neighborhood U of A in X, then we say that ϕ has the *selection neighborhood extension property*, or SNEP. As pointed out in [6], Theorem 1.1 can be strengthened to conclude that ϕ actually has the SEP.

Before stating Theorem 1.2, we recall the following concepts from [4, p. 565], where $n \geq -1$: A space S is C^n if every continuous image of an i-sphere $(i \leqq n)$ in S is contractible in S. A collec-

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[1] Observe that, for normal X, $\dim_X Z \leqq n$ is valid if either $\dim Z \leqq n$ or $\dim X \leqq n$.

tion $\mathscr{S}$ of subsets of a metric space Y is *uniformly equi-LC^n* if to every $\varepsilon > 0$ corresponds a $\nu(\varepsilon) > 0$ such that, for every $S \in \mathscr{S}$, every continuous image of an i-sphere $(i \leq n)$ in S of diameter $< \nu(\varepsilon)$ is contractible over a subset of S of diameter $< \varepsilon$. Since there is no (-1)-sphere, the above properties are always vacuously satisfied when $n = -1$, and hence Theorem 1.2 really generalizes Theorem 1.1.

THEOREM 1.2. *Let X be paracompact, Y a Banach space, $Z \subset X$ with $\dim_X Z \leq n + 1$, and $\phi : X \to \mathscr{F}(Y)$ l.s.c. with $\phi(x)$ convex for all $x \in X - Z$ and with $\{\phi(x) : x \in Z\}$ uniformly equi-LC^n. Then ϕ has the SNEP. If, moreover, $\phi(x)$ is C^n for every $x \in Z$, then ϕ has the SEP.*

Theorem 1.2 reduces to [3, Theorem 3.2″] when $Z = \varnothing$, and it implies [4, Theorem 1.2] when $Z = X$ (more generally, when Z is open in X and $\phi(x)$ is a singleton for $x \in X - Z$).

In our next theorem, a metric space Y is called an AR (resp. ANR) if it is a retract (resp. neighborhood retract) of every metric space E containing it as a closed subset. A collection $\mathscr{S}$ of subsets of a topological space Y is called *equi-LC^n in Y* if, for every $y \in Y$, every neighborhood V of y contains a neighborhood W of y such that, for any $S \in \mathscr{S}$, every continuous image of an i-sphere $(i \leq n)$ in $W \cap S$ is contractible in $V \cap S$. It is easy to see that every uniformly equi-LC^n collection of subsets of a metric space Y is equi-LC^n in Y; for a partial converse, see Lemma 6.1(b).

THEOREM 1.3. *Let X be paracompact, Y a complete metric ANR, $Z \subset X$ with $\dim_X Z \leq n + 1$, and $\phi : X \to \mathscr{F}(Y)$ l.s.c. with $\phi(x) = Y$ for $x \in X - Z$ and with $\{\phi(x) : x \in Z\}$ equi-LC^n in Y. Then ϕ has the SNEP. If moreover, Y is an AR and $\phi(x)$ is C^n for every $x \in Z$, then ϕ has the SEP.*

Observe that, when $Z = \varnothing$, Theorem 1.3 reduces to the known result that every complete metric AR (resp. ANR) has the extension property (resp. neighborhood extension property) with respect to paracompact spaces (see, for instance, C. H. Dowker [1]).

Our next result reduces to [4, Theorem 1.2] when $Z = \varnothing$.

THEOREM 1.4. *Let X be paracompact, $A \subset X$ closed with $\dim_X (X - A) \leq n + 1$, $Z \subset X - A$ with $\dim_X Z \leq m + 1$ (where $m \leq n$), Y complete metric, and $\phi : X \to \mathscr{F}(Y)$ l.s.c. such that $\{\phi(x) : x \in X - Z\}$ is equi-LC^n in Y and $\{\phi(x) : x \in Z\}$ is equi-LC^m in Y.*

Then ϕ *has the* SNEP *at* A.[2] *If, moreover,* $\phi(x)$ *is* C^n *for all* $x \in X - Z$ *and* C^m *for all* $x \in Z$, *then* ϕ *has the* SEP *at* A.

The paper is arranged as follows. Two known lemmas on l.s.c. maps are stated in §2, some general results on selections are established in §3, and a result which is implicitly contained in [4] is recorded in §4. After these preliminaries, Theorems 1.2, 1.3 and 1.4 are then proved in §§5, 6 and 7, respectively.

2. Two lemmas on l.s.c. maps. The following two known lemmas will be applied in the sequel.

LEMMA 2.1 [3, Example 1.3]. *Let* $\phi: X \to 2^Y$ *be l.s.c.,* $A \subset X$ *closed, and* g *a selection for* $\phi|A$. *Define* $\phi_g: X \to 2^Y$ *by* $\phi_g(x) = \phi(x)$ *if* $x \in X - A$ *and* $\phi_g(x) = \{g(x)\}$ *if* $x \in A$. *Then* ϕ_g *is also l.s.c.*

Henceforth, we shall freely refer to the map ϕ_g defined in Lemma 2.1.

LEMMA 2.2 [4, Lemma 11.3]. *Let* $\phi: X \to 2^Y$ *be l.s.c. with* Y *a metric space, let* $K \subset Y$ *be compact, and let* $\varepsilon > 0$. *Then* $\{x \in X: K \subset B_\varepsilon(\phi(x))\}$[3] *is open in* Y.

3. Two properties of set-valued maps. In this section we consider the following two properties of a map $\phi: X \to 2^Y$, where Y is a metric space.

(3.1) To every $\varepsilon > 0$ corresponds an $\alpha(\varepsilon) > 0$ with the following property: If $A \subset X$ is closed, g is a selection for $\phi|A$, and $h: X \to Y$ is continuous with $d(h, \phi_g) < \alpha(\varepsilon)$, then g extends to a selection f for ϕ with $d(f, h) < \varepsilon$.[4]

(3.2) To every $\varepsilon > 0$ corresponds a $\beta(\varepsilon) > 0$ with the following property: If $A \subset X$ is closed, g is a selection for $\phi|A$, $h: X \to Y$ is continuous with $d(h, \phi_g) < \beta(\varepsilon)$, and $\mu > 0$, then there exists a continuous $f: X \to Y$ with $d(f, h) < \varepsilon$ and $d(f, \phi_g) < \mu$.

Property (3.2) (often with $A = \varnothing$) is used—implicitly or explicitly—in the proofs of almost all general selection theorems known to the author, including those in this paper. It is clear that the slightly simpler property (3.1) (which is called the *selection approximation property*, or SAP, in [6, §5]) always implies (3.2); the

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[2] i.e., ϕ satisfies the definition of the SNEP for this particular A.

[3] $B_\varepsilon(S)$ denotes the ε-neighborhood of S.

[4] Here d is the metric on Y, and $d(h, \phi_g) < \alpha$ denotes that $d(h(x), \phi_g(x)) < \alpha$ for all $x \in X$. Similarly for $d(f, h) < \varepsilon$, etc.

following result shows that, under mild restrictions, the two properties are actually equivalent.

PROPOSITION 3.3. *If Y is a complete metric space, and if $\phi(x)$ is closed in Y for all $x \in X$, then (3.2) implies (3.1). Moreover, one can take $\alpha(\varepsilon) = \beta(\varepsilon/2)$; in particular, if one can take $\beta(\infty) = \infty$[5] then one can also take $\alpha(\infty) = \infty$.*

Proof. Assume that ϕ satisfies (3.2) and let us show that it satisfies (3.1) with $\alpha(\varepsilon) = \beta(\varepsilon/2)$. Let $f_1 = h$. By induction, use (3.2) to construct continuous maps $f_n: X \to Y$ such that $d(f_n, \phi_g) < \beta(2^{-n}\varepsilon)$ and $d(f_{n+1}, f_n) < 2^{-(n+1)}\varepsilon$ for all $n \geq 1$. Let $f = \lim_n f_n$. This f satisfies (3.1) with $\alpha(\varepsilon) = \beta(\varepsilon/2)$.

PROPOSITION 3.4. *Suppose $\phi: X \to 2^Y$, with Y metric. Then:*
   (a) *If ϕ satisfies (3.1) with $\alpha(\infty) = \infty$, then ϕ has the SEP.*
   (b) *If X is collectionwise normal[6], if Y is a Banach space, and if $\phi | B$ satisfies (3.1) for every closed $B \subset X$ with $\alpha(\infty)$ independent of B, then ϕ has the SNEP.*

Proof. (a) Clear (see Footnote 5).
   (b) Suppose $A \subset X$ is closed and g is a selection for $\phi | A$. By a theorem of C. H. Dowker [1], g can be extended to a continuous $h: X \to Y$. Let $U = \{x \in X: d(h, \phi) < \alpha(\infty)\}$. Then U is open in X (since ϕ is l.s.c.) and $A \subset U$, so there is an open V in X such that $A \subset V \subset \bar{V} \subset U$. Then $d(h | \bar{V}, \phi_g | \bar{V}) < \alpha(\infty)$, so by assumption g extends to a selection f for $\phi | \bar{V}$. That completes the proof.

4. **A lemma about nerves of coverings.** If $\mathscr{U}$ is a covering of X, then we write $N(\mathscr{U})$ for the nerve of $\mathscr{U}$ and $N^i(\mathscr{U})$ for the i-skeleton of $N(\mathscr{U})$. A function $u: N^i(\mathscr{U}) \to Y$ is called a *representation* if $u | \sigma$ is continuous for every simplex σ of $N^i(\mathscr{U})$.

LEMMA 4.1. *Let Y be a Banach space and $\mathscr{S} \subset 2^Y$ uniformly equi-LC^n. Then to every $\varepsilon > 0$ corresponds a $\gamma(\varepsilon) > 0$ with the following property: If X is paracompact, $\phi: X \to \mathscr{S}$ l.s.c., $h: X \to Y$ continuous with $d(h, \phi) < \gamma(\varepsilon)$, and $\mu > 0$, then there exists a locally finite open cover $\mathscr{U}$ of X and a representation $u: N^{n+1}(\mathscr{U}) \to Y$ such that, if σ is a simplex of $N^{n+1}(\mathscr{U})$ with vertices $U_1, \cdots, U_j$, and if $x \in U_1 \cap \cdots \cap U_j$, then*

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[5] That one can take $\alpha(\infty) = \infty$ simply means, in effect, that there exists a selection f for ϕ_g, with nothing of consequence being assumed or concluded about h. This useful shorthand, which was introduced in [4], will be used throughout this paper.

[6] This property lies between normality and paracompactness.

$$u(\sigma) \subset B_{\mu}(\phi(x)) \cap B_{\varepsilon}(h(x)) \ .$$

If every $S \in \mathscr{S}$ *is* C^n, *then one can take* $\gamma(\infty) = \infty$.

Proof. This result is, in effect, a slight generalization of [4, Lemma 6.1]. In that lemma, it was assumed—in addition to our present hypotheses—that dim $X \leq n + 1$, and it was concluded—in addition to our present conclusions—that the cover $\mathscr{U}$ is of order $\leq n + 2$ (so that $N(\mathscr{U}) = N^{n+1}(\mathscr{U})$). The proof of [4, Lemma 6.1] remains valid for our present result.

5. Proof of Theorem 1.2. By Propositions 3.3 and 3.4, it will suffice to show that ϕ satisfies condition (3.2),[7] and that one can take $\beta(\infty) = \infty$ in case $\phi(x)$ is C^n for every $x \in Z$. In fact, we will show that ϕ satisfies (3.2) with $\beta(\varepsilon) = \gamma(\varepsilon)$, where $\gamma(\varepsilon)$ is as in Lemma 4.1.

Let A, g and h be as in (3.2), with $\beta(\varepsilon) = \gamma(\varepsilon)$. Our hypotheses imply that $\{\phi_g(x): x \in X\}$ is uniformly equi-LC^n, so we can pick an open cover $\mathscr{U}$ of X and a representation $u: N^{n+1}(\mathscr{U}) \to Y$ as in Lemma 4.1 (applied to ϕ_g). Now recall that, by Dugundji's extension theorem, every convex subset of a Banach space has the extension property with respect to metric spaces. By inductively climbing up the i-skeletons of $N(\mathscr{U})$ with $i > n + 1$, we can therefore extend u to a representation $v: N(\mathscr{U}) \to Y$ such that $v(\sigma) \subset \operatorname{conv} v(\dot{\sigma})$ for every simplex σ of $N(\mathscr{U})$ with dim $\sigma > n + 1$. It follows that $v(\sigma) \subset \operatorname{conv} u(\sigma \cap N^{n+1}(\mathscr{U}))$ for every simplex σ of $N(\mathscr{U})$.

From now on, the proof will closely follow the proof of Theorem 1.1 given in [7].

Let $\{V_U: U \in \mathscr{U}\}$ be an open cover of X such that $\bar{V}_U \subset U$ for all $U \in \mathscr{U}$. For each $x \in X$, let $\sigma(x)$ be the simplex of $N(\mathscr{U})$ spanned by those $U \in \mathscr{U}$ for which $x \in \bar{V}_U$. Let $S = X - Z$. For each $s \in S$, let $H_s = \bigcup\{\bar{V}_U: s \notin \bar{V}_U\}$, and let

$$G_s = \{x \in X: v(\sigma_s) \subset [B_{\mu}(\phi_g(x)) \cap B_{\varepsilon}(h(x))]\} - H_s \ .$$

Then G_s is open in X by Lemmas 2.1 and 2.2, while $s \in G_s$ by the properties of $\mathscr{U}$ and u stated in Lemma 4.1, our construction of v, and the fact that $B_{\mu}(\phi_g(s))$ and $B_{\varepsilon}(h(s))$ are convex. Also, by the last part of the definition of G_s, $\sigma_x \subset \sigma_s$ for every $x \in G_s$.

Let $G = \bigcup_{s \in S} G_s$, and let $E = X - G$. Then $E \subset Z$ and E is closed in X, so dim $E \leq n + 1$. Now $\{V_U \cap E: U \in \mathscr{U}\}$ is a locally finite, relatively open cover of E, so it has a relatively open refinement $\{O_U: U \in \mathscr{U}\}$ of (indexed) order $\leq n + 2$ with $O_U \subset V_U$ for all

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[7] It is worth recording that this will also establish following result: If ϕ is as in Theorem 1.2, then ϕ has property (3.1).

$U \in \mathscr{U}$. Let $W_U = V_U \cap (O_U \cup G)$. Then $\{W_U \colon U \in \mathscr{U}\}$ is an open cover of X, $W_U \subset V_U \subset U$ for all $U \in \mathscr{U}$, and every $x \in E$ is in W_U for at most $n+2$ elements $U \in \mathscr{U}$. Let $\xi \colon X \to N(\mathscr{U})$ be a canonical map obtained by using a partition of unity on X subordinated to $\{W_U \colon U \in \mathscr{U}\}$, and define $f \colon X \to Y$ by $f = v \circ \xi$. Let us show that $d(f, \phi_g) < \mu$ and $d(f, h) < \varepsilon$, as required by (3.2).

If $x \in E$, then $\xi(x) \in N^{n+1}(\mathscr{U})$, so

$$f(x) = v(\xi(x)) = u(\xi(x)) \in u(\sigma_x) \subset B_\mu(\phi_g(x)) \cap B_\varepsilon(h(x)) \,,$$

where the last inclusion holds by our hypotheses on u. Now suppose $x \in G$. Then $x \in G_s$ for some $s \in S$. Hence $\sigma_x \subset \sigma_s$, so

$$f(x) = v(\xi(x)) \in v(\sigma_x) \subset v(\sigma_s) \subset B_\mu(\phi_g(x)) \cap B_\varepsilon(h(x)) \,,$$

where the last inclusion holds by the definition of G_s. That completes the proof.

6. **Proof of Theorem 1.3.** We begin by recalling a definition from [5]. A metric space Y is a *uniform* ANR if to every $\varepsilon > 0$ corresponds a $\delta(\varepsilon) > 0$ with the following property: If Y is embedded isometrically as a closed subset in a metric space (E, d), then there exists a retraction $r \colon B_{\delta(\infty)}(Y) \to Y$ such that $d(z, r(z)) < \varepsilon$ whenever $z \in E$ and $d(z, Y) < \delta(\varepsilon)$. If one can take $\delta(\infty) = \infty$ (so that the domain of r is always E), then Y is called a *uniform* AR.

Before proving Theorem 1.3, we need the following lemma.

LEMMA 6.1. *Let* (Y, d_0) *be a metric* ANR (*resp.* AR), *and let* $\mathscr{S} \subset 2^Y$ *be equi-LC^n in* Y. *Then there exists a compatible metric* $d \geqq d_0$ *on* Y *such that:*
  (a) (Y, d) *is a uniform* ANR (*resp. uniform* AR).
  (b) $\mathscr{S}$ *is uniformly* equi-LC^n *with respect to* d.

Proof. The existence of a metric d satisfying (a) was proved in [5, Theorem 1.1][3], and the existence of a metric d satisfying (b) was proved in [4, Proposition 2.1]. Both results were proved with the aid of [2, Theorem 1], and that theorem can similarly be applied to obtain a metric d satisfying (a) and (b) simultaneously. We omit the details.

Having established Lemma 6.1, we now proceed to the proof of Theorem 1.3. We begin by remetrizing Y with a metric d as in Lemma 6.1, with $\mathscr{S} = \{\phi(x) \colon x \in Z\}$; since the original metric d_0 on Y complete, so is the metric $d \geqq d_0$. Next, we embed (Y, d) isometrically in a Banach space (E, d). Let $\delta(\varepsilon)$ and $r \colon B_{\delta(\infty)}(Y) \to Y$ be as in

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[3] For AR's, it had previously been proved by H. Toruńczyk in [8, Proposition 2.2].

the above definition of a uniform ANR. We shall regard ϕ as a map from X to $\mathscr{F}(E)$.

By Propositions 3.3 and 3.4, it will suffice to show that ϕ satisfies condition (3.2),[9] and that one can take $\beta(\infty) = \infty$ in case Y is an AR and $\phi(x)$ is C^n for every $x \in Z$. In fact, we will show that ϕ satisfies (3.2) with $\beta(\varepsilon) = \gamma(\frac{1}{2}\delta(\varepsilon/2))$, where γ is as in Lemma 4.1 and δ is as in the above definition of a uniform ANR.

We proceed as in the proof of Theorem 1.2, with a few modifications. Let A, g and h be as in (3.2), with $\beta(\varepsilon) = \gamma(\frac{1}{2}\delta(\varepsilon/2))$. Our hypotheses imply that $\{\phi_g(x): x \in X\}$ is uniformly equi-LC^n, so we can choose an open cover $\mathscr{U}$ of X and a representation $u: N^{n+1}(\mathscr{U}) \to E$ as in Lemma 4.1 (applied to ϕ_g), but with ε replaced by $\varepsilon' = \frac{1}{2}\delta(\varepsilon/2)$ and with μ replaced by $\mu' = \delta(\mu/2)$. In terms of this ε', the above definition of $\beta(\varepsilon)$ becomes $\beta(\varepsilon) = \gamma(\varepsilon')$, so that $d(h, \phi_g) < \gamma(\varepsilon')$. Moreover, our choice of ε' and μ' yields the following two assertions, whose easy verification is left to the reader. (Recall that $r: B_{\delta(\infty)}(Y) \to Y$ is our retraction.)

(1) If $x \in X$, $e \in E$, and $e \in B_{\varepsilon'}(h(x))$, then $e \in B_{\delta(\varepsilon/2)}(Y)$ and $r(e) \in B_{\varepsilon}(h(x))$.

(2) If $x \in A$, $e \in E$ and $e \in B_{\mu'}(g(x))$, then $r(e) \in B_{\mu}(g(x))$.

Now extend the representation $u: N^{n+1}(\mathscr{U}) \to E$ to a representation $v: N(\mathscr{U}) \to E$ precisely as in the proof of Theorem 1.2. The properties of $\mathscr{U}$, u and v imply that, if σ is a simplex of $N(\mathscr{U})$ and if x is in the intersection of its vertices, then $v(\sigma) \subset B_{\varepsilon'}(h(x))$, so $v(\sigma) \subset B_{\delta(\infty)}(Y)$ by (1). It follows that $v(N(\mathscr{U}))$ is contained in the domain of r, and we can define $w = r \circ v$. Our proof now continues just like the proof of Theorem 1.2, except that v is replaced by w in the definitions of G_s and f. Everything goes through as before, except that it requires a bit more care to check that $s \in G_s$ and that $f(x) \in B_{\varepsilon}(h(x)) \cap B_{\mu}(\phi_g(x))$ when $x \in E$.

(a) $s \in G_s$: We must check that $w(\sigma_s) \subset B_{\varepsilon}(h(s))$ and that $w(\sigma_s) \subset B_{\mu}(\phi_g(s))$. For the first inclusion, observe that $v(\sigma_s) \subset B_{\varepsilon'}(h(s))$ by the previous paragraph, so $w(\sigma_s) \subset B_{\varepsilon}(h(s))$ by (1). For the second inclusion, we distinguish two cases: If $s \notin A$, then $\phi_g(s) = Y$, so our inclusion is clear because $w(\sigma_s) \subset Y$. If $s \in A$, then $\phi_g(s) = \{g(s)\}$; since $u(\sigma_s) \subset B_{\mu'}(\phi_g(s))$ by Lemma 4.1, we have $u(\sigma_s) \subset B_{\mu'}(g(s))$, hence $v(\sigma_s) \subset B_{\mu'}(g(s))$ by the construction of v, so (2) implies that $w(\sigma_s) \subset B_{\mu}(g(s)) = B_{\mu}(\phi_g(s))$.

(b) $f(x) \in B_{\varepsilon}(h(x)) \cap B_{\mu}(\phi_g(x))$ when $x \in E$: Just as in the proof of Theorem 1.2, we have $v(\xi(x)) \in B_{\varepsilon'}(h(x)) \cap B_{\mu'}(\phi_g(x))$ whenever $x \in E$; since $f(x) = r(v(\xi(x)))$, our assertion follows from (1) and (2).

That completes the proof.

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[9] It is worth recording that this will also establish the following result: If $\phi: X \to \mathscr{F}(Y)$ is as in Theorem 1.3, then Y can be remetrized so that ϕ has property (3.1).

REMARK. Since the above proof of Theorem 1.3 is fairly complicated, it may be worthwhile to outline an alternative proof for the important special case where $n = -1$: First of all, we now only need part (a) of Lemma 6.1. Next, let us show that (3.2) is satisfied with $\beta(\varepsilon) = \frac{1}{2}\delta(\varepsilon/2)$, where δ is as in the definition of a uniform ANR: With h and ϕ_g as in (3.2), define

$$\theta(x) = \phi_g(x) \cap B_{\beta(\varepsilon)}(h(x)) ,$$

$$\psi(x) = [\operatorname{conv}(\theta(x)) \cap r^{-1}(B_{\mu/2}(\phi(x)))]^- .$$

Standard facts about l.s.c. maps imply that ψ is l.s.c. Now $\psi(x) = [\operatorname{conv}(\theta(x))]^-$ for all $x \in Z$, so ψ has a selection f' by Theorem 1.1.[10] It is easy to check that $f'(x)$ is in $B_{\delta(\varepsilon/2)}(Y)$ — and thus in the domain of r — for all $x \in X$, so we can define $f = r \circ f'$. It is not hard to check that this f is an extension of g which satisfies all the requirements of (3.2).

7. **Proof of Theorem 1.4.** As observed in the introduction, this theorem reduces to [4, Theorem 1.2] when $Z = \varnothing$. The proof of Theorem 1.4 is very similar to that of the older result given in [4], so it will suffice to indicate the necessary modifications.

(1) Analogously to [4, Proposition 2.1], one proves that there exists a compatible complete metric d on Y which simultaneously makes $\{\phi(x): x \in X - Z\}$ uniformly LC^n and $\{\phi(x): x \in Z\}$ uniformly equi-LC^m. (Observe that $\{\phi(x): x \in X\}$ is therefore also uniformly equi-LC^m.)

(2) After the modification in (1) above, we can follow the proof of [4, Theorem 1.2] essentially without change until we get to the proof of [4, Lemma 7.3]. In that proof, the hypothesis that $\{\phi(x): x \in X\}$ is uniformly equi-LC^n was needed to construct the maps $u_{\sigma,x}$ (for $\dim \sigma = i + 1$) in the middle of p. 572 of [4]. Since $\{\phi(x): x \in X\}$ is uniformly equi-LC^m in our present situation, we can construct $u_{\sigma,x}$ as before when $i \leq m$. For $i > m$, however, $u_{\sigma,x}$ can be defined as before only if $x \in X - Z$, but not, in general, when $x \in Z$. To overcome this obstacle, it suffices to modify the proof of [4, Lemma 7.3] when $i > m$ by first of all shrinking the given open cover $\mathcal{U}$ of X to an open cover $\mathcal{U}' = \{U': U \in \mathcal{U}\}$ with the following two properties:

(a) $U' \subset U$ for all $U \in \mathcal{U}$.

(b) For the cover $\mathcal{U}'$, the construction of suitable maps $u'_{\sigma,x}$ is possible for *all* $x \in X$. More precisely: If σ is a simplex of

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[10] This is where we need the assumption that $n = -1$. For $n > -1$, there is no reason to suppose that ψ satisfies the hypotheses of Theorem 1.2. That explains why, in our proof of the general case of Theorem 1.3, we were unable to *apply* Theorem 1.2 and therefore had to *imitate its proof* instead.

$N^{i+1}(\mathscr{U})$ with $\dim \sigma = i + 1$ and with vertices $U_1, \cdots, U_{i+2}$, and if $x \in U'_1 \cap \cdots \cap U'_{i+2}$ then $u|\dot{\sigma}$ can be extended to a continuous $u'_{x,\sigma}$: $\sigma \to Y$ such that $u'_{x,\sigma}(\sigma) \subset B_\mu(\phi(x))$ and $\operatorname{diam} u'_{x,\sigma}(\sigma) < \varepsilon/2$.

We construct $\mathscr{U}'$ as follows. Let $S = X - Z$. If σ is a simplex of $N^{i+1}(\mathscr{U})$ with $\dim \sigma = i + 1$, and if $x \in S$ is in the intersection of the vertices of σ, then (since $x \notin Z$) we can define $u_{x,\sigma}$, $W_{x,\sigma}$ and W_x precisely as in the middle of p. 572 of [4]. Now pick an open cover $\{U^*\colon U \in \mathscr{U}\}$ of X such that $\bar{U}^* \subset U$ for all $U \in \mathscr{U}$, and for $s \in S$ let

$$G_s = W_s - \bigcup\{\bar{U}^*\colon U \in \mathscr{U}, s \notin \bar{U}^*\}\,.$$

Clearly G_s is open in X and $s \in G_s$. Let $G = \bigcup_{s \in S} G_s$, and let $E = X - G$. Then $E \subset Z$ and E is closed in X, so $\dim E \le m + 1$. Hence the relatively open cover $\{U^* \cap E\colon U \in \mathscr{U}\}$ of E has a relatively open refinement $\{O_U\colon U \in \mathscr{U}\}$ of (indexed) order $\le m + 2$ with $O_U \subset U^*$ for all $U \in \mathscr{U}$. Now let $U' = U^* \cap (O_U \cup G)$ for all $U \in \mathscr{U}$, and let $\mathscr{U}' = \{U'\colon U \in \mathscr{U}\}$. Clearly $\mathscr{U}'$ is an open cover of X, so let us verify that it satisfies conditions (a) and (b).

That (a) is satisfied is clear, so it remains to check (b). Now if $x \in E$, then (b) is trivially satisfied, since then $x \in U'$ for at most $m + 2 \le i + 1$ sets $U \in \mathscr{U}$. So suppose $x \in G$. Then we can choose $s \in S$ such that $x \in G_s$. Now if σ and $U_1, \cdots, U_{i+2}$ are as in (b), then $x \in U_1^* \cap \cdots \cap U_{i+2}^*$, so the definition of G_s implies that

$$s \in (\bar{U}_1^* \cap \cdots \cap \bar{U}_{i+2}^*) \subset (U_1 \cap \cdots \cap U_{i+2})\,.$$

Thus s is in the intersection of the vertices of σ, so $u_{s,\sigma}$ and $W_{s,\sigma}$ are defined and $W_s \subset W_{s,\sigma}$. Hence $x \in G \subset W_s \subset W_{s,\sigma}$. We now let $u'_{x,\sigma} = u_{s,\sigma}$. This choice of $u'_{x,\sigma}$ satisfies (b) because $x \in W_{s,\sigma}$. That completes the proof.

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