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CONSTRUCTION OF Z_p-ACTIONS ON MANIFOLDS

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This is one in a series of papers about PL topological actions of the prime order group Z_p on the m-dimentional ball B^m.

Recall that P. A. Smith [15] has shown that any fixed point set $K \subset B^m$ of an action $Z_p \times B^m \to B^m$ must satisfy

$$(1) \qquad \bar{H}_*(K, Z_p) = 0$$

$$(2) \qquad (K, K \cap \partial B^m) \text{ is a } Z_p\text{-homology manifold pair .}$$

More recently the author has shown [7] that if $K \cap B^m$ is the fixed point set of a PL topological action $Z_p \times B^m \to B^m$, and p odd, then K also satisfies

$$(3) \qquad h_*(K) = 0 .$$

Here

$$h_*(K) \in H_*(K, Z)$$

is a characteristic class defined for all rational-homology manifolds. There are many examples of K satisfying (1), (2) above, but not satisfying (3) above. Properties of $h_*(K)$, and of related characteristic classes, can be found in [7]. An important step in the verification of (3) above was a PL equivariant index theorem (see Theorem A below) which shall be described now.

Let G_p denote the Witt group of nondegenerate symmetric forms over the field Z_p: each element in G_p is represented by a symmetric matrix (a_{ij}) over Z_p with $\det(a_{ij}) \neq 0$; (a_{ij}) and (b_{ij}) are added in G_p by forming their direct sum $(a_{ij}) \oplus (b_{ij})$; the zero element of G_p is represented by any direct sum of hyperbolic planes $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. G_p also has a ring structure: if $\lambda\colon V \times V \to Z_p$, $\lambda'\colon V' \times V' \to Z_p$ are two symmetric forms representing $\alpha, \beta \in G_p$, then $\alpha \cdot \beta$ is represented by the tensor product $\lambda \oplus \lambda'\colon (V \oplus V') \times (V \oplus V') \to Z_p$.

If -1 is a square mod p then $G_p \cong Z_2 \oplus Z_2$: let $r\colon G_p \to Z_2$ map (a_{ij}) to its rank mod 2, and let $\det\colon G_p \to Z_2$ map (a_{ij}) to its determinant it the group of units Z_p mod the subgroup of square units $Z_p^*/(Z_p^*)^2 \cong Z_2$; then $r \oplus \det\colon G_p \to Z_2 \oplus Z_2$ is an isomorphism. If -1 is not a square mod p, then $G_p \cong Z_4$: [1] is a generator for G_p, and in G_p there are relations $[1] \oplus [1] \oplus [1] \sim [-1]$, $[1] \oplus [-1] \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Let A denote a finitely generated torsion-free module over the group ring $Z(Z_p)$. Following Wall [16, 17], a *Hermitian form* on A is a bilinear mapping $\lambda: A \times A \to Z(Z_p)$ satisfying $\lambda(\alpha\cdot x, \beta\cdot y) = \alpha\cdot\bar{\beta}\lambda(x, y)$ for $\alpha, \beta \in Z(Z_p)$, where if t is a generator for Z_p then $\overline{a_0 + a_1 t + a_2 t^2 + \cdots + a_{p-1}t^{p-1}} = a_0 + a_{p-1}t + a_{p-2}t^2 + \cdots + a_1 t^{p-1}$. To each Hermitian form λ on A is associated a symmetric bilinear form $\hat{\lambda}: A \times A \to Z$ over the integers by the rule $\lambda(x, y) = a_0 + a_1 t + \cdots + a_{p-1}t^{p-1} = \hat{\lambda}(x, y) = a_0$: elsewhere $\hat{\lambda}$ is called the transfer of λ. Set $A_s = \{x \in A \mid s\cdot x = 0\}$, $A_\eta = \{x \in A \mid \eta\cdot x = 0\}$, $\hat{\lambda}_s \equiv \hat{\lambda}|_{A_s \times A_s}$, $\hat{\lambda}_\eta \equiv \hat{\lambda}|_{A_\eta \times A_\eta}$, where $s = t - 1$ and $\eta = 1 + t + t^2 + \cdots + t^{p-1}$. Let $i(\lambda)$, $i_\eta(\lambda)$, $i_s(\lambda)$ denote the indices of $\hat{\lambda}$, $\hat{\lambda}_\eta$, $\hat{\lambda}_s$; note that over the rationals $\hat{\lambda} = \hat{\lambda}_s \oplus \hat{\lambda}_\eta$, so $i(\lambda) = i_\eta(\lambda) + i_s(\lambda)$.

Now let $r: Z_p \times M \to M$ be a PL action of the group Z_p on a closed oriented PL manifold, M, having dimension $4k$. $K \subset M$ denotes the fixed point set of r. We now define the invariants $i(M)$, $i_\eta(M)$, $i_s(M)$, $i_p(K)$.

The first three will denote i, i_η, i_s applied to the Z_p-equivariant intersection pairing

$$\frac{H_{2k}(M, Z)}{\mathrm{Tors}(M)} \times \frac{H_{2k}(M, Z)}{\mathrm{Tors}(M)} \longrightarrow Z(Z_p)$$

where $\mathrm{Tors}(M)$ denotes the torsion subgroup of $H_{2k}(M, Z)$. This intersection pairing is determined by the given orientaion, $[M]$, of M.

Let K_i, $i = 1, 2, , \cdots, q$ be the connected components of K. Each K_i is a closed Z_p-homology manifold (a P. A. Smith theorem). We assume in addition that each K_i is orientable over the integers, i.e., $H_{k_i}(K_i, Z) \cong Z$, $H_j(K_i, Z) = 0\ \forall j > k_i$, where $k_i = \dim(K_i)$.

If $k_i = 0 \bmod 4$, write $k_i = 4l_i$ and choose arbitrarily an integral orientation, $[K_i]$, for K_i. $[K_i]$ determines an intersection pairing

$$\frac{H_{2l_i}(K_i, Z)}{\mathrm{Tors}(K_i)} \times \frac{H_{2l_i}(K_i, Z)}{\mathrm{Tors}(K_i)}$$

where $\mathrm{Tors}(K_i)$ denotes the torsion subgroup of $H_{2l_i}(K_i, Z)$. The $\bmod p$ reduction of this pairing represents some $\alpha_i \in G_p$.

Define $i_p(K) \equiv \sum_i h_i(\alpha_i)$, where the summation runs over all i with $k_i = 0 \bmod 4$, and where the homomorphisms $h_i: G_p \to G_p$ are yet to be defined.

We now define $h_i: G_p \to G_p$: Choose for each K_i a "slice" $Z_p \times D^{4k-k_i} \to D^{4k-k_i}$ of the action $Z_p \times M \to M$ near the fixed point set K_i. These "slices" can be gotten by constructing a Z_p-equivariant cell structure "dual" to a Z_p-equivariant PL triangulation for $Z_p \times M \to M$, and then choosing $Z_p \times D^{4k-k_i} \to D^{4k-k_i}$ to be any equivariant cell dual to a k_i dimensional simplex of K_i. Note that the orbit space $\partial D^{4k-k_i}/Z_p$ is a homotopy lens space, which we'll denote by L_i. For the K_i under consideration, $k_i = 4l_i$. So dimension $(L_i) = 4(k - l_i) - 1$. Thus $H_{2(k-l_i)-1}(L_i, Z) \cong Z_p$ and links dually with itself. Note that there are two linking forms, corresponding to the two distinct integral orientations for L_i. A particular orientation $[L_i]$ is determined from $[M]$ and $[K_i]$ as follows. $[M]$ and $[K_i]$ determine an orientation $[D^{4k-k_i}]$ for the dual cell D^{4k-k_i} by requiring the equation $[M] = [D^{4k-k_i}] \times [K_i]$ to hold near D^{4k-k_i}. Since $L_i = \partial D^{4k-k_i}/Z_p$, $[D^{4k-k_i}]$ determines an orientation $[L_i]$. Now let link: $H_{2(k-l_i)-1}(L_i, Z) \times H_{2(k-l_i)-1}(L_i, Z) \to Q/Z$ denote the linking pairing associated to $[L_i]$. If for all $x \in H_{2(k-l_i)-1}(L_i, Z)$ we have link $(x, x) = a_x/p$ where the *integer* a_x *is a square* mod p then set $h_i \equiv 1$. Otherwise define h_i to send each symmetric matrix $[a_{ij}]$ to $[ba_{ij}]$ where b is some non-square element of Z_p.

Here is an equivalent definition of h_i. Note that link: $H_{2(k-l_i)-1}(L_i, Z) \times H_{2(k-l_i)-1}(L, Z) \to Q/Z$ maps into $Z_p \subset Q/Z$, and thus represents an element $\ln(L) \in G_p$. $h_i: G_p \to G_p$ is just left multiplication by $\ln(L)$.

THEOREM A. *Let* $Z_p \times M \to M$ *be a* PL *action of a group having odd prime order on an oriented, closed,* PL *manifold having dimension equal zero* mod 4.

Then $i_p(K)$, $i(M)$, *and* $i_\eta(M)$ *are related by the following tables.*

Table (1) if $p = 2q + 1$ with $q =$ odd,

$i_p(K)$	$i_\eta(M) + (p-1)i(M) \bmod 8$
[1]	$+2q$
$[1] \oplus [1]$	4
$[1] \oplus [1] \oplus [1]$	$-2q$
0	0

Table (2) if $p = 4q + 1$ with $q =$ odd,

$i_p(K)$	$i_\eta(M) + (p-1)i(M) \bmod 8$
[1]	4
[2]	0
$[1] \oplus [2]$	4
0	0

Table (3) if $p = 8q + 1$

$i_p(K)$	$i_\eta(M) \bmod 8$
[1]	0
[l]	4
$[1] \oplus [l]$	4
0	0

Here l denotes any integer which is not a square mod p.

REMARK. The entries under $i_p(K)$ in Tables 1, 2 completely exhaust the elements of the Witt group G_p. To see this note that by the quadratic reciprocity principle -1 is not a square mod $p \Leftrightarrow p = 2q + 1$ with $q = \text{odd}$. And 2 is not a square mod $p \Leftrightarrow p = 4q + 1$ with $q = \text{odd}$.

This theorem was first formulated and proven by the author, under the condition that 2 generates the group of units in the field Z_p (see [8]). A complete proof and significant generalization was later given by J. P. Alexander and G. C. Hamrick [1].

There is also a characteristic class version of Theorem A, due to the author [7], which follows directly from Theorem A. If $Z_p \times M \to M$ is a PL action on an oriented PL manifold M (M need not be closed), let K denote the fixed point set and $\psi\colon Z_p \times R \to R$ an equivariant regular neighborhood for K in $Z_p \times M \to M$. There are two characteristic classes

$$\sum_i \gamma^i(K) \in \sum_i H^{m-4i}((K, \partial K),\ W(Q) \otimes_Z Z_{(2)})$$

and

$$\sum_i \theta^i(\psi, R) \in \sum_i H^{m-4i}((K, \partial K),\ W(Q(Z_p)) \otimes_Z Z_{(2)}) .$$

Here $m = \text{dimension}\ (M)$, $Z_{(2)}$ denotes the integers locallized at 2, $W(Q)$ and $W(Q(Z_p))$ denote the Witt-Grothendiek group of nonsingular symmetric and hermetian forms over Q and $Q(Z_p)$ respectively. The class $\gamma^*(K)$ depends only on the topological type of $(K, \partial K)$. But the class $\theta^*(\psi, R)$ depends on the PL topological type of $Z_p \times R \to R$. There is a relation between $\gamma^*(K)$ and $\theta^*(\psi, R)$ which is determined directly by the tables in Theorem A. By exploiting this relation, one can prove that $h_*(K) = 0$ for any fixed point of a Z_p action on a PL manifold ($h_*(\ )$ is the characteristic class mentioned in (3) above) [see 7].

In this paper we prove the following related (to Thm. A) result.

THEOREM 0.1. *Let n be any positive integer, and $[a_{ij}]$ a square,* *symmetric matrix over the integers satisfying* $\det(a_{ij}) \neq 0 \bmod p$. *Let* Z_p *deno*?*e the integers localized at* p. *Then there is a* PL *group action* $Z_p \times M \to M$ *on a* $(4 + 4n)$*-dimensional almost paralizable manifold* M, *having a* 4*-dimensional fixed point set* K, *for which the intersection pairing* $H_2(K, Z) \times H_2(K, Z) \to Z$ *is equivalent over* $Z_{(p)}$ *to* $[a_{ij}] \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. *Furthermore* $i_p(K)$, $i(M)$, $i_\eta(M)$ *are related as in the tables given above in Theorem* A.

REMARK. If the action $Z_p \times M \to M$ is restricted to the boundary of a cell which is dual to a 4-dimensional simplex of K, the orbit space will be an exotic lens space, L, of dimension $4n - 1$. If L' is any other exotic lens space of dimension $4n - 1$, there is an action $Z_p \times M' \to M'$ satisfying 0.1, and for which L' is obtained form $Z_p \times M' \to M'$ in the manner just described.

Theorem 0.1 is related in several ways to the authors study of actions $Z_p \times B^m \to B^m$. Originally, Theorem 0.1 was the first step in proving Theorem A above. By forming the connected sum of $Z_p \times M \to M$ in Theorem A with some $CP^n \times (Z_p \times M' \to M')$ where $Z_p \times M' \to M'$ comes from 0.1, it reduces proving Theorem A to the special case when the fixed point set K has a mid-dimensional intersection form equivalent over the rationals to a hyperbolic form. Then a surgery procedure on K and $Z_p \times M \to M$ changed the homology of M and K (but not the value of $i_p(K)$, $i(M)$, $i_\eta(M)$, so that Theorem A becomes obvious.

Another use of Theorem 0.1 (one which shall be followed in a later paper) is to aid in completing surgery on a complicated surgery problem. To any triangulated subset $K \subset B^m$ satisfying (1), (2) above, there is associated a complicated surgery problem $\mathscr{S}$, and surgery can be completed on $\mathscr{S}$ (i.e. $\mathscr{S} = 0$ in a surgery group) iff $K \subset B^m$ is the fixed point set of a PL action $Z_p \times B^m \to B^m$ [see 8]. It is known $\mathscr{S}$ lies in the 2-torsion supgroup of a complicated surgery group, and that $\mathscr{S} = 0$ if $\bar{H}_*$ $(K, Z_2) = 0$ (see [8]). To analyze $\mathscr{S}$ further it is important to relate it to some intrinsic invariant of K. The intrinsic invariant which works is the characteristic class $h_*(K) \in H_*(K, Z_2)$ in (3) above. In further papers it will be shown that surgery can be completed on $\mathscr{S}$ iff $h_*(K) = 0$, thus proving that (1), (2), (3) above are both necessary and sufficient conditions for K to be a fixed point set of some $Z_p \times B^m \to B^m$. In proving the implication

$$h_*(K) = 0 \longmapsto \mathscr{S} = 0 \ ,$$

Theorem 0.1 will be used *in detail*. That is, not just the statement of 0.1 will be used, but also some further properties (see 3.1 below) of $Z_p \times M \to M$ constructed in the proof of 0.1 will be used.

The rest of this paper is organized as follows:

Section 1. Shows that certain torsion free $Z(Z_p)$-modules are actually stably free.

Section 2. Computes $i_\eta(\lambda)$ for certain Hermitian forms $\lambda: Z(Z_p) \times Z(Z_p) \to Z(Z_p)$, and gives an algebraic version of the computations in the tables of Theorem A.

Section 3. Constructs the actions $Z_p \times M \to M$, and examines homology properties.

Section 4. Verifies Tables 1, 2, 3 for the actions constructed in § 3.

It is a pleasure to thank G. Hamrick for some suggestions about this paper. Thanks also to J. Milgram for help with the calculations in § 2, and to the referee his suggestion of Lemma 2.2 to simplify the computations in § 4.

1. Let $0 \to C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} C_{m-2} \xrightarrow{\partial_{m-2}} \cdots C_0$ be a finitely generated free $Z(Z_n)$-chain complex. Suppose the homology groups $H_i(C)$ are torsion prime to n for $i \neq j$, and the torsion part of $H_i(C)$—denoted $\mathrm{Tors}(H_j)$—is also prime to n. It is a well-known fact that under these restrictions the quotient $H_j(C)/\mathrm{Tors}(H_j)$ must be a projective $Z(Z_n)$-module.

THEOREM 1.0. *$H_j(C)/\mathrm{Tors}(H_j)$ is a stably free $Z(Z_n)$-module if one of the following holds*:

(a) *Z_n acts trivially on each of the modules $H_i(C)(i \neq j)$*, $\mathrm{Tors}(H_j)$.

(b) *n is a prime; p is a prime which generates the group of units in the field Z_n; each of the modules $H_i(C)(i \neq j)$*, $\mathrm{Tors}(H_j)$ *has order equal to a power of p.*

Proof.

Step 1. For M equal any of the modules $H_i(C)(i \neq j)$, $\mathrm{Tors}(H_j)$, there is an exact sequence of $Z(Z_n)$-modules

$$0 \longrightarrow F \longrightarrow F' \longrightarrow M \longrightarrow 0$$

where F, F' are finitely generated stably free $Z(Z_n)$-modules.

If the chain complex satisfies (a), then this follows Lemma 1.1 [4].

Suppose (b) is satisfied. First consider the case when $H_i(C)$ $(i \neq j)$, and $\mathrm{Tors}(H_j)$ are Z_p-vector spaces. Set $\Gamma \equiv Z_p(Z_n)$. Γ is the direct sum of two fields $Z_p \otimes Z_p(\theta)$ where θ is a primitive nth root of unity: $Z_p(\theta)$ is a field because by the hypothesis of (b) $1 + x + x^2 + \cdots + x^{n-1}$ is an irreducible polynomial over the field Z_p. So as Γ-modules $H_i(C)(i \neq j)$ and $\mathrm{Tors}(H_j)$ as the direct sum of a finite number of copies of Z_p and $Z_p(\theta)$. The kernels of the first two of the following natural projections

$$Z(Z_n) \longrightarrow Z_p$$
$$Z(Z_n) \longrightarrow \Gamma$$
$$Z(Z_n) \longrightarrow Z_p(\theta)$$

are free $Z(Z_n)$-modules (see § 1 in [3]); hence the kernel of the third projection must be stably free (Schanuel's Lemma). Now it is seen that the kernel of $F' \xrightarrow{\alpha} H_i(C)$ (or of $F' \xrightarrow{\alpha} \mathrm{Tors}(H_j)$) must be stably free, where α is any surjection from a finitely generated, stably free $Z(Z_n)$-module F' (again, Schanuel's Lemma).

To establish Step 1 in general, split $H_i(C)$ (or $\mathrm{Tors}(H_j)$ into its primary components, filter each of the by submodules $A_p^1 \subset A_p^2 \subset A_p^3 \subset \cdots \subset H_i(C)_p$ where A_p^{i+1}/A_p^i is a Z_p-vector space. Using the induction, and the special case just considered, argue that each A_p^i has a length two finite generated free resolution. Clearly then the same holds for $H_i(C) = \otimes_p H_i(C)_p$.

Step 2. There is no loss in supposing $j \neq m$. Consider the exact sequence

$$0 \longrightarrow K_i \longrightarrow C_i \longrightarrow K_{i-1} \longrightarrow H_{i-1}(C) \longrightarrow 0\,,$$

where $K_i \equiv \mathrm{kernel}(\partial_i)$. Using this sequence, Schanuel's Lemma, and Step 1, argue that K_i is stably free for all $i \leqq j$. In particular, $H_j(C)$ has a finitely stably free resolution

$$0 \longrightarrow C_m \longrightarrow \cdots \longrightarrow C_{j+1} \longrightarrow K_j \longrightarrow H_j(C) \longrightarrow 0\,.$$

$\mathrm{Tors}(H_j)$ has a finite stably free resolution by Step 1. It follows that the quotient of these two modules, $H_j(C)/\mathrm{Tors}(H_j)$, also has a finite stably free resolution. But since this last module is torsion free, and with finite projective dimension, it must be projective, [see 5.1 in 13]. Finally a projective module with a finite stably free resolution must be stably free, so $H_j(C)/\mathrm{Tors}(H_j)$ is stably free as claimed.

2. Notations. For two primes p, q, $\left(\frac{q}{p}\right)$ equals $+1$ if q is a square mod p, and equals -1 otherwise.

The calculations needed to prove 0.1, which are purely algebraic in nature, are gathered together in the following lemma.

Any Hermitian for $\lambda: Z(Z_p) \times Z(Z_p) \to Z(Z_p)$ is represented by a matrix $[\alpha]$, where $\alpha \in Z(Z_p)$ with $\bar{\alpha} = \alpha$. This means that $\lambda(1, 1) = \alpha$, so $\lambda(x, y) = x\bar{y}\alpha$ follows from the properties of a hermetian form. For specific α, $i_\eta([\alpha])$ calculates as follows:

LEMMA 2.0.

(a) $i_\eta([t^{(p-1)/2} + t^{(p-1)/2+1}]) = 0$ *for* $p = 4q + 1$.

(b) $i_\eta([1 + a\eta]) = 2q$ *for* $p = 2q + 1$, $a =$ *any integer*.

(c) $i_\eta([\alpha_l]) = 4 \bmod 8$, *for* $p = 8q + 1$, l *an* odd *positive prime integer less than* $p/2$ *with* $\left(\frac{l}{p}\right) = 1$, *and*

$$\alpha_l = t^{(p-2l+1)/2} + t^{(p-2l+1)/2+1} + \cdots + t^{(p-2l+1)/2+2l-1} .$$

Proof of 2.0.

Part (a). Let $W \subset (Z(Z_p))$ be generated over Z by $\{t - 1, t^2 - t, t^3 - t^2, \cdots, t^{(p-1)/2-1} - t^{(p-1)/2-2}\}$. Then with respect to the transfer pairing, $W \perp W$, and $\dim_Q(W^\perp/W) = 2$. A Q-basis for $W^\perp/W$ is given by

$$\{v, w\} \equiv \left\{\left(\sum_{i=(p-1)/4}^{(p-1)/2} t^{2i} - \left(\frac{p-1}{4} + 1\right)\right), \left(\sum_{i=(p-1)/4}^{(p-1)/2} t^{2i+1} - \left(\frac{p-1}{4}\right)\right)\right\} .$$

Then

$$\begin{bmatrix} \langle v, v\rangle & \langle v, w\rangle \\ \langle w, v\rangle & \langle w, w\rangle \end{bmatrix} = \begin{bmatrix} -\left(\frac{p-1}{2}\right) & -\left(\frac{p-1}{2}\right) - 1 \\ -\left(\frac{p-1}{2}\right) - 1 & -\left(\frac{p-1}{2}\right) \end{bmatrix}$$

where $\langle\ \rangle$ is the transfer pairing associated to $[t^{(p-1)/2} + t^{(p-1)/2+1}]$. Since the latter matrix has index zero (see pages 7-11 of [3]), there is $v' \in W^\perp/W \otimes_Z R$ so that $\langle v', v'\rangle = 0$ ($R =$ real numbers). Now $W' = \{v' \cup W \otimes_Z R\}$ satisfies: $W' \perp W'$; $\dim(W') = 1/2 \dim((Z(Z_p))_\eta \otimes_Y R)$. So the index in question is zero.

Part (b). With respect to the real basis $\{t - 1, t^2 - t, t^3 - t^2, \cdots \cdots, t^{p-1} - t^{p-2}\}$, the index in question is computed from the $(p - 1) \times (p - 1)$ matrix

$$\begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & -1 & 2 & & & \\ & & & & \ddots & & \\ & & & & & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}$$

which has 2's down the diagonal, -1's on either side of the diagonal and zero's elsewhere. The index of this matrix is $p-1$ (see pages 7-11 of [3]).

Part (c). Set $\lambda = [\alpha_l]$. Then $i_\eta(\lambda) = i(\hat{\lambda}) - i_s(\lambda)$. $\hat{\lambda}_s$ is represented by $[2lp]$ so $i_s(\lambda) = +1$. Thus it suffices to show that $i(\hat{\lambda}) = +5 \bmod 8$. To do this we'll need the results of Appendix 4 in [2]. In the notation of that Appendix L will denote the Z-module $Z(\boldsymbol{Z}_p)$ provided with the Z-valued symmetric bilinear form $\hat{\lambda}$. Note that $\hat{\lambda}(t^i, t^i) = 0 \forall\ i$ where t is a generator for $\boldsymbol{Z}_p$, thus $\hat{\lambda}(x, x) = 0 \bmod 2$ $\forall x \in Z(\boldsymbol{Z}_p)$. So L is of type II and Milgrams Theorem (pg. 127 [2]) applies. $L^\#$ is the subgroup of $L \otimes Q$ generated by L and $\eta/2l$, where $\eta = 1 + t + t^2 + \cdots + t^{p-1}$. Thus $L^\#/L \cong Z_{2l}$ and has $x \equiv \eta/2l$ for generator,

$$\varphi(x) \equiv (1/2)\hat{\lambda}(\eta/2l, \eta/2l) = (1/2)\cdot[1/(4l^2)]\cdot 2lp = \frac{p}{4l} .$$

So by Milgram's Theorem $i(\hat{\lambda}) = +5 \bmod 8$ if and only if $\sum_{k=0}^{2l-1} \exp(\pi i k^2(p/4l))$ is a positive real multiple of $\exp(2\pi i\ 5/8)$.

We will show this in several steps. First we claim

$$\sum_{k=0}^{2l-1} \exp\left(2\pi i k^2 \cdot \frac{p}{4l}\right) = \underbrace{\left(\sum_{k=0}^{1} \exp\left(2\pi i k^2 \cdot \frac{lp}{4}\right)\right)}_{\alpha} \cdot \underbrace{\left(\sum_{k=0}^{l-1} \exp\left(2\pi i k^2 \cdot \frac{p}{l}\right)\right)}_{\beta} .$$

This follows by splitting $L^\#/L$ into its primary components

$$L^\#/L \cong Z_2 \oplus Z_l .$$

Note, this is an orthognal with respect to the quadratic function φ, i.e.,

$$\varphi(x \oplus y) = \varphi(x \oplus 0) + \varphi(0 \oplus y)$$

for any $x \in Z_2$, $y \in Z_l$. So

$$\begin{aligned} &\sum_{k=0}^{2l-1} \exp(2\pi i k^2 \cdot p/4l) \\ &= \sum_{x \in Z_2} \sum_y \exp(2\pi i \varphi(x \oplus y)) \\ &= \underbrace{\Big(\sum_{x \in Z_2} \exp(2\pi i \varphi(x \oplus 0))\Big)}_{\alpha} \cdot \underbrace{\Big(\sum_{y \in Z_l} \exp(2\pi i)\varphi(0 \oplus y)\Big)}_{\beta} \end{aligned}$$

In the notation of pg. 85 [9], $\beta = G(p, l)$. The computations of pages 85, 87 [9], show

$$G(p, l) = \begin{cases} \left(\begin{matrix} p \\ l \end{matrix}\right)\cdot(\sqrt{l}) & \text{if } l = 1 \bmod 4 \\ \left(\begin{matrix} p \\ l \end{matrix}\right)\cdot(i\sqrt{l}) & \text{if } l = 3 \bmod 4 . \end{cases}$$

Also, since $p = 8q + 1 \Rightarrow lp = l \bmod 4$; so

$$\alpha = \begin{cases} 1 + \exp\left(\frac{2li}{4}\right) & \text{if } l = 1 \bmod 4 \\ 1 + \exp(2\pi i \cdot 3/4) & \text{if } l = 3 \bmod 4 . \end{cases}$$

From these calculations for β and α it follows that $\alpha\cdot\beta = \left(\begin{matrix} p \\ l \end{matrix}\right)\sqrt{l}$ $(1 + i)$. Since $p = 8q + 1$ and $\left(\begin{matrix} l \\ p \end{matrix}\right) = -1$, by quadratic reciprocity (p. 78 [9]) we get $\left(\begin{matrix} p \\ l \end{matrix}\right) = -1$. So $\alpha\cdot\beta = -\sqrt{l}(1+i) = \sqrt{2l}\exp(2\pi i 5/8)$ as desired.

LEMMA 2.1. *Let* $[m_{ij}]$ *denote a symmetric matrix with integer entries, satisfying: index* $([m_{ij}]) = k$, $\det([m_{ij}]) = \pm 1$. *Then the* mod$p$ *reduction of* $[m_{ij}]$ *equals* $k[1]$ *in* G_p.

Proof of 2.1. $[m_{ij}]$, $[m_{ij}] \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ are equal in G_p. Since the latter matrix is indefinite it is congruent over $Z(1/2)$ to

$$\underbrace{[1] \oplus [1] \oplus \cdots \oplus [1]}_{k_2\text{-fold}} \oplus \underbrace{[-1] \oplus [-1] \oplus \cdots \oplus [-1]}_{k_2\text{-fold}}$$

where $k_1 - k_2 = k$ (see Theorems 1, 2 in [11]).

In the remainder of this section we give a algebraic version of Theorem 0.1. We need the following notation. If R denotes a module over the group ring $Z(Z_p)$, then $(\ )\otimes_{ZZ_p} R$ will denote the tensor product over $Z(Z_p)$ with R. For example each of $Z(\theta)$, Z, Z_p is a module over $Z(Z_p)$ via the following scheme of ring homorphisms

$$\begin{array}{ccc} Z(Z_p) & \longrightarrow & Z(\theta) \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z_p . \end{array}$$

Here $Z(\theta)$ is the ring of integers augmented by a primitive pth root of unity θ, and the group Z_p acts trivially on each of the $Z(Z_p)$-modules Z and Z_p. For any hermetian form $\lambda\colon F \times F \to Z(Z_p)$ defined on a $Z(Z_p)$-module F, there are the hermetian forms

$$\lambda \otimes_{ZZ_p} Z(\theta)$$
$$\lambda \otimes_{ZZ_p} Z$$
$$\lambda \otimes_{ZZ_p} Z_p \ ,$$

over the rings $Z(\theta)$, Z, Z_p.

Let $\bar{\lambda}: \bar{F} \times \bar{F} \to Z(\theta)$ denote a hermetian form defined on a free $Z(\theta)$-module $\bar{F}$. We define and index-type invariant, $\operatorname{ind}(\bar{\lambda})$, to be the index (as a symmetric form over Z) of the composition

$$\bar{F} \times \bar{F} \xrightarrow{\bar{\lambda}} Z(\theta) \xrightarrow{\psi} Z$$

where $\psi(a_0 + a_1\theta + \cdots + a_{p-1}\theta^{p-1}) \equiv pa_0 - (a_0 + a_1 + \cdots + a_{p-1})$. Then note, for λ a hermetian form defined on a free $Z(Z_p)$-module F,

$$i_\eta(\lambda) = \operatorname{ind}(\lambda \otimes_{ZZ_p} Z(\theta)) \ . \tag{2.2}$$

PROPOSITION 2.3. *If* $\bar{\lambda}: \bar{F} \times \bar{F} \to Z(\theta)$ *is a nonsingular hermetian form* ($\bar{F}$ *a free* $Z(\theta)$*-module*), *and* $\lambda \otimes_{Z(\theta)} Z_p = 0$ *in* G_p, *then* $\operatorname{ind}(\bar{\lambda}) = 0$ mod 8.

COROLLARY 2.4. *In general, if* $\bar{\lambda}: \bar{F} \times \bar{F} \to Z(\theta)$ *is a nonsingular hermetian form over the free* $Z(\theta)$*-module* $\bar{F}$, *then* $\operatorname{ind}(\bar{\lambda})$ *depends* (mod 8) *only on the value of* $\lambda \otimes_{Z(\theta)} Z_p$ *in* G_p.

Before proving 2.3, we derive from it and 2.4 some tables establishing the relations between $\operatorname{ind}(\bar{\lambda})$ mod 8 and the value of $\bar{\lambda} \otimes_{Z(\theta)} Z_p$ in G_p.

PROPOSITION 2.5. $\bar{F}$ *denotes a free* $Z(\theta)$*-module, and* $\bar{\lambda}: \bar{F} \times \bar{F} \to Z(\theta)$ *a nonsingular hermetian form. Then the following relations always hold between* $\operatorname{ind}(\bar{\lambda})$(mod 8) *and the value of* $\bar{\lambda} \otimes_{Z(\theta)} Z_p$ *in* G_p.

TABLE 1. If $p = 2q + 1$, q = odd.

$\bar{\lambda} \otimes_{Z(\theta)} Z_p$ in G_p	$\operatorname{ind}(\bar{\lambda})$ mod 8
[1]	$2q$
$[1] \oplus [1]$	4
$[1] \oplus [1] \oplus [1]$	$-2q$
0	0

TABLE 2. If $p = 4q + 1$, q = odd.

$\bar{\lambda} \otimes_{Z(\theta)} Z_p$ in G_p	$\text{ind}(\bar{\lambda}) \bmod 8$
[1]	4
[2]	0
$[1] \oplus [2]$	4
0	0

TABLE 3. $p = 8q + 1$, $q = \text{odd}$.

$\bar{\lambda} \otimes_{Z(\theta)} Z_p$ in G_p	$\text{ind}(\bar{\lambda}) \bmod 8$
[1]	0
$[l]$	4
$[1] \oplus [l]$	4
0	0

Here l is any positive which is not a square mod p.

Proof of Proposition 2.5. It suffices (see 2.4) to check that the values in the tables hold for some choices of $\bar{\lambda}$. We shall choose $\bar{\lambda} = \lambda \otimes_{ZZ_p} Z(\theta)$, where $\lambda: F \times F \to Z(Z_p)$ is one of the hermetian forms given in 2.0 above.

Table 1. To verify line 1, set $\lambda = [1]$, $\bar{\lambda} \equiv \lambda \otimes_{ZZ_p} Z(\theta)$. Then $\text{ind}(\bar{\lambda}) = i_\eta(\lambda)$ (see 2.2), and $i_\eta(\lambda) = 2q$ (see 2.0(b)). Obviously $\bar{\lambda} \otimes_{Z(\theta)} Z_p = [1]$ in G_p. This verifies the first line. The other three lines can be deduced from line one and the addativity of the invariants $\lambda \otimes_{Z(\theta)} Z_p$, $\text{ind}(\bar{\lambda})$.

Table 2. To verify line 1, set $\lambda = [1]$ and $\bar{\lambda} = \lambda \otimes_{ZZ_p} Z(\theta)$. Then $\text{ind}(\bar{\lambda}) = i_\eta(\lambda)$ (see 2.2), and $i_\eta(\lambda) = 4q$ (see 2.0 (b)). So $\text{ind}(\bar{\lambda}) = 4 \bmod 8$ ($q = \text{odd}$). Obviously $\bar{\lambda} \otimes_{Z(\theta)} Z_p = [1]$ in G_p. This verifies line 1 in Table 2.

To verify line 2 in Table 2, set $\lambda = [t^{(p-1)/2} + t^{(p-1)/2+1}]$, $\bar{\lambda} \equiv \lambda \otimes_{ZZ_p} Z(\theta)$. Then $\text{ind}(\bar{\lambda}) = i_\eta(\lambda)$ (see 2.2), and $i_\eta(\lambda) = 0$ (see 2.0 (a)). Obviously $\lambda \otimes_{Z(\theta)} Z_p = [2]$ in G_p. This verifies line 2.

The other lines in Table 2 follows from lines one and two, and the addativity of the invariants $\bar{\lambda} \otimes_{Z(\theta)} Z_p$, $\text{ind}(\bar{\lambda})$.

TABLE 3. To verify line one, set $\lambda = [1]$, $\bar{\lambda} \equiv \lambda \otimes_{ZZ_p} Z(\theta)$. Then $\text{ind}(\bar{\lambda}) = i_\eta(\lambda)$ (see 2.2), and $i_\eta(\lambda) = 8q$ (see 2.0(b)). Obviously $\bar{\lambda} \otimes_{Z(\theta)} Z_p = [1]$ in G_p. This verifies line one.

To verify line two, set $\lambda = [\alpha_l]$, where α_l is given in 2.0 (c), and $\bar{\lambda} \equiv \lambda \otimes_{ZZ_p} Z(\theta)$. Then $\text{ind}(\bar{\lambda}) = i_\eta(\lambda)$ (see 2.2), and $i_\eta(\lambda) = 4$

mod 8 (see 2.0 (c)). Also $\bar{\lambda} \otimes_{Z(\theta)} Z_p = [2l] = [l]$ in G_p (because 2 is a square mod p). This verifies line two.

The other lines are deduced from lines one and two and the additivity of the invariants $\bar{\lambda} \otimes_{Z(\theta)} Z_p$, ind$(\bar{\lambda})$.

This completes the proof of 2.5.

Proof of Proposition 2.3. Assume $\bar{\lambda} \otimes_{Z(\theta)} Z_p = 0$ in G_p. Then $\det(\bar{\lambda} \otimes_{Z(\theta)} Z_p) = a^2$ for some $a \in Z_p$. Choose $b \in \{1, 2, 3, \cdots, (p-1)/2\}$ so that a^{-1} equals the mod p reduction of b. Set $\alpha = 1 + \theta + \theta^2 + \cdots + \theta^{b-1}$. Note α is a unit in $Z(\theta)$ (see 1.1 in [4]). Thus $\bar{\alpha}$ is also a unit in $Z(\theta)$. But $\det(\bar{\lambda})$ is a unit in $Z(\theta)$ by the hypothesis of 2.3, so $\det(\bar{\lambda})$ is also a unit in $Z(\theta)$, where

$$\bar{\lambda} \equiv \bar{\lambda} \otimes \begin{bmatrix} 0 & \alpha \\ \bar{\alpha} & 0 \end{bmatrix} .$$

Set

$$[m_{ij}] = \bar{\lambda} \otimes_{Z(\theta)} Z_p .$$

By assumption $\bar{\lambda} \otimes_{Z(\theta)} Z_p$ equals zero in G_p, so there is $T \in GL(Z_p)$. Satisfying

$$T^t[m_{ij}]T = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \cdots \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{n\text{-fold}} .$$

We claim that T is the reduction mod p of an integral matrix s with $\det(S) = \pm 1$. In fact $\det[m_{ij}] = 1$, so $\det(T) = \pm 1$ in Z_p. So T is the product of elementary matrices, permutation matrices, and diagonal matrices having ± 1 down the diagonal. These matrices are the mod p reduction of integral valued elementary matrices, permutation matrices, and diagonal matrices having ± 1 entries down the diagonal. Multiply these last matrices together to get S.

Now recall that $Z(Z_p)$ can be displayed as the fiber product of $Z(\theta)$ and Z over Z_p, by the diagram

$$\begin{array}{ccc} Z(Z_p) & \xrightarrow{g} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle j} \\ Z(\theta) & \xrightarrow{h} & Z_p \end{array}$$

where

$$f(\sum_i a_i t^i) = \sum_i a_i \theta^i$$

$$h(\sum_i a_i \theta^i) = \sum_i a_i \bmod p$$

$$j(n) = n \bmod p$$
$$g\left(\sum a_i t^i\right) = \sum_i a_i \,.$$

In the same way the nonsingular hermetian forms over $Z(Z_p)$ are a fiber product of the nonsingular hermetian forms over $Z(\theta)$ and Z over the nonsingular hermetian forms over the field Z_p. In particular the diagram

2.6.

$$\begin{array}{ccc}
[\beta_{ij}] & \dashrightarrow & (S^t)^{-1}\underbrace{\left[\begin{bmatrix}0&1\\1&0\end{bmatrix}\oplus\cdots\oplus\begin{bmatrix}0&1\\1&0\end{bmatrix}\right]}_{n\text{-fold}}S^{-1} \\
\Big\downarrow & & \Big\downarrow \\
\bar{\lambda}\oplus\begin{bmatrix}0&\alpha\\\bar{\alpha}&0\end{bmatrix} & \longrightarrow & [m_{ij}]
\end{array}$$

defines a nonsingular hermetian form $[\beta_{ij}]$ over $Z(Z_p)$, as the fiber product of $\bar{\lambda}\oplus\begin{bmatrix}0&\alpha\\\bar{\alpha}&0\end{bmatrix}$ with $(S^t)^{-1}\left[\begin{bmatrix}0&1\\1&0\end{bmatrix}\oplus\cdots\oplus\begin{bmatrix}0&1\\1&0\end{bmatrix}\right]S^{-1}$ over $[m_{ij}]$. Note that all of the last three forms are nonsingular because $\det(S)$ ± 1, α is a unit in $Z(\theta)$, and $\bar{\lambda}$ is nonsingular by the hypothesis of 2.3.

We remark that any nonsingular hermetian form λ over $Z(Z_p)$ is even if $\lambda \otimes_{ZZ_p} Z$ is even. By 2.6,

$$[\beta_{ij}] \otimes_{ZZ_p} Z = (S^t)^{-1}\left[\begin{bmatrix}0&1\\1&0\end{bmatrix}\oplus\cdots\oplus\begin{bmatrix}0&1\\1&0\end{bmatrix}\right]S^{-1}$$

which is even. So $[\beta_{ij}]$ is even.

Now we complete the proof of 2.3. There are the following equalities.

2.7.

(a) $\operatorname{ind}(\bar{\lambda}) = i_\eta[\beta_{ij}]$

(b) $i_s[\beta_{ij}] + i_\eta[\beta_{ij}] = i[\widehat{\beta_{ij}}]$

(c) $i[\widehat{\beta_{ij}}] = 0 \bmod 8$

(d) $i_s[\beta_{ij}] = 0$.

Note that 2.3 follows direct from 2.7.

To verify 2.7 apply 2.2 and 2.6.

To verify 2.7 (b) review the definitions of i_s, i_η, i.

To verify 2.7 (c), recall that $[\beta_{ij}]$ is even and nonsingular. So $[\widehat{\beta_{ij}}]$ is even and nonsingular over the integers.

To prove 2.7(d) note $i_s[\beta_{ij}] = i([\beta_{ij}] \otimes_{ZZ_p} Z)$. By 2.6

$$[\beta_{ij}] \otimes_{ZZ_p} Z = (S^t)^{-1}\left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right] S^{-1} .$$

This completes the proof of 2.3.

LEMMA 2.8. *Let* $\lambda\colon F \times F \to Z(Z_p)$ *be a hermetian form, with* F *a finitely generated free module over* $Z(Z_{(p)})$. *Then* mod 8 *we have*

$$i_\eta(\lambda) = -(p-1)i(\hat{\lambda}) \bmod 8 .$$

Proof. We first remark that the equality holds for λ of the form $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or [1]. Also, the equation holds for a general λ iff it holds for $\lambda \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $\lambda \oplus [1]$. So without loss of generality we may assume

(2.9) $\quad \operatorname{rank}(\lambda) = \text{any convenient number} \bmod 4 .$

$$\det(\lambda \otimes_{ZZ_p} Z) = \pm 1 \quad \text{in} \quad Z_p .$$

Next, we remark that by 2.2 we have

(2.10) $\quad \operatorname{ind}(\bar{\lambda}) = i(\lambda) \bmod 8$

where

$$\bar{\lambda} \equiv \lambda \otimes_{ZZ_p} Z(\theta) .$$

Case 1. $p = 4q + 1$. We take $\operatorname{rank}(\lambda) = 0 \bmod 2$ (2.9). Then $i(\hat{\lambda}) = 0 \bmod 2$. So the equation in 2.8 take the form $i_\eta(\lambda) = 0$ mod 8. By 2.10, 2.5 takes 2, 3, it suffices to show $\bar{\lambda} \otimes_{Z(\theta)} Z_p$ equals zero in G_p. This follows from 2.9 and the choice $\operatorname{rank}(\lambda) = 0 \bmod 2$.

Case 2. $p = 2q + 1$, $q =$ odd. We choose $\operatorname{rank}(\lambda) = 0 \bmod 4$. Then the equation in 2.8 takes the form $i_\eta(\lambda) = 0 \bmod 8$. By 2.10, 2.5 table, it suffices to show $\bar{\mu} \otimes_{Z(\theta)} Z_p = 0$ in G_p. Because $\operatorname{rank}(\lambda) = 0$ mod 4, $\operatorname{rank}(\bar{\lambda} \otimes_{Z(\theta)} Z_p) = 0 \bmod 4$. Moreover $\det(\bar{\lambda} \otimes_{Z(\theta)} Z_Z) = +1$ (2.9). This implies $\bar{\lambda} \otimes_{Z(\theta)} Z_p = 0$ in G_p.

This completes the proof of 2.8.

3. In this section, the actions $Z_p \times M \to M$ are constructed,and their homological properties are determined.

Let $[e_{ij}]$ be a symmetric matrix with integral entries, so that the $\det[a_{ij}]$ is a unit mod p. Note that $[a_{ij}] \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is congruent over the integers localized at p to $[b_{ii}]$ with $b_{ii} = 0$ ($p =$ odd). Using plumbing, construct a framed cobordism extension W of D^4 which realizes $[b_{ij}]$ as its 2-dimensional intersection pairing: there are framed embeddings $S_j^2 \times D^2 \subset W - \partial W j = 1, 2, \cdots, l$ representing a

Z-basis for $H_2(W, Z)$ with respect to which the intersection pairing is represented by $[b_{ij}]$. $[W]$ will denote the orientation of $(W, \partial W)$ which gives this intersection pairing. Note that ∂W is a Z_p-homology 3-sphere. L will denote an arbitrary homotopy lens space of dimension $4k-1 (k \geqq 1)$ with $\pi_1(L) \cong Z_p$. L comes equipped with an integral orientation $[L]$. An invariant $\alpha([L]) \in Z_2$ of the oriented lens space is defined as follows. Let link: $H_{2k-1}(L, Z) \times H_{2k-1}(L, Z) \rightarrow Q/Z$ denote the linking pairing determined by the orientation $[L]$. Then $\alpha([L]) = 0$ if $\text{link}(x, x) = a_x/p$ where the integer a_x is a square $\bmod p$ $\forall x \in H_{2k-1}(L, Z)$. Set $\alpha([L]) = 1$ otherwise. Note that if $\alpha([L]) = 1$, then $\text{link}(x, x) = a_x/p$ with $a_x = \text{square} \bmod p$ holds only if $x = 0$.

If $\alpha([L]) = 0$ then we can find, for any $l > 0$, pairwise disjoint embeddings $\{L_i \subset L \backslash i = 1, 2, \cdots, l\}$ of $(2k-1)$-dimensional lens spaces L_i in L, which induce isomorphism on fundamental groups, and so that the universal cover inclusions $\hat{L}_i \subset \hat{L}$, $\hat{L}_j \subset \hat{L}$ link in $+1 \forall i, j \leqq l$ with $i \neq j$. If, on the other hand $\alpha([L]) = 1$, then for any l we can find embeddings $\{L_i \subset L | i = 1, 2, \cdots, l\}$ as before but with all the universal cover inclusions $\hat{L}_i \subset \hat{L}$, $\hat{L}_j \subset L$ linking in $b \forall i, j \leqq l$ with $i \neq j$, where b is an integer which is not square $\bmod p$.

Consider the surgery problem $(W, \partial W) \times L \xrightarrow{g \times 1} (D^4, S^3) \times L$, where $g: (W, \partial W) \rightarrow (D^4, S^3)$ is a degree 1 map which map each of $S_j^2 \times D^2 \subset W$ to the point $b_0 \in (D^4, S^3)$. Let the $\{\tau_j | j = 1, 2, \cdots, l\}$ denote the normal bundle to L_j in L.

First extend $g \times 1$ by adding framed 2-handles to $\partial W \times L$ to kill $\pi_1(\partial W \times L)$. This changes $H_2(\partial W \times L)$ from $\{0\}$ to a free $Z(Z_p)$-modulo F (use Theorem 1.0, and recall the ∂W is a Z_p-homology 3-sphere).

Now add framed three handles to a $Z(Z_p)$-basis for F, killing F, and extending $g \times 1$ to a normal map $f: (X, \partial X) \rightarrow (D^4, S^3) \times L$ with $f|_{\partial X}$ a homotopy equivalence. Set

$$\bar{X} \equiv X \times I \cup \{U_{j=1}^l D_j^3 \times D^2 \times \tau_j\}$$
$$\bar{X}_\partial \equiv \partial X \times I$$

where the union is taken along $S_j^2 \times D^2 \times \tau_j \times 1 \sim \partial D_j^3 \times D^2 \times \tau_j$. And extend f to $\bar{f}: (\bar{X}, \bar{X}_\partial) \rightarrow (D^4, S^3) \times L$ by letting $\bar{f}_{|D_j^3 \times D^2 \times \tau_k}$ the projection to $b_0 \times \tau_j \subset D^4 \times L$. Finally extend $\bar{f}$ to $h: (Y, Y_\partial) \rightarrow (D^4, S^3) \times L$ by first completing surgery on $\bar{f}: (\partial_+ \bar{X}, \partial_+ \bar{X}_\partial) \rightarrow (D^4, S^3) \times L$ mod $\bar{f}|_{\partial_+ \bar{X}_\partial}$; and then surgering $\bar{f}: \bar{X} \rightarrow D^4 \times L$ up to the middle dimension by performing interior surgeries away from the polyhedra $(S_j^2 \times 0 \times L_j) \times I \cup D_j^3 \times 0 \times L_j$, $j = 1, 2, \cdots, l$: completing surgery on $f_{|(\partial_+ \bar{X}, \partial_+ \bar{X}_\partial)}$ mod $f_{|\partial_+ \bar{X}_\partial}$ requires the calculation $L_3^h(Z_p) = 0$, which is a direct consequence of R. Lee's calculation $L_3^S(Z_p) = 0$ [10], of Lemma 6.7 in

[12], and of the Rothenberg exact sequence (see [14]).

Note that the original $\bar{X}$ is equal $(X\times I)\cup\{U_{j=1}^{l}D_j^3\times D^2\times\tau_j\}$ and so is contained in Y. Add to both the original $\bar{X}$, and to Y, the set $W\times\mathrm{cone}(L)$. Then the resulting spaces are the orbit spaces of PL actions, $Z_p\times A\rightarrow A$ and $Z_p\times B\rightarrow B$ respectively, both having W for fixed point set.

The $Z_p\times M\rightarrow M$ considered in this section are now obtained by adding to $Z_p\times B\rightarrow B$ the cone action cone $(Z_p\times\partial B\rightarrow\partial B)$. The fixed point set, K, equals $W\cup\mathrm{cone}(\partial W)$, which is a Z_p-homology manifold$\cdot[b\cdot b_{ij}]$ represents $i_p(K)$ in G_p, where $b=+1$ if $\alpha([L])=0$, and b is an integer which is not a square mod p if $\alpha([L])=1$. Note that $\partial B=S^{4k+3}$, so M is a PL manifold required.

That $\partial B=S^{4k+3}$ requires some proof. A careful inspection of the construction of B makes clear that $\partial B=(\partial W\times\mathrm{cone}(\hat{L})\cup C\cup\hat{\bar{X}}_\partial\cup\partial_+\hat{\bar{X}}$: where (C,∂_+C) is the surgery cobordism gotten by doing equivariant 1 and 2-surgeries on $g\times 1{:}\,\partial W\times\hat{L}\rightarrow S^3\times\hat{L}$ to get the homotopy equivalent $f{:}\,\partial\hat{\bar{X}}\rightarrow S^3\times\hat{L}$ of the last paragraph: X_∂, $\partial_+\bar{X}$ are defined in the last paragraph; and the unions are taken along $\partial_-C=\partial W\times\hat{L}$, $\partial_+C=\partial_-(\hat{\bar{X}}_\partial)$, $\partial(\partial_+\hat{\bar{X}})=\partial_+(\hat{\bar{X}})$. Note that by construction $\partial_+\bar{X}$ is homotopy equivalent to $D^4\times L$, $\bar{X}_\partial$ is the product cobordism. Now a Van Kampen argument (using the above decomposition for ∂B) shows that $\pi_1(\partial B)=\{1\}$, and a Mayor-Viotoris argument shows that $\bar{H}_*(\partial B)=H_{4k+3}(\partial B)=Z$. So $\partial B=S^{4k+3}$ by the PL Poincare conjecture.

Note that the action $Z_p\times A\rightarrow A$ is a subset of the action $Z_p\times B\rightarrow B$. Calculations shows that $H_{2k+2}(A,Z)$ is a free Z-module with Z-basis represented by the polyhedra $P_j=(S_j^2\times\mathrm{cone}(\hat{L}_j)\cup D_j^3\times\hat{L}_j)1\leqq j\leqq l$; $\bar{H}_i(A,Z)=0$ for $i\neq 2k+2$ or 2; $H_2(A,Z)$ is a torsion group of order prime to p on which Z_p acts trivially. The P_j are Z_p-invariant, so Z_p acts trivially on $H_{2k+2}(A,Z)$ also.

Finally consider the intersection number $[P_j]\cap[P_i]$ of homology classes. I claim that $[P_j]\cap[P_i]=b([S_j^2]\cap[S_i^2])$, where $[S_j^2]\cap[S_i^2]$ is computed in the 4-dimensional manifold W. To see this, choose PL isotopies ϕ_t^j; $\mathrm{cone}(\hat{L})\rightarrow\mathrm{cone}(\hat{L})$ satisfying

(a) $\phi_t^j|_{\hat{L}}=$ identity

(b) the $\phi_1^j(\mathrm{cone}(\hat{L}_j))$, $\phi_1^i(\mathrm{cone}(\hat{L}_i))$ intersect pairwise transversely in a finite number of pts.

By the choice of the embeddings $L_i, L_j\subset L$ the $\hat{L}_i$, L_j are pairwise disjoint and have linking number in $\hat{L}$ equal b. So the intersection number of $\phi_1^j(\mathrm{cone}(\hat{L}_j))$, $\phi_1^i(\mathrm{cone}(\hat{L}_i))$ in $\mathrm{cone}(\hat{L})$ must be b if $i\neq j$. Set $P_j'=(S_j^2\times\phi_1^j(\mathrm{cone}(\hat{L}_j)))\cup D_j^3\times\hat{L}_j$. Then P_j, P_j' represent the same homology class, and the P_j', P_i' intersect transversely with intersection number $b\cdot([S_j^2]\cap[S_j^2])$. Thus $[P_j]\cap[P_i]=b\cdot([S_j^2]\cap[S_i^2])$ as claimed. So the Z_p-intersection pairing

$$H_{2k+2}(A, Z) \times H_{2k+2}(A, Z) \longrightarrow Z(Z_p)$$

is *represented* by $[b \cdot b_{ij} \cdot \eta]$, where $b = +1$ if $\alpha([L]) = 0$ and b is an integer which is nonsquare mod p if $\alpha([L]) = 1$.

Concerning $Z_p \times B \to B$: $\bar{H}_*(B, Z) = H_{2k+2}(B, Z)$, which is necessarily a free abelian group. Also, in the exact sequence

$$H_{2k+3}((B, A), Z) \xrightarrow{\partial} H_{2k+2}(A, Z) \underset{j}{\overset{i}{\rightleftarrows}} H_{2k+2}(B, Z) \longrightarrow H_{2k+2}((B, A), Z)$$

there exists a left inverse, j, to the map i modulo the class of p^*-torsion groups $\mathscr{L}_{p*}$: this is so because $\det[b_{ij}]$ is a unit mod p. So mod $\mathscr{L}_{p*} H_{2k+2}((B, A), Z)$ is a free abelian group and $H_i((B, A), Z) = 0$ for $i \neq 2k + 2$. Moreover, Z_p acts trivially on the torsion subgroup of $H_{2k+2}((B, A), Z)$ and $H_i((B, A), Z)$ for $i \neq 2k + 2$, because Z_p acts trivially on $H_*(A, Z)$. So Theorem 1.0, as applied to the $Z(Z_p)$-free cellular chain complex of the relative space (B, A), shows that $H_{2k+2}((B, A), Z)/\mathrm{Tors}(B, A)$ is a stabily free $Z(Z_p)$-module, where $\mathrm{Tors}(B, A)$ denotes the torsion subgroup of $H_{2k+2}((B, A), Z)$. $H_{2k+2}(A, Z) \subset U$ has quotient equal an element of $\mathscr{L}_{p*}$. Z_p acts trivially on U because it does on $H_{2k+2}(A, Z)$, and the restriction to U of the Z-intersection pairing in B is congruent mod p to $[b \cdot b_{ij}]$.

I'll end this section with a list of the things to remember.

We have constructed a PL group action $\psi: Z_p \times M \to M$, on a $4(k + 1)$-dimensional manifold M, satisfying

3.1. (a) M is an almost parallizable, oriented PL manifold, and ψ is orientation preserving.

(b) The fixed point set of ψ is $W \cup \mathrm{cone}(\partial W)$, so it has mid-dimensional intersection pairing represented by $[b_{ij}]$, which is congruent over $Z_{(p)}$ to $[a_{ij}] \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ where $[a_{ij}]$ comes from 0.1.

(c) The orbit space of the *slice* action of ψ is the homotopy lens space L.

(d) $H_{2k+2}(M, Z) = U \oplus F$, where F is a free $Z(Z_p)$-module, and Z_p acts trivially on the torsion free module U.

(e) The restriction to U of the intersection form on M is represented by $\eta[c_{ij}]$, where $\eta = 1 + t + t^2 + \cdots + t^{p-1}$, c_{ij} are integers, $[c_{ij}]$ is congruent mod p to $b \cdot [a_{ij}] \oplus \left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right]$. Here $[a_{ij}]$ comes from 0.1 and $b = 1$ if $\alpha[L] = 0$ and b is a positive integer not equal a square mod p if $\alpha[L] \neq 0$.

(f) Note that the intersection form (over Z) is nonsingular on $U \oplus F$ (because M is a PL manifold) and is an even form (because

M is almost parallizable). Thus the index of this form equals 0 mod 8.

4. In this section we shall complete the proof of Theorem 0.1. The proof will depend on the following two lemmas.

NOTATION. Let $\lambda: (U \oplus F) \times (U \oplus F) \to Z(Z_p)$ denote the intersection for $Z_p \times B \to B$. Let λ_F, λ_U denote its restriction to $F \times F$, $U \times U$, etc. Let $Z(\theta)$ denote the integers with the primitive pth root of unity θ adjoined. Each of Z, $Z(\theta)$, Z_p is a $Z(Z_p)$-module, and Z_p is a Z and $Z(\theta)$-module, via the augmentations

$$Z(Z_p) \rightrightarrows \begin{matrix} Z(\theta) \\ \longrightarrow \\ Z \end{matrix} \rightrightarrows Z_p .$$ Thus there are the tensor product forms

$\lambda_F \otimes_{ZZ_p} Z(\theta)$, $\lambda_F \otimes_{ZZ_p} Z$, $\lambda_F \otimes_{ZZ_p} Z_p$, etc.

LEMMA 4.1. *$\det \lambda_F \otimes_{ZZ_p} Z(\theta)$ is a unit in $Z(\theta)$.*

LEMMA 4.2. *$i_p(K)$ equals $\lambda_F \otimes_{ZZ_p} Z_p$ in G_p.*

Before proving 4.1, 4.2 we shall complete the proof of Theorem 0.1.

Proof of Theorem 0.1. Set $\bar{\lambda} = \lambda_F \otimes_{ZZ_p} Z(\theta)$. By 2.2 we have $\operatorname{ind}(\bar{\lambda}) = i_\eta(\lambda_F)$. Also.

$$i_\eta(M) = i_\eta(\lambda_{U \otimes F}) = i_\eta(\lambda_F) .$$

So

4.3(a). $\operatorname{ind}(\bar{\lambda}) = i_\eta(M)$

By 4.2 we have

4.3 (b). $i_p(K) = \bar{\lambda} \otimes_{Z(\theta)} Z_p$ in G_p.

Now by, by 4.1 $\bar{\lambda}$ must be nonsingular, so Proposition 2.5 is applicable to $\bar{\lambda}$. Applying 2.5, 4.3(a), (b) gives the tables in 0.1.

Thic sompletes the proof of Theorem I.1.

Proof of 4.1. Note that $\det(\hat{\lambda}) = \pm 1$ (see 3.1(f)), so if $x_1, x_2, \cdots, x_n$ are a $Z(Z_p)$-basis for $F \exists$ a dual free $Z(Z_p)$-module $F^* \subset U \oplus F$

generated by $x_1^*, x_2^*, \cdots, x_n^*$ satisfying $[\lambda(x_i, x_j^*)] = \begin{bmatrix} 1 & & & \\ & 1 & 0 & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix}$. Choose $r_j \in U$ so that $x_j^* - r_j \in F \forall_j$. Then $[\lambda_F(x_i, x_j^* - r_j)] = \begin{bmatrix} 1 & & & \\ & 1 & 0 & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix} +$ $\eta[c_{ij}]$ for some integral matrix (c_{ij}), which gives

$$[\lambda_F \otimes_{Z_p} Z(\theta)(x_i, x_j^* - r_j)] = \begin{bmatrix} 1 & & & \\ & 1 & 0 & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix}.$$

This completes the proof of 4.1.

Proof of 4.2. We divide the proof into two cases, depending on whether $\left(\frac{-1}{p}\right) = +1$ or $\left(\frac{-1}{p}\right) = -1$.

Case 1. Suppose $p = 4q + 1$. Such p satisfy $\left(\frac{-1}{p}\right) = +1$. In this case, to show $i_p(K) = \lambda_F \otimes_{ZZ_p} Z_p$ in G_p, it suffices to show that

$$\begin{aligned} \operatorname{rank}(i_p(K)) &= \operatorname{rank}(\lambda_F \otimes_{ZZ_p} Z_p) \bmod 2 \\ \det(i_p(K)) &= \det(\lambda_F \otimes_{ZZ_p} Z_p) \quad \text{in} \quad Z_p^*/(Z_p^*)^2 . \end{aligned}$$

Note that if $\operatorname{rank}(i_p(K)) = 1 \bmod 2$ then the Z-rank of U is odd, and consequently the $Z(Z_p)$-rank of F is odd (see 3.1(f)). Thus the ranks of $i_p(K)$ and $\lambda_F \otimes_{ZZ_p} Z_p$ are equal mod 2. Next note that for any Z-basis $y_1, y_2, \cdots, y$ for U we have

$$[\lambda(y_i, x_j)] = \eta \cdot [d_{ij}]$$

where this equality defines the integral matrix $[d_{ij}]$. So $U' \perp F$, where U' is generated by the $y_i' = y_i - \sum_j \eta \cdot d_{ij} \cdot x_j^*$. Consequently $\det(\hat{\lambda}_{U' \oplus F}) = \det(\hat{\lambda}_{U'}) \cdot \det(\hat{\lambda}_F)$. Moreover

$$\lambda(y_i', y_j') = \lambda(y_i, y_j) - p\eta \cdot \alpha_{ij} ,$$

for some $\alpha_{ij} \in Z(Z_p)$; so $\hat{\lambda}_U$ equals $\hat{\lambda}_U$ mod p. In particular $\det(i_p(K)) = \det(\hat{\lambda}_{U'})$ in $Z_p^*/(Z_p^*)$ (see 3.1(c), and the definition of $i_p(K)$ $U' \oplus F$ generates $U \oplus F \bmod \mathscr{L}_{P^*}$, so $\det(\hat{\lambda}_{U' \oplus F}) = \pm\alpha^2$, where α is a unit mod p. Using the last three determinant equalities, and $\left(\frac{-1}{p}\right) = +1$, one gets $\det(i_p(K)) = \det(\hat{\lambda}_F)$ in $Z_p^*/(Z_p^*)^2$. It remains to see $\det(\hat{\lambda}_F) = \det(\lambda_F \otimes_{ZZ_p} Z_p)$ in $Z_p^*/(Z_p^*)^2$. Let $[\alpha_{ij}]$ be a matrix representation for λ_F, then there is an exact sequence

$$0 \longrightarrow F \xrightarrow{\varepsilon} F \longrightarrow X \longrightarrow 0$$

of $Z(Z_p)$-modules where ε is given by $[\alpha_{ij}]$ and X is a finite p^*-torsion module on which Z_p acts trivially ($\det[\alpha_{ij}]$ is a unit in $Z(\theta)$ by 4.1)). Tensoring with Z over $Z(Z_p)$ gives the exact sequence of Z-modules

$$0 \longrightarrow F \otimes_{ZZ_p} Z \xrightarrow{\varepsilon'} F \otimes_{ZZ_p} Z \longrightarrow X \longrightarrow 0$$

where ε' has $[\mathrm{aug}(\alpha_{ij})]$ for associated matrix. The first of the above sequences displays $\det(\lambda_F) = \pm|X|$, while the second displays $\det(\lambda_F \otimes_{ZZ_p} Z) = \pm|X|$. Thus $\det(\lambda_F \otimes_{ZZ_p} Z_p) = \det(\hat{\lambda}_F)$ in $Z_p^*/(Z_p^*)^2$.

Case 2. Suppose $p = 2q + 1$, $q = $ odd. For such p we have $\left(\frac{-1}{p}\right) = -1$.

We begin by constructing from $(\lambda, U \oplus F)$ another hermetian form $(\lambda', U' \oplus F')$ as follows.

Choose $V \subset U$ so that U/V is all p^*-torsion, and $\det(\hat{\lambda}_V) = \pm 1$ mod p. Then, if $[m_{ij}]$ represents $\hat{\lambda}_V$ with respect to a Z-basis $y_1, y_2, \cdots, y_l$ of V, there will exist $T \in GL(Z_p)$ so that $T^t[\bar{m}_{ij}]T$ and $i_p(K)$ are related by the table

$T^t[\bar{m}_{ij}]T$	$i_p(K)$
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$[1] \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$[1]$
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
$[-1] \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$[-1]$

Here $[\bar{m}_{j}]$ denotes the mod production of $[m_{ij}]$. Necessarily $\det(T) = \pm 1$, so T is the mod p reduction of a matrix S, with integral entries satisfying $\det(S) = \pm 1$. S represents a transformation of $(y_1, \cdots, y_l)$ to a new Z-basis $(y'_1, \cdots, \cdots, y'_l)$ for V with respect to which $\hat{\lambda}_V$ has $S^t[m_{ij}]S$ for matrix representation. $S^t[m_{ij}]S$ can be written as $[d_{ij}] - p \cdot [c_{ij}]$ where $[d_{ij}]$ is the appropriate one of the unimodular matrices listed in the left hand column of the above table. Consider the $Z(Z_p)$-Hermitian form λ_1 defined on the $Z(Z_p)$ free-module $F_1 \oplus F_1^*$

which, with respect to a basis $x_1, x_2, \cdots, x_l$ for F_1 and a "dual" basis $x_1^*, x_2^*, x_3^*, \cdots, x_l^*$ for F_1^*, has $\begin{bmatrix}[c_{ij}] & 1\\ 1 & 0\end{bmatrix}$ for matrix representation. Set $\lambda' \equiv \lambda \oplus \lambda_1$; set $U' \equiv \oplus_i [y_i'']$ where $y_i'' = y_i' - \eta \cdot x_i$; set $F_2 \equiv F \oplus F_1 \oplus F_1^*$. If $F_2 \to (U \oplus F_2)/U'$ denotes the composite $F_2 \subset U \oplus F_2 \to (U \oplus F_2)/U'$, there will be an exact sequence

$$0 \longrightarrow F_2 \longrightarrow (U \oplus F_2)/U' \longrightarrow U/V \longrightarrow 0 .$$

From this we deduce that $(U \oplus F_2)/U'$ is a stably free $Z(Z_p)$-module as follows. U/V is all p^*-torsion, and $(U \oplus F_2)/U'$ is a torsion free Z-module (in fact a calculation shows $\det(\hat{\lambda}'_{U'}) = \pm 1$): so $\exists$ a $Z(Z_p)$-module homomorphism $h: (U \oplus F_2)/U' \to F_2$ having finite cokernel X of order prime to p and having zero kernel. This shows that $(U + F_2)/U'$ is a projective module. Since Z_p acts trivially on U/V, Schanuel's lemma in conjunction with Theorem 1.0 and the exact sequence immediately above show that $(U \oplus F_2)/U'$ is (stably) free as a $Z(Z_p)$-module. Thus $U \oplus F_2 = U' \oplus F'$ with $F' \cong (U \oplus F_2)/U'$. This completes the definition of $(\lambda', U' \oplus F')$.

Now we shall list the properties of $(\lambda', U' \oplus F')$ which we shall need.

4.4. (a) $\lambda'_{F'} \oplus_{ZZ_p} Z_p = \lambda_F \otimes_{ZZ_p} Z_p$.
(b) $\hat{\lambda}'_{U'}$ and $i_p(K)$ are related by the following table.

$\hat{\lambda}'_{U'}$	$i_p(K)$ in G_p
$\begin{bmatrix}0&1\\1&0\end{bmatrix} \oplus \cdots \oplus \begin{bmatrix}0&1\\1&0\end{bmatrix}$	$\begin{bmatrix}0&1\\1&0\end{bmatrix}$
$[1] \oplus \begin{bmatrix}0&1\\1&0\end{bmatrix} \oplus \cdots \oplus \begin{bmatrix}0&1\\1&0\end{bmatrix}$	$[1]$
$\begin{bmatrix}1&0\\0&1\end{bmatrix} \oplus \begin{bmatrix}0&1\\1&0\end{bmatrix} \oplus \cdots \oplus \begin{bmatrix}0&1\\1&0\end{bmatrix}$	$\begin{bmatrix}1&0\\0&1\end{bmatrix}$
$[-1] \oplus \begin{bmatrix}0&1\\1&0\end{bmatrix} \oplus \cdots \oplus \begin{bmatrix}0&1\\1&0\end{bmatrix}$	$[-1]$

(c) $\det(\lambda'_{F'})$ is a unit in $Z(Z_p)$.

To see 4.4(a), note that $\lambda_F \otimes_{ZZ_p} Z_p = (\lambda_{F \oplus F_1 \oplus F_1^*}) \otimes_{ZZ_p} Z_p$ because $\lambda_{F \oplus F_1 \oplus F_1^*} \equiv \lambda_F \oplus \begin{bmatrix}[c_{ij}] & 1\\ 1 & 0\end{bmatrix}$. Let $h: F \oplus F_1 \oplus F_1^* \to F'$ denote the composition

$$F \oplus F_1 \oplus F_1^* \subset U \oplus F \oplus F_1 \oplus F_1^* = U' \oplus F' \to F' .$$

Note that h induces an isomorphism

$$\lambda_{F\oplus F_1\oplus F_1^*} \otimes_{ZZ_p} Z(\theta) \cong \lambda_{F'} \otimes_{ZZ_p} Z(\theta) .$$

So

$$\lambda_{F\oplus F_1\oplus F_1^*} \otimes_{ZZ_p} Z_p = \lambda_{F'} \otimes_{ZZ_p} Z_p \quad \text{in} \quad G_p .$$

We leave the routine verification of 4.4(b) to the reader.

To see 4.4(c), consider the form $[b_{ij}]$ over Z defined by $\eta[b_{ij}] \equiv \lambda'(y_i'', x_j')$, where $x_1', x_j', \cdots, x_q'$ is a $Z(Z_p)$-basis for F'. If $[b_{ij}] = [0]$, then $\det(\hat{\lambda}') = \det(\lambda_U') \cdot \det(\hat{\lambda}_F')$. Because both $\det(\hat{\lambda}')$ and $\det(\hat{\lambda}_{U'}')$ one units in Z it follows that $\det(\hat{\lambda}_{F'}')$ is also. So $\det(\hat{\lambda}_{F'}')$ must be a unit in $Z(Z_p)$. Now if $[b_{ij}] \neq [0]$, choose $z_1', z_2', \cdots, z_q'$ in U' satisfying $\lambda'(y_i'', z_j') = -\eta[b_{ij}]$. Then replace F' by the $Z(Z_p)$-module having $x_1' + z_1', x_2' + z_2', \cdots, x_q' + x_q'$ for basis, and note that the above argument can now be carried out.

We can now complete the proof of 4.2. We do this by considering each of the possible values for $i_p(K)$ in G_p.

$i_p(K) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ in G_p. In this case $[c_{ij}]$ above will be an *even* symmetric matrix over Z. So $\lambda' = \lambda \oplus \begin{bmatrix} [c_{ij}] & 1 \\ 1 & 0 \end{bmatrix}$ will be an even hermetian form. Thus $\lambda_{F'} \otimes_{ZZ_p} Z$ will be an even symmetric form over Z with unit determinant (4.4c). So index $(\lambda_{F'} \otimes_{ZZ_p} Z) = 0 \bmod 8$. Now it follows from Lemma 2.1 that $\lambda_{F'} \otimes_{ZZ_p} Z_p = 0$ in G_p. So $i_p(K) = \lambda_{F'} \otimes_{ZZ_p} Z_p$ in G_p.

$i_p(K) = [1]$ in G_p. Recall that $\hat{\lambda}$ has even rank, because it is an even symmetric form over Z with unit determinant (3.3). So $\lambda' = \lambda \oplus \begin{bmatrix} c_{ij} & 1 \\ 1 & 0 \end{bmatrix}$ also has even rank. By 4.4(b) $\hat{\lambda}_{U'}'$ must have odd rank. So $\lambda_{F'}' \otimes_{ZZ_p} Z$ must also have odd rank, implying that index $(\lambda_{F'}' \otimes_{ZZ_p} Z) = \pm 1 \bmod 4$. If this index is $+1 \bmod 4$, then $\lambda_{F'}' \otimes_{ZZ_p} Z_p$ equals $[1]$ in G_p by Lemma 2.1. So $i_p(K) = \lambda_{F'}' \otimes_{ZZ_p} Z_p$ in G_p as claimed. Now suppose index $(\lambda_{F'}' \otimes_{ZZ_p} Z) = -1 \bmod 4$. We will derive a contradiction from this assumption. By Lemma 2.1 and $\operatorname{ind}(\lambda_{F'}' \otimes_{ZZ_p} Z) = -1$ we get $\lambda_{F'}' \otimes_{ZZ_p} Z_p = [-1]$ in G_p. From 2.2, 2.5 Table 1, and this last equality, we get $i_\eta(\lambda_{F'}') = -2q \bmod 8$, where $p = 2q + 1$. From 4.4(b) we get $i(\hat{\lambda}_{U'}') = +1$. But $i(\hat{\lambda}_{U'}') + i(\lambda_{F'}' \otimes_{ZZ_p} Z) + i_\eta(\lambda_{F'}') = i(\hat{\lambda}')$ and $i(\lambda') = i(\lambda) = 0 \bmod 8$. This leads to

$$1 + (-1 + 4m) + (-2q) = 0 \bmod 8$$

which is impossible when q is odd.

$$i_p(k) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or } [-1] \text{ in } G_p .$$

These two cases can be settled by arguments similar to the previous cases. The remaining details are left to the reader.

This completes the proof of 4.2.

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