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HOPF-C^*-ALGEBRAS AND LOCALLY COMPACT GROUPS

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We define Hopf-C^*-algebras and show one can associate to each locally compact group G a cocommutative Hopf-C^*-algebras $\{C^*(G), d\}$ (here $C^*(G)$ is the C^*-algebra of G) with *involution and coidentity whose intrinsic group is isomorphic* and homeomorphic to G. We also show that if the associated Hopf-C^*-algebras are isomorphic then the groups are isomorphic and homeomorphic.

The problem of finding dual objects for a locally compact group G has been extensively studied. As far as we know, the first one to use Hopf algebras in this context was J. Ernest [5]. We should mention also the work of M. Enock and J. M. Schwartz [4] who, working with Kac algebras, established a duality between categories. Our research is based on the work of P. Eymard [6] and M. E. Walter [11]. We show that the C^*-algebra $C^*(G)$ has a natural Hopf structure (although in general it is not a Hopf algebra in the usual sense) and we recover G from $C^*(G)$ using this Hopf structure. Based in this example we define the general concept of Hopf-C^*-algebras. We hope we will be able to characterize all C^*-algebras coming from groups using Hopf-C^*-algebras in the near future.

In §1 we establish some notation and prove some elementary results needed in the sequel. In §2 we define the concept of Hopf-C^*-algebra and prove that two isomorphic Hopf-C^*-algebras have isomorphic intrinsic groups. We end up this section stating without proof a theorem that characterizes all isometric (algebra) isomorphism between the duals (as Banach spaces) of two cocommutative Hopf-C^*-algebras. The last section is devoted to proving that we can associate to each locally compact group G a Hopf-C^*-algebra whose intrinsic group is isomorphic and homeomorphic to G.

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1. Notation and preliminaries. If X is a set and K is a subset of X, we denote by K^c the set theoretic complement of K; if $f: X \rightarrow Z$ is a function from X into a set Z the restriction of f to K is denoted by $f|K$. By χ_K we mean the characteristic function of K. If X is a locally compact Hausdorff space, we denote by $C_0(X)$ (respectively, $C_\infty(X)$) the algebra of all complex-valued continuous function on X with compact support (respectively vanishing at infinity).

All vector spaces are over the complex numbers. If B is a Banach space, we denote by B^* its dual space. If $\Phi: B_1 \to B_2$ is a linear map between Banach spaces, the transpose of Φ, denoted Φ^t, is the map $\Phi^t: \Psi \in B_2^* \to \Psi\Phi \in B_1^*$. If B is a commutative Banach algebra we denote its spectrum by $\sigma(B)$; as a set we take it to be the set of all nonzero complex-valued homomorphisms of B.

If $\mathscr{H}$ is a Hilbert space, we denote by $\mathscr{B}(\mathscr{H})$ the algebra of all bounded operators on $\mathscr{H}$. Inner products are always denoted by $(\cdot|\cdot)$ or by $(\cdot|\cdot)_{\mathscr{H}}$ when we want to emphasize the Hilbert space we are working in. If $X \subseteq \mathscr{B}(\mathscr{H})$, we denote by X' its commutant.

If A is a C^*-algebra and π is a *-representation of A in some Hilbert space, we denote this Hilbert space by $\mathscr{H}_\pi$. The set of all positive elements of A is written A^+. If A has an identity element, we denote it by 1_A. If A is a von Neumann algebra, then A_*, A_r, A_u denote the predual of A, the set of all invertible elements in A and the group of all unitaries of A respectively.

All topological groups are assumed to be Hausdorff. Approximate identities are always assumed to have norms bounded by 1. By an isomorphism we mean a bijective homomorphism.

Let A be a C^*-algebra. We denote by $M(A)$ the algebra of all double centralizers on A (see [2] for the definition and properties of double centralizers). It is well known that $M(A)$ can be identified with the idealizer of A in A^{**} (i.e., the largest C^*-subalgebra of A^{**} in which A is a two-sided ideal) and we will use this identification whenever convenient. The strict topology of $M(A)$, denoted $S(M(A): A)$, is the locally convex topology generated by the family of pseudonorms $\{\lambda_a, \rho_a: a \in A\}$, where $\lambda_a(b) = \|ab\|$ and $\rho_a(b) = \|ba\|$, $b \in A$. It follows (cf. [2]) that $S(M(A): A)$ is a Hausdorff topology, A is $S(M(A): A)$ dense in $M(A)$ and that $M(A)$ is $S(M(A): A)$-complete. It is easy to see that any net in $M(A)$ coverging strictly (i.e., in the strict topology) must converge in the weak* topology of A^{**}.

PROPOSITION 1.1. *Let A, B be C^*-algebras and let $\Psi: A \to M(B)$ be a *-homomorphism. Then:*

(i) *There exists a unique weakly* continuous *-homomorphism $\tilde{\Psi}: A^{**} \to B^{**}$ extending Ψ.*

(ii) *If there exists a net $\{e_\lambda\} \subseteq A$ such that $\Psi(e_\lambda) \to 1_{M(B)}$ in the strict topology of $M(B)$ then $\tilde{\Psi}(M(A)) \subseteq M(B)$, $\tilde{\Psi}|M(A)$ is the unique *-homomorphism from $M(A)$ into $M(B)$ extending Ψ and*

$$\tilde{\Psi}(x) = strict - \lim \Psi(xe_\lambda) = strict - \lim \Psi(e_\lambda x) \tag{1.1}$$

for all $x \in M(A)$. Moreover, if (e_λ) is an approximate identity for A, then $\tilde{\Psi}(1_{M(A)}) = 1_{M(B)}$.

(iii) *If* $\Psi(A) \supseteq B$, *then there exists a net* $\{e_\lambda\} \subseteq A$ *with* $\Psi(e_\lambda) \to 1_{M(B)}$ *strictly.*

(iv) $\tilde{\Psi}|_{M(A)}$ *is* 1-1 *if and only if* Ψ *is* 1-1.

(v) *If* $\Psi: A \to B$ *is a* $*$*-isomorphism, then* $\tilde{\Psi}|_{M(A)}: M(A) \to M(B)$ *and* $\tilde{\Psi}: A^{**} \to B^{**}$ *are also* $*$*-isomorphisms.*

REMARKS. (1) Assuming the existence of a net $\{e_\lambda\}$ with $\Psi(e_\lambda) \to 1_{M(B)}$ strictly one can prove directly (i.e., without using double duals) that Ψ can be extended to a $*$-homomorphism $\bar{\Psi}: M(A) \to M(B)$ by

$$\bar{\Psi}(x)b = \lim \Psi(xe_\lambda)b \ ,$$
$$b\bar{\Psi}(x) = \lim b\Psi(e_\lambda x) \ ,$$

for $x \in M(A)$, $b \in B$. Moreover this extension is unique. The proof is very easy once one knows that any $*$-homomorphism $\Psi: A \to M(B)$ satisfies

$$||\Psi(xa_1)b - \Psi(xa_2)b|| \leqq ||x||\,||\Psi(a_1)b - \Psi(a_2)b|| \ ,$$
$$||b\Psi(a_1x) - b\Psi(a_2x)|| \leqq ||x||\,||b\Psi(a_1) - b\Psi(a_2)|| \ ,$$

for all $a_1, a_2 \in A$, $b \in B$, $x \in M(A)$.

(2) The existence of a net as above is necessary even for the commutative case if we wish to get a unique extension $\bar{\Psi}: M(A) \to M(B)$. E.g., let $A = C_\infty(\boldsymbol{R})$, $B = \boldsymbol{C} = M(\boldsymbol{C})$, $\Psi: A \to M(B)$ the zero map. Of course the zero map $M(A) \to M(B)$ extends Ψ. Let us show there are other $*$-homomorphisms extending Ψ. Let $\beta\boldsymbol{R}$ be the Stone-Čech compactification of $\boldsymbol{R}$. Then $M(A) = C(\beta\boldsymbol{R}) \cong C_b(\boldsymbol{R})$, where $C_b(\boldsymbol{R})$ is the set of all bounded continuous functions from $\boldsymbol{R}$ into $\boldsymbol{C}$. Let $x_0 \in \boldsymbol{R}^c$. Every function $f \in A$ extends uniquely to $\tilde{f} \in M(A)$ with $\tilde{f}(x_0) = 0$. Define $\bar{\Psi}: M(A) \to M(B) = \boldsymbol{C}$ by $\bar{\Psi}(f) = \tilde{f}(x_0)$. Then $\bar{\Psi}$ is a $*$-homomorphism extending Ψ and $\bar{\Psi} \neq 0$.

Proof of proposition 1.1. (i) follows easily from [3; 2.1]. As for (ii), assume $\{e_\lambda\} \subseteq A$ is a net with $\Psi(e_\lambda) \to 1_{M(B)}$ strictly. Let $z \in \tilde{\Psi}(M(A))$, say $z = \tilde{\Psi}(x)$, $x \in M(A)$. Then, for all $b \in B$,

$$zb = z \lim \Psi(e_\lambda)b = \lim \tilde{\Psi}(x)\Psi(e_\lambda)b = \lim \Psi(xe_\lambda)b \in B$$

(since $xe_\lambda \in A$). Similarly,

$$bz = \lim b\Psi(e_\lambda x) \in B \ .$$

Thus $z \in M(B)$; moreover, for all $b \in B$,

$$\lim b\Psi(xe_\lambda) = \lim (bz)\Psi(e_\lambda) = bz \ ,$$
$$\lim \Psi(e_\lambda x)b = \lim \Psi(e_\lambda)(zb) = zb \ ,$$

so that (1.1) holds. The uniqueness is an easy corollary of (1.1). Now assume that $\{e_\lambda\}$ is an approximate identity for A. Then $e_\lambda \to 1_{M(A)}$ strictly, so in particular $e_\lambda \to 1_{M(A)}$ in the weak* topology of A^{**}; but $\tilde{\Psi}$ is weakly* continuous, so $\tilde{\Psi}(e_\lambda) \to \tilde{\Psi}(1_{M(A)})$ weakly. Since $\Psi(e_\lambda) \to 1_{M(B)}$ strictly, we get $\tilde{\Psi}(1_{M(A)}) = 1_{M(B)}$ and (ii) holds.

If $\Psi(A) \supseteq B$, let $\{u_\lambda : \lambda \in \Lambda\} \subseteq B$ be an approximate identity for B. Choose $e_\lambda \in A$ such that $\Psi(e_\lambda) = u_\lambda$ for all $\lambda \in \Lambda$. Then $\{e_\lambda : \lambda \in \Lambda\}$ is the desired net and (iii) follow.

As for (iv), clearly if $\tilde{\Psi}|_{M(A)}$ is 1-1, then Ψ is 1-1. Suppose that Ψ is 1-1: if $x \in M(A)$ and $\tilde{\Psi}(x) = 0$, then xa, $ax \in A$ and $\Psi(xa) = \tilde{\Psi}(x)\Psi(a) = 0 = \Psi(ax)$ for all $a \in A$, so that $x = 0$.

Suppose $\Psi: A \to B$ is a $*$-isomorphism. Applying (ii) and (iii) to Ψ and Ψ^{-1} and using the uniqueness of the extensions given by (i), we get $\tilde{\Psi}(M(A)) \subseteq M(B)$, $(\Psi^{-1})^{\sim}(M(B)) \subseteq M(A)$ and $\tilde{\Psi}(\Psi^{-1})^{\sim}$, $(\Psi^{-1})^{\sim}\tilde{\Psi}$ are the identity maps on B^{**}, A^{**} respectively. Hence the proposition follows. $\square$

REMARK 1.2. The above proposition remains true if we change all "homomorphisms" to "anti-homomorphisms": just consider the C^*-algebra B^0 opposed to B (i.e., B^0 has the same underlying Banach space but its multiplication is given by $(x, y) \to yx$) and compose Ψ with the natural map $M(B) \to M(B^0)$.

Let A and B be C^*-algebras and let $A \odot B$ be their algebraic tensor product. If α is a norm on $A \odot B$, the completion of $A \odot B$ with respect to α will be denoted $A \otimes_\alpha B$; the dual norm of α [9] will be denoted by α^*. We will denote by α_0 the least C^*-norm among all C^*-norms on $A \odot B$ having finite dual norms. It follows that α_0 and α_0^* are cross norms and α_0 is equal to Guichardet's $*$-norm (cf [9], [7]). We will denote by ν the l.u.b. of all C^*-sub-cross pseudonorms on $A \odot B$; ν is in fact a C^*-cross norm on $A \odot B$ [7]. Since we will use this norm most of the time, we will denote the completion of $A \odot B$ with respect to it simply by $A \otimes B$. Guichardet [7] proved that if G is a locally compact group with C^*-algebra $C^*(G)$, then $C^*(G \times G) \cong C^*(G) \otimes C^*(G)$.

If A and B are von Neumann algebras, we will denote their usual tensor product by $A \bar{\otimes} B$. It follows that

$$(A \bar{\otimes} B)_* = A_* \otimes_{\alpha_0^*} B_* [9] .$$

PROPOSITION 1.3. *Let A and B be C^*-algebras. Then there exists a unique weakly* continuous surjective $*$-homomorphism $\gamma: (A \otimes B)^{**} \to A^{**} \bar{\otimes} B^{**}$ extending the identity on $A \odot B$.*

Proof. Let $C = A \otimes B$ and let ρ, λ, π be the universal representations [3; 2.7.6] of A, B, C respectively, generating the von Neumann algebras $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$. Then $\rho \otimes \lambda$ extends to a nondegenerate $*$-representation, also denoted $\rho \otimes \lambda$, of C on $\mathscr{H}_\rho \otimes \mathscr{H}_\lambda$ generating the von Neumann algebra $\mathfrak{A} \bar{\otimes} \mathfrak{B}$ [7]. We then have the following commutative diagrams (cf [3; 12.11]):

$$\begin{tikzcd} A^{**} & \mathfrak{A} \arrow[l, "\theta"'] \\ A \arrow[u, hook] \arrow[ur, "\rho"'] & \end{tikzcd} \qquad \begin{tikzcd} B^{**} & \mathfrak{B} \arrow[l, "\tau"'] \\ B \arrow[u, hook] \arrow[ur, "\lambda"'] & \end{tikzcd} \qquad \begin{tikzcd} C^*{}_* \arrow[r, "\sigma"] & \mathfrak{C} \arrow[d, "(\rho\otimes\lambda)^\sim"] \\ C \arrow[u, hook] \arrow[ur, "\pi"'] \arrow[r, "(\rho\otimes\lambda)"'] & \mathfrak{A} \bar{\otimes} \mathfrak{B} \end{tikzcd}$$

where θ, τ, σ are weakly* continuous $*$-isomorphism and $(\rho \otimes \lambda)^\sim$ is the unique surjective normal $*$-homomorphism satisfying $(\rho \otimes \lambda)^\sim \pi = \rho \otimes \lambda$. Also, $\theta \otimes \tau$ defines a weakly* continuous $*$-isomorphism $\theta \bar{\otimes} \tau: \mathfrak{A} \bar{\otimes} \mathfrak{B} \to A^{**} \bar{\otimes} B^{**}$. Hence we can define a surjective weakly* continuous $*$-homomorphism γ by $\gamma = (\theta \bar{\otimes} \tau)(\rho \otimes \lambda)^\sim \sigma^{-1}: C^{**} = (A \otimes B)^{**} \to A^{**} \bar{\otimes} B^{**}$. It is clear that $\gamma|_{A \odot B}$ is just the identity map. The uniqueness follows from the weak* density of $A \odot B$ in $(A \otimes B)^{**}$ and the weakly* continuity of γ. $\square$

Again let A and B be C^*-algebras. Since A and B can be naturally identified with two-sided ideals in $M(A)$ and $M(B)$ respectively, $A \otimes B$ can be identified with a two-sided ideal in $M(A) \otimes M(B)$ [7, Corollary 5, page 31]. Thus, by Proposition 3.7 of [2], there is a unique $*$-homomorphism $\mu: M(A) \otimes M(B) \to M(A \otimes B)$ such that $\mu|_{A \otimes B}$ coincide with the natural embedding $A \otimes B \hookrightarrow M(A \otimes B)$. At this point it should be remarked that we do not know when μ will be injective; we believe that $\ker \mu \neq \{0\}$ in general. Of course, if A and B are commutative, then μ is injective since in this case $\nu = \alpha_0$ (see [9]). Even in this case μ will not be surjective [1]. In any case,

$$\operatorname{Ker} \mu = \{x \in M(A) \otimes M(B): x(A \otimes B) = 0\}$$

and μ is continuous with respect to the strict topologies $S(M(A) \otimes M(B): A \otimes B)$ and $S(M(A \otimes B): A \otimes B)$.

2. Hopf-C^*-algebras. In this section we define the concept of Hopf-C^*-algebras and study some of their properties. We remark that our "Hopf-C^*-algebras" are not Hopf algebras in general since the comultiplication takes values in $M(A \otimes A)$ instead of $A \otimes A$ as is the case for Hopf algebras.

Let A be a C^*-algebra and let $d: A \to M(A \otimes A)$ be a $*$-homomorphism. As we have seen in the end of § 1, there are $*$-homomorphism

$$\Psi_1\colon M(A \otimes A) \otimes M(A) \longrightarrow M(A \otimes A \otimes A)\ ,$$
$$\Psi_2\colon M(A) \otimes M(A \otimes A) \longrightarrow M(A \otimes A \otimes A)\ .$$

Since A is a two-sided ideal in $M(A)$, $M(A \otimes A) \otimes A$ and $A \otimes M(A \otimes A)$ can be identified with two-sided ideals in $M(A \otimes A) \otimes M(A)$ and $M(A) \otimes M(A \otimes A)$ respectively, [7, Corollary 5, page 31], so we can define

$$\Psi_1(d \otimes I),\ \Psi_2(I \otimes d)\colon\ A \otimes A \longrightarrow M(A \otimes A \otimes A)\ ,$$

where $I\colon A \to A$ is the identity map. By Proposition 1.1, these maps extend uniquely to normal $*$-homomorphisms

$$[\Psi_1(d \otimes I)]^{\sim},\ [\Psi_2(I \otimes d)]^{\sim}\colon (A \otimes A)^{**} \longrightarrow (A \otimes A \otimes A)^{**}\ .$$

LEMMA 2.1. *Let A be C^*-algebra and let Ψ_1, Ψ_2, d be as above. Assume there is an approximate identity $\{e_\lambda\colon \lambda \in \Lambda\} \subseteq A$ with $d(e_\lambda) \to 1_{M(A\otimes A)}$ strictly. Let $\Gamma = \Lambda \times \Lambda$ and define a partial order in Γ by*

$$(\lambda_1, \lambda_2) \leqq (\lambda_3, \lambda_4) \text{ if and only if } \lambda_1 \leqq \lambda_2 \text{ and } \lambda_3 \leqq \lambda_4\ .$$

For $\gamma = (\lambda_1, \lambda_2) \in \Gamma$, let $u_\gamma = e_{\lambda_1} \otimes e_{\lambda_2}$. Then $\{u_\gamma\colon \gamma \in \Gamma\}$ is an approximate identity for $A \otimes A$ and $\Psi_1(d \otimes I)u_\gamma$, $\Psi_2(I \otimes d)u_\gamma \to 1_{M(A\otimes A\otimes A)}$ strictly. In particular

$$[\Psi_1(d \otimes I)]^{\sim}(M(A \otimes A)) \subseteq M(A \otimes A \otimes A)\ ,$$
$$[\Psi_2(I \otimes d)]^{\sim}(M(A \otimes A)) \subseteq M(A \otimes A \otimes A)\ .$$

Proof. The proof that $\{u_\gamma\colon \gamma \in \Gamma\}$ is an approximate identity for $A \otimes A$ and that $\Psi_1(d \otimes I)u_\gamma$, $\Psi_2(I \otimes d)u_\gamma \to 1_{M(A\otimes A\otimes A)}$ strictly is straightforward. The last part of the lemma is just part (ii) of Proposition 1.1. □

Let $\tau\colon A \otimes A \to A \otimes A$ be the automorphism defined by $\tau(a \otimes b) = b \otimes a$, $a, b \in A$. Let $\tilde{\tau}\colon (A \otimes A)^{**} \to (A \otimes A)^{**}$ be the extension of τ given by Proposition 1.1. Then $\tilde{\tau}$ is a weakly* continuous $*$-isomorphism with $\tilde{\tau}^2$ equal to the identity map and $\tilde{\tau}(M(A \otimes A)) = M(A \otimes A)$.

REMARK ON NOTATION. From now on, if $\Psi\colon A \to B$ (or $\Psi\colon A \to M(B)$) is a $*$-homomorphism or a $*$-anti-homomorphism, A and B C^*-algebras, we will denote by $\tilde{\Psi}$ the extension to the double duals given by Proposition 1.1 or Remark 1.2; if $\tilde{\Psi}(M(A)) \subseteq M(B)$, we will denote by $\bar{\Psi}$ the map $\tilde{\Psi}|_{M(A)}\colon M(A) \to M(B)$.

DEFINITION 2.2. Let A be a C^*-algebra and let Ψ_1, Ψ_2 be as above. We say that $\{A, d\}$ is a Hopf-C^*-algebra if and only if:

(i) $d: A \rightarrow M(A \otimes A)$ is a 1-1 $*$-homomorphism;

(ii) there exists an approximate identity $\{e_\lambda: \lambda \in \Lambda\} \subseteqq A$ with $d(e_\lambda) \rightarrow 1_{M(A \otimes A)}$ strictly;

(iii) (coassociativity) the following diagram is commutative:

$$\begin{array}{ccc} A & \xleftarrow{\quad d \quad} & M(A \otimes A) \\ {\scriptstyle d}\big\downarrow & & \big\downarrow{\scriptstyle [\Psi_1(d \otimes I)]^-} \\ M(A \otimes A) & \xrightarrow[{[\Psi^2(I \otimes d)]^-}]{} & M(A \otimes A \otimes A) \,. \end{array} \tag{2.1}$$

We call d a *comultiplication*. By an *involution* of $\{A, d\}$ we mean $*$-anti-isomorphism $j: A \rightarrow A$ of period two satisfying

$$(j \otimes j)^- d = \bar{\tau} d j \,. \tag{2.2}$$

If there is such a j, we say that $\{A, d, j\}$ is an *involutive* Hopf-C^*-algebra. We say that $\{A, d\}$ is *cocommutative* (also called *symmetric* by some authors) if and only if $\bar{\tau} d = d$. If $\varepsilon: A \rightarrow \mathbf{C}$ is a nonzero $*$-homomorphism satisfying

$$(\varepsilon \otimes I)^- d = I = (I \otimes \varepsilon)^- d \,, \tag{2.3}$$

we say that ε is a *coidentity* for $\{A, d\}$. In the case $\{A, d\}$ has both an involution j and a coidentity ε, we write $\{A, d, j, \varepsilon\}$.

REMARK. Condition (ii) of the above definition insures that $\tilde{d}: A^{**} \rightarrow (A \otimes A)$ preserves identities, that $\bar{d}: M(A) \rightarrow M(A \otimes A)$ is the unique $*$-homomorphism extending d and that $[\Psi_1(d \otimes I)]^\sim(M(A \otimes A)) \subseteqq M(A \otimes A \otimes A)$, $[\Psi_2(I \otimes d)]^\sim(M(A \otimes A)) \subseteqq M(A \otimes A \otimes A)$ (these inclusions are not needed in the sequel; we include them here simply to show that we can stay in the double centralizers algebras). In the case of Hopf-von Neumann algebras, the comultiplication preserves identities by assumption (see [5]).

DEFINITION 2.3. Two Hopf-C^*-algebras $\{A_1, d_1\}$, $\{A_2, d_2\}$ are *isomorphic* if and only if there exists a $*$-isomorphism $\theta: A_1 \rightarrow A_2$ with $(\theta \otimes \theta)^- d_1 = d_2 \theta$. If $\{A_1, d_1, j_1\}$ and $\{A_2, d_2, j_2\}$ are involutive Hopf-C^*-algebras we require further that $\theta j_1 = j_2 \theta$. If the Hopf-C^*-algebras have coidentities $\varepsilon_1, \varepsilon_2$, we also assume that $\varepsilon_1 = \varepsilon_2 \theta$.

PROPOSITION 2.4. *Let $\{A, d\}$ be a Hopf-C^*-algebra. If $\psi, \eta \in A^*$, define a linear functional $\psi\eta$ on A by $\langle a, \psi\eta \rangle = \langle d(a), \psi \otimes \eta \rangle$, $a \in A$. Then the map $m: (\psi, \eta) \in A^* \times A^* \rightarrow \psi\eta \in A^*$ defines a multiplication in A^* making it a Banach algebra. Moreover, if $\{A, d\}$ is cocom-*

mutative, A^* is commutative and if $\{A, d\}$ has a coidentity ε, ε is the identity of A^*.

Proof. If $\psi, \eta \in A^*$, then $\psi \otimes \eta \in (A \otimes A)^*$ with $\|\psi \otimes \eta\| = \|\psi\eta\|$, so it makes sense to define $\psi\eta$ as above. It follows that $\psi\eta \in A^*$ with $\|\psi\eta\| \leqq \|\psi\| \|\eta\|$. Since m is clearly a bilinear map, to show that A^* is a Banach algebra it is enough to prove the associativity of m. We claim that for all $x \in (A \otimes A)^{**}$, $\psi, \phi, \eta \in A^*$, we have

$$\langle x, \psi\phi \otimes \eta\rangle = \langle [\Psi_1(d \otimes I)]^{\sim} x, \psi \otimes \phi \otimes \eta\rangle , \tag{2.4}$$

$$\langle x, \psi \otimes \phi\eta\rangle = \langle [\Psi_2(I \otimes d)]^{\sim} x, \psi \otimes \phi \otimes \eta\rangle . \tag{2.5}$$

We will prove (2.4); (2.5) can be handled in a similar way. Since $[\Psi_1(d \otimes I)]^{\sim}$ is weakly* continuous and $A \odot A$ is weakly* dense in $(A \otimes A)^{**}$, it is enough to show (2.4) holds for $x \in A \odot A$; by linearity, it suffices to consider $x = a \otimes b$, $a, b \in A$. So let $x = a \otimes b$, $a, b \in A$, and let $\psi, \phi, \eta \in A^*$. By Theorem 2.1 of [10], we can write $\psi = f \cdot \psi'$, $\phi = g \cdot \phi'$, $\eta = h \cdot \eta'$, where $f, g, h \in A$ and $\psi', \phi', \eta' \in A^*$. Then:

$$\begin{aligned}\langle x, \psi\phi \otimes \eta\rangle &= \langle a, \psi\phi\rangle\langle b, \eta\rangle = \langle d(a), \psi \otimes \phi\rangle\langle b, \eta\rangle \\ &= \langle d(a), f \cdot \psi' \otimes g \cdot \phi'\rangle\langle b, h \cdot \eta'\rangle = \langle d(a)(f \otimes g), \psi' \otimes \phi'\rangle\langle bh, \eta'\rangle \\ &= \langle d(a)(f \otimes g) \otimes bh, \psi' \otimes \phi' \otimes \eta'\rangle = \langle [\Psi_1(d \otimes I)(a \otimes b)] \\ &\qquad (f \otimes g \otimes h), \psi' \otimes \phi' \otimes \eta'\rangle \\ &= \langle \Psi_1(d \otimes I)(a \otimes b), f \cdot \psi' \otimes g \cdot \phi' \otimes h \cdot \eta'\rangle \\ &= \langle [\Psi_1(d \otimes I)]^{\sim} x, \psi \otimes \phi \otimes \eta\rangle .\end{aligned}$$

Hence (2.4) follows. We are now ready to show the associativity of m. Let $a \in A$, $\psi, \phi \in A^*$. Using (2.1), (2.4) and (2.5), we get:

$$\begin{aligned}\langle a, (\psi\phi)\eta\rangle &= \langle d(a), \psi\phi \otimes \eta\rangle = \langle [\Psi_1(d \otimes I)]^{-} d(a), \psi \otimes \phi \otimes \eta\rangle \\ &= \langle [\Psi_2(I \otimes d)]^{-} d(a), \psi \otimes \phi \otimes \eta\rangle = \langle d(a), \psi \otimes \phi \otimes \eta\rangle \\ &= \langle a, \psi(\phi\eta)\rangle .\end{aligned}$$

Thus $(\psi\phi)\eta = \psi(\phi\eta)$ for all $\psi, \phi, \eta \in A^*$, so m is associative and A^* is a Banach algebra.

Suppose $\{A, d\}$ is cocommutative. It is easy to see that $\langle x, \psi \otimes \eta\rangle = \langle \tilde{\tau}(x), \eta \otimes \psi\rangle$ for all $x \in (A \otimes A)^{**}$, $\psi, \eta \in A^*$. Hence, if $a \in A$ and $\psi, \eta \in A^*$,

$$\begin{aligned}\langle a, \psi\eta\rangle &= \langle d(a), \psi \otimes \eta\rangle = \langle \tilde{\tau} d(a), \eta \otimes \psi\rangle = \langle d(a), \eta \otimes \psi\rangle \\ &= \langle a, \eta\psi\rangle ,\end{aligned}$$

so that $\psi\eta = \eta\psi$ for all $\psi, \eta \in A^*$ and thus A^* is commutative.

Finally assume $\{A, d\}$ has a coidentity ε. We claim that

$$\langle x, \varepsilon \otimes \psi \rangle = \langle (\varepsilon \otimes I)^{\sim} x, \psi \rangle , \tag{2.6}$$

$$\langle x, \psi \otimes \varepsilon \rangle = \langle (I \otimes \varepsilon)^{\sim} x, \psi \rangle , \tag{2.7}$$

for all $x \in (A \otimes A)^{**}$, $\psi \in A^*$. Again we will only prove (2.6) and it suffices to work with $x = a \otimes b$, $a, b \in A$. But in this case,

$$\begin{aligned}\langle x, \varepsilon \otimes \psi \rangle &= \langle a, \varepsilon \rangle \langle b, \psi \rangle = \langle \varepsilon(a) b, \psi \rangle = \langle (\varepsilon \otimes I)(a \otimes b), \psi \rangle \\ &= \langle (\varepsilon \otimes I)^{\sim} x, \psi \rangle ,\end{aligned}$$

so (2.6) follows. Now, if $a \in A$ and $\psi \in A^*$, using (2.3), (2.6) and (2.7), we get:

$$\begin{aligned}\langle a, \varepsilon\psi \rangle &= \langle d(a), \varepsilon \otimes \psi \rangle = \langle (\varepsilon \otimes I)^{-} d(a), \psi \rangle = \langle a, \psi \rangle , \\ \langle a, \psi\varepsilon \rangle &= \langle d(a), \psi \otimes \varepsilon \rangle = \langle (I \otimes \varepsilon)^{-} d(a), \psi \rangle = \langle a, \psi \rangle ,\end{aligned}$$

so that $\varepsilon\psi = \psi = \psi\varepsilon$ for all $\psi \in A^*$, i.e., ε is the identity of A^*. □

REMARK 2.5. If $x \in A^{**}$ and $\psi, \eta \in A^*$, then $\langle x, \psi\eta \rangle = \langle \tilde{d}(x), \psi \otimes \eta \rangle$.

Let $\{A, d\}$ be a cocommutative Hopf-C^*-algebra. It is well known that the spectrum $\sigma(A^*)$ of A^* is locally compact and it is contained in the closed unit ball of A^{**}. If ε is a coidentity for $\{A, d\}$, then $\sigma(A^*)$ is compact and contained in the unit sphere of A^{**}. Note that in this case $\langle x, \varepsilon \rangle = 1$ for all $x \in \sigma(A^*)$: indeed, $\langle x, \varepsilon \rangle = \langle x, \varepsilon^2 \rangle = \langle x, \varepsilon \rangle^2$, so $\langle x, \varepsilon \rangle$ is either 0 or 1; but if $\langle x, \varepsilon \rangle = 0$ we have, for all $\psi \in A^*$, $\langle x, \psi \rangle = \langle x, \psi\varepsilon \rangle = \langle x, \psi \rangle \langle x, \varepsilon \rangle = 0$, so $x = 0$, a contradiction; hence $\langle x, \varepsilon \rangle = 1$.

Denote by e the identity element of A^{**}. Let $\gamma: (A \otimes A)^{**} \to A^{**} \bar{\otimes} A^{**}$ be the weakly* continuous surjective $*$-homomorphism given by Proposition 1.3.

LEMMA 2.6. *Let* $x \in (A \otimes A)^{**}$, $\psi, \eta \in A^*$. *Then* $\langle x, \psi \otimes_\nu \eta \rangle = \langle \gamma(x), \psi \otimes \eta \rangle$, *where* $\psi \otimes_\nu \eta \in (A \otimes A)^{**}$ *and* $\psi \otimes \eta \in (A^{**} \bar{\otimes} A^{**})_* = A^* \otimes_{*\alpha_0} A^*$[9] *are the natural extensions of* $\psi \otimes \eta: A \odot A \to \boldsymbol{C}$.

Proof. Let $x \in (A \otimes A)^{**}$. Pick a net $\{x_\lambda\} \subseteq A \odot A$ converging to x in the weak* topology of $(A \otimes A)^{**}$. Since γ is weakly* continuous and coincides with the identity map $A \odot A$, we get:

$$\langle x, \psi \otimes_\nu \eta \rangle = \lim \langle x_\lambda, \psi \otimes_\nu \eta \rangle = \lim \langle \gamma(x_\lambda), \psi \otimes \eta \rangle = \langle \gamma(x), \psi \otimes \eta \rangle .$$

PROPOSITION 2.7. $\sigma(A^*) = \{x \in A^{**}: x \neq 0$ *and* $\gamma\tilde{d}(x) = x \otimes x\}$. *It follows then that* $\sigma(A^*) \cup \{0\}$ *is a monoid* (i.e., *a semigroup with identity*) *under the multiplication inherited from* A^{**}. *Also, if*

$x \in \sigma(A^*)$, then $x^* \in \sigma(A^*)$ and if $\{A, d\}$ has a coidentity, then $\sigma(A^*)$ is also a monoid.

Proof. Let $x \in \sigma(A^*)$ and let $\Phi \in A^* \odot A^*$, say $\Phi = \sum_{i=1}^{n} \psi_i \otimes \eta_i$. Applying Lemma 2.6 and Remark 2.5, we get

$$\begin{aligned}\langle \gamma\tilde{d}(x), \Phi \rangle &= \sum \langle \gamma\tilde{d}(x), \psi_i \otimes \eta_i \rangle = \sum \langle \tilde{d}(x), \psi_i \underset{\nu}{\otimes} \eta_i \rangle \\ &= \sum \langle x, \psi_i \eta_i \rangle = \sum \langle x, \psi_i \rangle \langle x, \eta_i \rangle = \sum \langle x \otimes x, \psi_i \otimes \eta_i \rangle \\ &= \langle x \otimes x, \Phi \rangle \ .\end{aligned}$$

Thus $\gamma\tilde{d}(x)$ and $x \otimes x$, considered as linear functionals on $(A^{**} \bar{\otimes} A^{**})_*$, agree on $A^* \odot A^*$; but $(A^{**} \bar{\otimes} A^{**})_* = A^* \otimes_{\alpha_0^*} A^*$ so $A^* \odot A^*$ is norm dense in $(A^{**} \bar{\otimes} A^{**})_*$ and hence $\gamma\tilde{d}(x) = x \otimes x$. Conversely, if $\tilde{\gamma}d(x) = x \otimes x$, applying again Remark 2.5 and Lemma 2.6 we get

$$\begin{aligned}\langle x, \psi\eta \rangle &= \langle \tilde{d}(x), \psi \underset{\nu}{\otimes} \eta \rangle = \langle \gamma\tilde{d}(x), \psi \otimes \eta \rangle = \langle x \otimes x, \psi \otimes \eta \rangle \\ &= \langle x, \psi \rangle \langle x, \eta \rangle\end{aligned}$$

for all $\psi, \eta \in A^*$; thus, if $x \neq 0$, $x \in \sigma(A^*)$. Also, if $x \in \sigma(A^*)$, then $x^* \neq 0$ and $\gamma\tilde{d}(x^*) = d(x)^* = (x \otimes x)^* = x^* \otimes x^*$, so $x^* \in \sigma(A^*)$.

As remarked before, property (ii) of Definition 2.1 implies that $\tilde{d}(\mathrm{e}) = 1_{M(A \otimes A)}$; since γ is surjective, $\gamma(1_{M(A \otimes A)}) = \mathrm{e} \otimes \mathrm{e}$, so $\gamma\tilde{d}(\mathrm{e}) = \mathrm{e} \otimes \mathrm{e}$ and therefore $\mathrm{e} \in \sigma(A^*)$. If $x, y \in \sigma(A^*)$, then

$$\gamma\tilde{d}(xy) = \gamma\tilde{d}(x)\gamma\tilde{d}(y) = (x \otimes x)(y \otimes y) = xy \otimes xy \ ,$$

so $xy \in \sigma(A^*) \cup \{0\}$. Thus $\sigma(A^*) \cup \{0\}$ is a monoid. Moreover, if ε is a coidentity for $\{A, d\}$, ε is a $*$-homormorphism, so

$$\langle xy, \varepsilon \rangle = \tilde{\varepsilon}(xy) = \tilde{\varepsilon}(x)\tilde{\varepsilon}(y) = 1$$

for all $x, y \in \sigma(A^*)$. Thus in this case $xy \neq 0$ and $\sigma(A^*)$ is a monoid. □

LEMMA 2.8. *If $x \in \sigma(A^*)$ is invertible, then x is unitary.*

Proof. Let us show first that $x^{-1} \in \sigma(A^*)$. We have:

$$\gamma\tilde{d}(x^{-1}) = (\gamma\tilde{d}(x))^{-1} = (x \otimes x)^{-1} = x^{-1} \otimes x^{-1}$$

so $x^{-1} \in \sigma(A^*)$. Now x^*x is also invertible, so $x^*x, (x^*x)^{-1} \in \sigma(A^*)$; hence $\|x^*x\| \leqq 1$ and $\|(x^*x)^{-1}\| \leqq 1$, so $\|x^*x\| = 1 = \|(x^*x)^{-1}\|$. Considering the commutative C^*-algebra generated by x^*x and $(x^*x)^{-1}$, we see that x^*x correspond to a strictly positive function f with $\|f\|_\infty = 1 = \|f^{-1}\|_\infty$. But then $f = 1$, so $x^*x = \mathrm{e}$. Similarly we get $\|xx^*\| = 1 = \|(xx^*)^{-1}\|$ and so $xx^* = \mathrm{e}$. Hence x is unitary. □

Now let $H = \sigma(A^*) \cap A_r^{**}$. By the preceding lemma, $H = \sigma(A^*) \cap A_u^{**}$. It follows that H is a topological group with respect to the weak* topology of A^{**}. By Proposition 2.7, we get

$$H = \{x \in A_u^{**} \colon \gamma\tilde{d}(x) = x \otimes x\} . \tag{2.8}$$

H is called the *intrinsic group* of $\{A, d\}$. We remark that even if the Hopf-C^*-algebra $\{A, d\}$ is not cocommutative we can define its intrinsic group H by (2.8); it is always a topological group. We then have the following proposition.

PROPOSITION 2.9. *If $\{A_1, d_1\}$, $\{A_2, d_2\}$ are two isomorphic Hopf-C^*-algebras with intrinsic groups H_1, H_2 respectively, then H_1 is isomorphic and homeomorphic to H_2.*

Proof. Let $\theta\colon A_1 \to A_2$ be a $*$-isomorphism with $(\theta \otimes \theta)^{-} d_1 = d_2\theta$. Then $(\theta \otimes \theta)^{\sim}\tilde{d}_1 = \tilde{d}_2\tilde{\theta}$ and $\tilde{\theta}\colon A_1^{**} \to A_2^{**}$ is a weakly* continuous $*$-isomorphism. It is clear that $\tilde{\theta}|_{H_1}\colon H_1 \to \tilde{\theta}(H_1)$ is a homeomorphic isomorphism, so all we have to show is taht $\tilde{\theta}(H_1) = H_2$. Let $\gamma_i\colon (A_i \otimes A_i)^{**} \to A_i^{**} \bar{\otimes} A_i^{**}$, $i = 1, 2$ be the surjective $*$-homomorphisms given by Proposition 1.3. It is easy to see that $\gamma_2(\theta \otimes \theta)^{\sim} = (\tilde{\theta} \bar{\otimes} \tilde{\theta})\gamma_1$. Now if $x \in H_1$, then $\gamma_1\tilde{d}_1(x) = x \otimes x$, so

$$\gamma_2\tilde{d}_2\tilde{\theta}(x) = \gamma_2(\theta \otimes \theta)^{\sim}\tilde{d}_1(x) = (\tilde{\theta} \bar{\otimes} \tilde{\theta})\gamma_1\tilde{d}_1(x) = \tilde{\theta}(x) \otimes \tilde{\theta}(x) ;$$

also $\tilde{\theta}(x)$ is unitary (since $\tilde{\theta}$ is a $*$-isomorphism and x is unitary), so $\tilde{\theta}(x) \in H_2$. Hence $\tilde{\theta}(H_1) \subseteq H_2$. Applying the same argument to θ^{-1}, we get $\tilde{\theta}(H_1) = H_2$. □

The argument used in part (i) of the following proposition was suggested to us by Marc A. Rieffel.

PROPOSITION 2.10. *Let $\{A, d\}$ be a cocommutative Hopf-C^*-algebra. Then:*

(i) *$\sigma(A^*)$ is a linearly independent set in A^{**}.*

(ii) *If A^* is semisimple, then $\sigma(A^*)$ generates A^{**} as a von Neumann algebra.*

(iii) *If j is an involution for $\{A, d\}$, then j^t is an isometric automorphism of A^*.*

Proof. Suppose (i) does not hold. Let

$$x = \sum_{i=1}^{n} \alpha_i x_i$$

be a dependency relation of shortest length among elements of

$\sigma(A^*)$, $x \neq x_i$ and $\alpha_i \neq 0$ for all $i = 1, \cdots, n$. Then, by Proposition 2.7,

$$\gamma\tilde{d}(x) = \sum \alpha_i \gamma\tilde{d}(x_i) = \sum \alpha_i x_i \otimes x_i$$
$$\gamma\tilde{d}(x) = x \otimes x = \sum_{i,j} \alpha_i \alpha_j x_i \otimes x_j .$$

Since the x_i are linearly independent, so are the $x_i \otimes x_j$; hence $\alpha_i\alpha_j = 0$ if $i \neq j$ and $\alpha_i^2 = \alpha_i$ for all $i = 1, \cdots, n$. Since $\alpha_i \neq 0$ for all i, we get $n = 1$ and $\alpha_1 = 1$, so that $x = x_1$ acontradiction. This proves (i).

As for (ii), represent A^{**} faithfully as a von Neumann algebra in some Hilbert space $\mathscr{H}$, i.e., $A^{**} \subseteq \mathscr{B}(\mathscr{H})$. Since $\sigma(A^*)$ is self-adjoint, the von Neumann algebra generated by $\sigma(A^*)$ in $\mathscr{B}(\mathscr{H})$ is $\sigma(A^*)''$, the double commutant of $\sigma(A^*)$. Clearly $\sigma(A^*)'' \subseteq A^{**}$. We want to show $A^{**} \subseteq \sigma(A^*)''$. Let $y \in \sigma(A^*)'$ and let $\psi \in \mathscr{B}(\mathscr{H})_*$. Defide two linear functionals ψ_1, ψ_2 on A^{**} by

$$\langle x, \psi_1 \rangle = \langle xy, \psi \rangle \quad \text{and} \quad \langle x, \psi_2 \rangle = \langle yx, \psi \rangle$$

for all $x \in A^{**}$. Since ψ is σ-weakly continuous, $\psi_1, \psi_2 \in A^*$. If $x \in \sigma(A^*)$, then $\langle x, \psi_1 \rangle = \langle xy, \psi \rangle = \langle yx, \psi \rangle = \langle x, \psi_2 \rangle$, so $\langle x, \psi_1 - \psi_2 \rangle = 0$ for all $x \in \sigma(A^*)$. Since A^* is semisimple, $\psi_1 = \psi_2$ and $\langle xy, \psi \rangle = \langle yx, \psi \rangle$ for all $x \in A^*$. But $\psi \in \mathscr{B}(\mathscr{H})_*$ was arbitrary, so $xy = yx$ for all $x \in A^*$. Hence $A^{**} \subseteq \sigma(A^*)''$. This proves (ii).

Finally let j be an involution for $\{A, d\}$. Since j is a bijective linear map, so is j^t. Also, by cocommutativity of $\{A, d\}$, equation (2.2) becomes

$$(j \otimes j)^{-} d = dj . \tag{2.9}$$

Now let $a \in A$, $\psi, \eta \in A^*$. Using (2.9) we get:

$$\begin{aligned} \langle a, j^t(\psi\eta) \rangle &= \langle j(a), \psi\eta \rangle = \langle dj(a), \psi \otimes \eta \rangle \\ &= \langle (j \otimes j)^{-} d(a), \psi \otimes \eta \rangle = \langle d(a), (j \otimes j)^t(\psi \otimes \eta) \rangle \\ &= \langle d(a), j^t(\psi) \otimes j^t(\eta) \rangle = \langle a, j^t(\psi) j^t(\eta) \rangle . \end{aligned}$$

Hence $j^t(\psi\eta) = j^t(\psi) j^t(\eta)$ for all $\psi, \eta \in A^*$, so j^t is an automorphism of A^*. It is in fact an isometry since j is a surjective isometry. $\square$

REMARK. We have also characterized all isometric (algebra) isomorphisms between the duals (as Banach spaces) of two cocommutative Hopf-C^*-algebras. Since we will not need this result in the sequel, we will state the theorem without proof. We remark the theorem can be proved using an argument due to Martin Walter; he proved the same type of theorem when the algebras are Fourier-Stieltjes algebras of locally compact groups (cf. [11; Theorem 2]).

THEOREM 2.11. *Let* $\{A_i, d_i\}$, $i = 1, 2$, *be cocommutative Hopf-*C^**-algebras and let* $\Phi: A_1^* \to A_2^*$ *be an isometric isomorphism. Denote by* H_i *the intrinsic group of* $\{A_i, d_i\}$, *by* $\sigma(A_i^*)$ *the spectrum of* A_i^* *and by* e_i *the identity element of* A_i^{**}. *Then* $x_0 = \Phi^t(\mathrm{e}_2) \in H_1$ *and there exists a* $*$*-preserving weak** *homeomorphism* $\alpha: A_2^{**} \to A_1^{**}$ *such that*:

(i) $\langle x, \Phi(\psi)\rangle = \langle x_0\alpha(x), \psi\rangle$ *for all* $x \in A_2^{**}$, $\psi \in A_1^*$;

(ii) $\alpha(\sigma(A_2^*)) = \sigma(A_1^*)$;

(iii) $\alpha(H_2) = H_1$;

(iv) $\alpha|_{H_2}: H_2 \to H_1$ *is either an isomorphism or an anti-isomorphism.*

3. The algebra $C^*(G)$ as a Hopf-C^*-algebra. Let G be a locally compact group and let $\mathscr{M}(G)$ be the algebra of all regular Borel measures on G. It is well known that $\mathscr{M}(G)$, as a Banach space, is isometrically isomorphic to $C_\infty(G)^*$. Whenever convenient, we will identify $\mathscr{M}(G)$ with $C_\infty(G)^*$. We fix a left Haar measure on G. Integration with respect to it will be denoted by $d\mathfrak{s}$ and integrals without explicit domains of integration are to be taken over G. We identify $L^1(G)$ with the measures in $\mathscr{M}(G)$ which are absolutely continuous with respect to the Haar measure.

Let $\Sigma(G)$ denote the family of all strongly continuous unitary representations of G. It is well known that every $\pi \in \Sigma(G)$ extends to a $*$-representation of $\mathscr{M}(G)$, also denoted by π, which is nondegenerate when restricted to $L^1(G)$, defined by

$$(\pi(\mu)\xi|\eta) = \int(\pi(\mathfrak{s})\xi|\eta) \quad d\mu(\mathfrak{s}) \,, \tag{3.1}$$

$\mu \in \mathscr{M}(G)$, $\xi, \eta \in \mathscr{H}_\pi$. If $\mu \in \mathscr{M}(G)$, define

$$\|\mu\|_{c^*} = \sup\{\|\pi(\mu)\|: \pi \in \Sigma(G)\} \,. \tag{3.2}$$

Then $\|\cdot\|_{c^*}$ define a seminorm in $\mathscr{M}(G)$ which is actually a norm, since the left regular representation of G is faithful on $\mathscr{M}(G)$. It is in fact a C^*-norm, since it is the supremum of C^*-norms. The completion of $L^1(G)$ with respect to this norm is called the *C^*-algebra* of G and denoted by $C^*(G)$. We will denote by $C^*(\mathscr{M}(G))$ the completion of $\mathscr{M}(G)$ with respect to $\|\cdot\|_{c^*}$. Note that $\|\cdot\|_{c^*} \leqq \|\cdot\|_{\mathscr{M}(G)}$.

As we have remarked before, Guichardet [7] proved that $C^*(G\times G) \cong C^*(G) \otimes C^*(G)$. Our goal in this section is to show that we can define a comultiplication $d: C^*(G) \to M(C^*(G \times G))$, an involution $j: C^*(G) \to C^*(G)$ and a coidentity $\varepsilon: C^*(G) \to \boldsymbol{C}$ so that $\{C^*(G), d, j, \varepsilon\}$ is a cocommutative involutive Hopf-C^*-algebra with coidentity. We will show also that G is isomorphic and homeomorphic to the intrinic group H of $\{C^*(G), d\}$, so that this Hopf-C^*-algebra determines the group. We remark that we need neither the involution nor the coidentity to show that $G \cong H$, although we believe they will be needed in the characterization problem.

Let Λ be the family of all neighborhoods of the identity $\mathfrak{e}$ of G. We fix an approximate identity $\{\mathfrak{e}_\lambda: \lambda \in \Lambda\} \subseteq L^1(G)$ for $L^1(G)$ (in the L^1-norm) satisfying

$$(3.3) \quad \begin{aligned} &(\mathrm{i}) \quad \mathfrak{e}_\lambda \geqq 0; \\ &(\mathrm{ii}) \quad \mathfrak{e}_\lambda = 0 \text{ on } \lambda^c; \\ &(\mathrm{iii}) \quad \int \mathfrak{e}_\lambda(\mathfrak{s})d\mathfrak{s} = 1. \end{aligned}$$

It is clear that $\{\mathfrak{e}_\lambda\}$ is also an approximate identity for $C^*(G)$.

If $\mu \in \mathscr{M}(G)$, it is easy to see that the maps $L_\mu, R_\mu: L^1(G) \to L^1(G)$ given by

$$L_\mu(f) = \mu * f \quad \text{and} \quad R_\mu(f) = f * \mu \ , \quad f \in L^1(G) \ ,$$

extend to bounded linear maps $L_\mu, R_\mu: C^*(G) \to C^*(G)$ with $\|L_\mu\| \leqq \|\mu\|_{c^*}$ and $\|R_\mu\| \leqq \|\mu\|_{c^*}$. It follows that $T_\mu = (L_\mu, R_\mu) \in M(C^*(G))$ and that $T: \mu \in \mathscr{M}(G) \to T_\mu \in M(C^*(G))$ is a $*$-homomorphism. Thus $\|T_\mu\| = \|L_\mu\| = \|R_\mu\| \leqq \|\mu\|_{c^*}$. A standard argument shows that T is in fact an isometry with respect to the norm $\|\cdot\|_{c^*}$ on $\mathscr{M}(G)$, so it extends to an isometric $*$-isomorphism $T: C^*(\mathscr{M}(G)) \to M(C^*(G))$ preserving identities. Hence we can identify $\mathscr{M}(G)$ and $C^*(\mathscr{M}(G))$ with subalgebras of $M(C^*(G))$, and we will do so whenever convenient.

If K is a closed subgroup of a locally compact group H, then $\mathscr{M}(K)$ can be naturally embedded in $\mathscr{M}(H)$ as a $*$-subalgebra: given $\mu \in \mathscr{M}(K)$ define $\tilde{\mu} \in \mathscr{M}(H)$ by $\tilde{\mu}(E) = \mu(E \cap K)$ for all Borel sets $E \subseteq H$. If $\mu \in \mathscr{M}(K)$, $\pi \in \Sigma(H)$ and $\rho \in \Sigma(K)$ is the restriction of π to K, then

$$(3.4) \qquad \begin{aligned} (\pi(\tilde{\mu})\xi \mid \eta) &= \int_H (\pi(\mathfrak{s})\xi \mid \eta) d\tilde{\mu}(\mathfrak{s}) = \int_K (\rho(\mathfrak{s})\xi \mid \eta) d\mu(\mathfrak{s}) \\ &= (\rho(\mu)\xi \mid \eta) \end{aligned}$$

for all $\xi, \eta \in \mathscr{H}_\pi = \mathscr{H}_\rho$. It follows that the embedding $\mu \to \tilde{\mu}$ is continuous (in fact norm decreasing) with respect to the C^*-norms of $\mathscr{M}(K)$ and $\mathscr{M}(H)$. Moreover, if every $\rho \in \Sigma(K)$ is the restriction of some $\pi \in \Sigma(H)$, then (3.4) implies that the map $\mu \to \tilde{\mu}$ is in fact an isometry with respect to the C^*-norms of $\mathscr{M}(K)$ and $\mathscr{M}(H)$, so it extends to an injective $*$-homomorphism $C^*(\mathscr{M}(K)) \to C^*(\mathscr{M}(H)) \subseteq M(C^*(H))$.

Now suppose $H = G \times G$ and $K = \{(\mathfrak{x}, \mathfrak{x}): \mathfrak{x} \in G\}$. Note that in this case every $\rho \in \Sigma(K)$ is the restriction of some $\pi \in \Sigma(H)$ (e.g., define $\pi(\mathfrak{x}, \mathfrak{y}) = \rho(\mathfrak{x}, \mathfrak{x}), \mathfrak{x}, \mathfrak{y} \in G)$. Since K is isomorphic and homeomorphic to G we get a 1-1 $*$-homomorphism $d: C^*(\mathscr{M}(G)) \to M(C^*(G \times G))$ with $d(\mathscr{M}(G)) \subseteqq \mathscr{M}(G \times G)$. To simplify notation, let $A = C^*(G)$. Since $A \otimes A \cong C^*(G \times G)$ and A is a C^*-subalgebra of $C^*(\mathscr{M}(G))$, restricting d to A we get an injective $*$-homomorphism $d: A \to M(A \otimes A)$.

Let $\mu \in \mathscr{M}(G)$ and let $\tilde{\mu} = d(\mu) \in \mathscr{M}(G \times G)$. It is easy to check that for all $f \in L^1(G \times G)$ and for a.e. $(\mathfrak{x}, \mathfrak{y}) \in G \times G$ we have:

$$(3.5) \quad \begin{cases} [d(\mu f)](\mathfrak{x}, \mathfrak{y}) = (\tilde{\mu} * f)(\mathfrak{x}, \mathfrak{y}) = \int f(\mathfrak{s}^{-1}\mathfrak{x}, \mathfrak{s}^{-1}\mathfrak{y}) d\mu(\mathfrak{s}) , \\ [f d(\mu)](\mathfrak{x}, \mathfrak{y}) = (\tilde{\mu} * f)(\mathfrak{x}, \mathfrak{y}) = \int \Delta(\mathfrak{s}^{-1})^2 f(\mathfrak{x}\mathfrak{s}^{-1}, \mathfrak{y}\mathfrak{s}^{-1}) d\mu(\mathfrak{s}) , \end{cases}$$

where Δ is the modular function of G. (In the last equation of (3.5) we get $\Delta(\mathfrak{s}^{-1})^2$ because $\Delta_{G \times G}(\mathfrak{x}, \mathfrak{h}) = (\mathfrak{x})\Delta(\mathfrak{y})$ for all $\mathfrak{x}, \mathfrak{y} \in G$.)

Let $\Psi_1: M(A \otimes A) \otimes M(A) \to M(A \otimes A \otimes A)$ and $\Psi_2: M(A) \otimes M(A \otimes A) \to M(A \otimes A \otimes A)$ be the (unique) $*$-homomorphism extending the natural injection $A \otimes A \otimes A \hookrightarrow M(A \otimes A \otimes A)$ (see end of §1). If $\mu \in \mathscr{M}(G) \subseteqq M(A)$ and $\nu \in \mathscr{M}(G \times G) \subseteqq M(A \otimes A)$, it is easy to check that $\Psi_1(\nu \otimes \mu) = \nu \times \mu$ (the product measure) and $\Psi_2(\mu \otimes \nu) = \mu \times \nu$.

LEMMA 3.1. *If $g \in L^1(G \times G)$, then $\Psi_1(d \otimes I)g$, $\Psi_2(I \otimes d)g \in \mathscr{M}(G \times G \times G)$, where $I: A \to A$ is the identity map (recall that $A = C^*(G)$). Moreover, if $F \in L^1(G \times G \times G)$ we have, for* a.e. $(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) \in G \times G \times G$:

$$(3.6) \quad \begin{cases} [(\Psi_1(d \otimes I)g) * F](\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) = \iint F(\mathfrak{s}^{-1}\mathfrak{x}, \mathfrak{s}^{-1}, \mathfrak{y}, \mathfrak{t}^{-1}\mathfrak{z}) g(\mathfrak{s}, \mathfrak{t}) d\mathfrak{s} d\mathfrak{t} \\ [F * (\Psi_1(d \otimes I)g)](\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) = \iint \delta(\mathfrak{s}, \mathfrak{t}) F(\mathfrak{x}\mathfrak{s}^{-1}, \mathfrak{y}\mathfrak{s}^{-1}, \mathfrak{z}\mathfrak{t}^{-1}) g(\mathfrak{s}, \mathfrak{t}) d\mathfrak{s} d\mathfrak{t} \\ [(\Psi_2(I \otimes d)g) * F](\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) = \iint F(\mathfrak{s}^{-1}\mathfrak{x}, \mathfrak{t}^{-1}\mathfrak{y}, \mathfrak{t}^{-1}\mathfrak{z}) g(\mathfrak{s}, \mathfrak{t}) d\mathfrak{s} d\mathfrak{t} \\ [F * (\Psi_2(I \otimes d)g)](\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) = \iint \delta(\mathfrak{t}, \mathfrak{s}) F(\mathfrak{x}\mathfrak{s}^{-1}, \mathfrak{y}\mathfrak{t}^{-1}, \mathfrak{z}\mathfrak{t}^{-1}) g(\mathfrak{s}, \mathfrak{t}) d\mathfrak{s} d\mathfrak{t} \end{cases}$$

where $\delta(\mathfrak{s}, \mathfrak{t}) = \Delta(\mathfrak{s}^{-1})^2 \Delta(\mathfrak{t}^{-1})$ for all $\mathfrak{s}, \mathfrak{t} \in G$ and Δ is the modular function of G.

Proof. We will prove the first two equalities; the other ones can be handled similarly. To simplify notation, let $\Psi = \Psi_1(d \otimes I)$. Given $g \in L^1(G \times G)$, approximate g in the L^1-norm by a sequence $\{g_n\} \subseteqq L^1(G) \odot L^1(G)$. Then $g_n \to g$ in $A \otimes A$ so $\psi(g_n) \to \psi(g)$ in $M(A \otimes A)$. Define a linear functional θ on $C_\infty(G \times G \times G)$ by

$$\langle P, \theta \rangle = \iint P(\mathfrak{s}, \mathfrak{s}, \mathfrak{t}) g(\mathfrak{s}, \mathfrak{t}) d\mathfrak{s} d\mathfrak{t} ,$$

$P \in C_\infty(G \times G \times G)$. It is easy to see that this defines a bounded linear functional with $\|\theta\| \leqq \|g\|_1$. Under the identification $\mathscr{M}(G\times G\times G) \cong C_\infty(G \times G \times G)^*$, θ corresponds to a regular Borel measure thal we will denote again θ. Using equations (3.5) it follows easily that

$$\|\psi(g_n)*F - \theta*F\|_1 \leqq \|F\|_1 \|g_n - g\|_1 \longrightarrow 0$$
$$\|F*\psi(g_n) - F*\theta\|_1 \leqq \|F\|_1 \|g_n - g\|_1 \longrightarrow 0$$

for all $F \in L^1(G \times G \times G)$, so that $\psi(g_n) \to \theta$ strictly. Hence $\psi(g) = \theta \in \mathscr{M}(G \times G)$. The first two equations now follow from the definition of θ. $\square$

COROLLARY 3.2. *If $\nu \in \mathscr{M}(G \times G)$, $g \in L^1(G \times G)$ and $F \in L^1(G\times G\times G)$, then, for* a.e. $(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) \in G \times G \times G$, *we have:*

$$(3.7)\quad \begin{cases} [\Psi_1(d \otimes I)(\nu*g)*F](\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) \\ \qquad = \int_{G\times G} [\Psi_1(d \otimes I)g*F](\mathfrak{s}^{-1}\mathfrak{x}, \mathfrak{s}^{-1}\mathfrak{y}, \mathfrak{t}^{-1}\mathfrak{z})d\nu(\mathfrak{s}, \mathfrak{t}) \\ [F*\Psi_1(d \otimes I)(\nu*g)](\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) \\ \qquad = \int_{G\times G} \delta(\mathfrak{s}, \mathfrak{t})[F*\Psi_1(d \otimes I)g](\mathfrak{x}\mathfrak{s}^{-1}, \mathfrak{y}\mathfrak{s}^{-1}, \mathfrak{z}\mathfrak{t}^{-1})d\nu(\mathfrak{s}, \mathfrak{t}) \\ [\Psi_2(I \otimes d)(\nu*g)*F](\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) \\ \qquad = \int_{G\times G} [\Psi_2(I \otimes d)g*F](\mathfrak{s}^{-1}\mathfrak{x}, \mathfrak{t}^{-1}\mathfrak{y}, \mathfrak{t}^{-1}\mathfrak{z})d\nu(\mathfrak{s}, \mathfrak{t}) \\ [F*\Psi_2(I \otimes d)(\nu*g)](\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) \\ \qquad = \int_{G\times G} \delta(\mathfrak{t}, \mathfrak{s})[F*\Psi_2(I \otimes d)g](\mathfrak{x}\mathfrak{s}^{-1}, \mathfrak{y}\mathfrak{t}^{-1}, \mathfrak{z}\mathfrak{t}^{-1})d\nu(\mathfrak{s}, \mathfrak{t}) \end{cases}$$

where $\delta(\mathfrak{s}, \mathfrak{t}) = \Delta(\mathfrak{s}^{-1})^2\Delta(\mathfrak{t}^{-1})$ *for all* $\mathfrak{s}, \mathfrak{t} \in G$.

LEMMA 3.3. $(\mathfrak{e}_\lambda)$ *converges in the strict topology of $M(A \otimes A)$ to the point mass $\delta_{(\mathscr{E},\mathscr{E})}$ at the identity $(\mathfrak{e}, \mathfrak{e})$ of $G \times G$.*

Proof. Using the definition of d, (3.3) and Theorem 20.15 of [8], it follows that $d(\mathfrak{e}_\lambda)*g \to g$ and $g*d(\mathfrak{e}_\lambda) \to g$ in the L^1-norm for all $g \in L^1(G\times G)$. Since $\|\cdot\|_{c^*} \leqq \|\cdot\|_1$ and $L^1(G\times G)$ is dense in $C^*(G\times G)$ we get the desired result. $\square$

LEMMA 3.4. *If $\nu \in \mathscr{M}(G \times G)$, then $(\Psi_1(d \otimes I))^-(\nu)$, $(\Psi_2(I \otimes d))^-(\nu) \in \mathscr{M}(G \times G \times G)$. Moreover, if $F \in L^1(G \times G \times G)$, we have for* a.e. $(\mathfrak{x}, \mathfrak{y}, \mathfrak{z}) \in G \times G \times G$

$$(3.8)\quad \begin{cases} [(\Psi_1(d\otimes I))^{-}(\nu)*F](\mathfrak{x},\mathfrak{y},\mathfrak{z}) = \int_{G\times G} F(\mathfrak{s}^{-1}\mathfrak{x},\mathfrak{s}^{-1}\mathfrak{y},\mathfrak{t}^{-1}\mathfrak{z})d\nu(\mathfrak{s},\mathfrak{t}) \\ [F*(\Psi_1(d\otimes I))^{-}(\nu)](\mathfrak{x},\mathfrak{y},\mathfrak{z}) = \int_{G\times G} \delta(\mathfrak{s},\mathfrak{t})F(\mathfrak{x}\mathfrak{s}^{-1},\mathfrak{y}\mathfrak{s}^{-1},\mathfrak{z}\mathfrak{t}^{-1})d\nu(\mathfrak{s},\mathfrak{t}) \\ [(\Psi_2(I\otimes d))^{-}(\nu)*F](\mathfrak{x},\mathfrak{y},\mathfrak{z}) = \int_{G\times G} F(\mathfrak{s}^{-1}\mathfrak{x},\mathfrak{t}^{-1}\mathfrak{y},\mathfrak{t}^{-1}\mathfrak{z})d\nu(\mathfrak{s},\mathfrak{t}) \\ [F*(\Psi_2(I\otimes d))^{-}(\nu)](\mathfrak{x},\mathfrak{y},\mathfrak{z}) = \int_{G\times G} \delta(\mathfrak{t},\mathfrak{s})F(\mathfrak{x}\mathfrak{s}^{-1},\mathfrak{y}\mathfrak{t}^{-1},\mathfrak{z}\mathfrak{t}^{-1})d\nu(\mathfrak{s},\mathfrak{t}) \end{cases}$$

where $\delta(\mathfrak{s},\mathfrak{t}) = \Delta(\mathfrak{s}^{-1})^2\Delta(\mathfrak{t}^{-1})$ *for all* $\mathfrak{s},\mathfrak{t}\in G$.

Proof. Again we will just prove the statements for $\psi = \Psi_1(d\otimes I)$. Let $\{e_\lambda\colon \lambda\in\Lambda\}\subseteq L^1(G)$ be as before and let $\Gamma=\Lambda\times\Lambda$. For $\gamma=(\lambda_1,\lambda_2)\in\Gamma$, let $u_\gamma = e_{\lambda_1}\otimes e_{\lambda_2}$. By Lemma 2.1, $\{u_\gamma\}$ is an approximate identity for $A\otimes A\cong C^*(G\times G)$ and $\psi(u_\gamma)\to\delta_{(e,e,e)}$ strictly in $M(A\otimes A\otimes A)$. By part (ii) of Proposition 1.1, we get

$$\tilde{\psi}(\nu) = \text{strict} - \lim \psi(\nu*u_\gamma) = \text{strict} - \lim \psi(u_\gamma*\nu)\ . \tag{3.9}$$

Also, as we have seen in the proof of Lemma 3.3, $d(e_\lambda)*g\to g$ and $g*d(e_\lambda)\to g$ in the L^1-norm for all $g\in L^1(G\times G)$. Then it follows easily that $\psi(u_\gamma)*F\to F$ and $F*\psi(u_\gamma)\to F$ in the L^1-norm for all $F\in L^1(G\times G\times G)$.

Now let $\nu\in\mathscr{M}(G\times G)$ and define $\tilde{\nu}\in\mathscr{M}(G\times G\times G)\cong C_\infty(G\times G\times G)^*$ via the linear functional

$$\langle P,\tilde{\nu}\rangle = \int_{G\times G} P(\mathfrak{s},\mathfrak{s},\mathfrak{t})d\nu(\mathfrak{s},\mathfrak{t})\ ,$$

$P\in C_\infty(G\times G\times G)$. Using Corollary 3.2, the definition of $\tilde{\nu}$ and that $\psi(u_\gamma)\to\delta_{(e,e,e)}$ strictly we get

$$\|\psi(\nu*u_\gamma)*F - \tilde{\nu}*F\|_1 \longrightarrow 0 \quad\text{and}\quad \|F*\psi(u_\gamma*\nu) - F*\tilde{\nu}\|_1\longrightarrow 0$$

for all $F\in L^1(G\times G\times G)$. Hence $\psi(\nu*u_\gamma)F\to\tilde{\nu}F$ and $F\psi(\nu*u_\gamma)\to F\tilde{\nu}$ for all $F\in A\otimes A\otimes A$. Thus, by (3.9)

$$\tilde{\psi}(\nu) = \tilde{\nu}\in\mathscr{M}(G\times G\times G)\ .$$

The first two equations in (3.8) now follow firom the difinition of $\tilde{\nu}$. □

COROLLARY 3.5. $[\Psi_2(I\otimes d)]^{-}d = [\Psi_1(d\otimes I)]^{-}d$.

Recall that $\tau\colon A\otimes A\to A\otimes A$ is the automorphism given by $\tau(a\otimes b) = b\otimes a$, $a, b\in A$.

LEMMA 3.6. *If* $\nu\in\mathscr{M}(G\times G)$, *then* $\tilde{\tau}(\nu)\in\mathscr{M}(G\times G)$ *and* $\tilde{\tau}(\nu)(E) = \nu(E^0)$ *for all Borel sets* $E\subseteq G\times G$, *where* $E^0 = \{(\mathfrak{y},\mathfrak{x})\colon(\mathfrak{x},\mathfrak{y})\in E\}$.

Proof. Let $\{u_\gamma: \gamma \in \Gamma\}$ be as in the proof of Lemma 3.4. It follows that $\{\tau(u_\gamma): \gamma \in \Gamma\}$ is an approximate identity for both $L^1(G\times G)$ (in the L^1-norm) and $C^*(G \times G)$. Applying Proposition 1.1 we get

$$\bar{\tau}(\nu) = \text{strict} - \lim \tau(\nu * u_\gamma) = \text{strict} - \lim \tau(u_\gamma * \nu)$$

for all $\nu \in \mathscr{M}(G\times G)$. On the other hand it is easy to see that, for all $g \in L^1(G \times G)$, □

$$||\tau(\nu * u_\gamma) * g - \nu^0 * g||_1 \longrightarrow 0 \text{ and } ||g * \tau(u_\gamma * \nu) - g * \nu^0||_1 \longrightarrow 0 ,$$

where $\nu^0 \in \mathscr{M}(G\times G)$ is the measure defined by $\nu^0(E) = \nu(E^0)$ for all Borel sets $E \subseteq G\times G$. Thus, since $||\cdot||_{c^*} \leqq ||\cdot||_1$ and $L^1(G\times G)$ is dense in $C^*(G\times G)$, we must have $\bar{\tau}(\nu) = \nu^0 \in \mathscr{M}(G \times G)$.

LEMMA 3.7. *The map $j: h \in L^1(G) \to \bar{h}^* \in L^1(G)$ extends to a $*$-anti-isomorphism $j: A \to A$ of period two.*

Proof. $j: L^1(G) \to L^1(G)$ is clearly a $*$-anti-isomorphism of period two. Thus, if we can show that j is an isometry with respect to the C^*-norm, the lemma follows. Let $\pi \in \Sigma(G)$. We denote $\mathscr{H}_\pi$ simply by $\mathscr{H}$. Let $\mathscr{H}^*$ be the dual space of $\mathscr{H}$ and consider the natural conjugate linear isometry $\theta: \xi \in \mathscr{H} \to (\cdot|\xi)_{\mathscr{H}} \in \mathscr{H}^*$. Then θ is surjective and

$$(\phi|\psi)_{\mathscr{H}^*} = (\theta^{-1}(\psi)|\theta^{-1}(\phi))_{\mathscr{H}}$$

for all $\phi, \psi \in \mathscr{H}^*$ or, equivalently,

$$(\xi|\eta)_{\mathscr{H}} = (\theta(\eta)|\theta(\xi))_{\mathscr{H}^*}$$

for all $\xi, \eta \in \mathscr{H}$. We will drop the subscripts $\mathscr{H}, \mathscr{H}^*$ from the inner products in what follows. Define $\bar{\pi}: \mathfrak{x} \in G \to \theta\pi(\mathfrak{x})\theta^{-1} \in \mathscr{B}(\mathscr{H}^*)$. It is easy to see that $\bar{\pi} \in \Sigma(G)$ and, identifying $\mathscr{H}$ with $\mathscr{H}^{**}$, $\bar{\bar{\pi}} = \pi$. If $h \in L^1(G)$ and $\xi, \eta \in \mathscr{H}$, we haue:

$$\begin{aligned}
(\xi|\pi(j(h)\eta)) &= (\pi(\bar{h})\xi|\eta) \\
&= \int \bar{h}(\mathfrak{s})(\pi(\mathfrak{s})\xi|\eta)d\mathfrak{s} \\
&= \int \bar{h}(\mathfrak{s})(\theta^{-1}\bar{\pi}(\mathfrak{s})\theta\xi|\eta)d\mathfrak{s} \\
&= \int \bar{h}(\mathfrak{s})(\bar{\pi}(\mathfrak{s}^{-1})\theta\eta|\theta\xi)d\mathfrak{s} \\
&= \int \Delta(\mathfrak{s}^{-1})\bar{h}(\mathfrak{s}^{-1})(\bar{\pi}(\mathfrak{s})\theta\eta|\theta\xi)d\mathfrak{s} \\
&= (\bar{\pi}(h^*)\theta\eta|\theta\xi) = (\xi|\theta^{-1}\bar{\pi}(h^*)\theta\eta) ,
\end{aligned}$$

so that $\pi(j(h)) = \theta^{-1}\bar{\pi}(h^*)\theta$. But then, since θ is an isometry onto, $||\pi(j(h))|| = ||\bar{\pi}(h^*)||$. Since $\pi \in \Sigma(G)$ was arbitrary and $\pi \leftrightarrow \bar{\pi}$ is a 1-1 correspondence, we get:

$$||j(h)||_{c^*} = \sup\{||\bar{\pi}(h^*)||: \pi \in \Sigma(G)\} = ||h^*||_{c^*} = ||h||_{c^*} .$$

Hence j is an isometry with respect to $||\cdot||_{c^*}$. □

LEMMA 3.8. *If* $\nu \in \mathscr{M}(G\times G)$, *then* $(j \otimes j)^{-}(\nu) = \bar{\nu}^* \in \mathscr{M}(G\times G)$.

Proof. First assume that the lemma is true for $\nu = g \in L^1(G\times G)$: then we can apply the argument used in the proofs of Lemmas 3.4 and 3.6 to get the desired result. So it is enough to show $(j \otimes j)(g) = \bar{g}^*$ for all $g \in L^1(G\times G)$. But the map $g \in L^1(G\times G) \to \bar{g}^*$ is continuous with respect to $||\cdot||_{c^*}$ (by Lemma 3.7, changing G by $G\times G$) and $j \otimes j$ is also continuous, so it is enough to show these maps agree on the $||\cdot||_{c^*}$-dense subset $L^1(G) \odot L^1(G)$. An easy computation shows this is indeed the case. □

THEOREM 3.9. *Let G be a locally compact group, $A = C^*(G)$, $d: A \to M(A \otimes A)$ as before satisfying* (3.5), *$j: A \to A$ as in Lemma 3.7. Then $\{A, d, j\}$ is an involutive cocommutative Hopf-C^*-algebra with coidentity ε, where*

$$\varepsilon(h) = \int h(\mathfrak{s})d\mathfrak{s} \tag{3.10}$$

for all $h \in L^1(G)$.

Proof. We have already shown that $\{A, d\}$ is a Hopf-C^*-algebra (Lemma 3.3 and Corollary 3.5). Let $h \in L^1(G)$. By Lemma 3.6,

$$\bar{\tau}d(h)(E) = d(h)(E^0) = \int_{\chi_{E^0}}(\mathfrak{s}, \mathfrak{s})h(\mathfrak{s})d\mathfrak{s}) = \int_{\chi_E}(\mathfrak{s}, \mathfrak{s})h(\mathfrak{s})d\mathfrak{s} = d(h)(E)$$

for all Borel sets $E \subseteqq G\times G$. Hence $\bar{\tau}d = d$ and $\{A, d\}$ is cocommutative.

We have already shown (Lemma 3.7) that j is a $*$-anti-isomorphism of period two, so to show that j is an involution for $\{A, d\}$ all we have to prove is that $(j \otimes j)^{-}d = \bar{\tau}dj$. Since $\{A, d\}$ is cocommutative, we have to show in fact that $(j \otimes j)^{-}d = dj$. Let $h \in L^1(G)$. By Lemma 3.8,

$$(j \otimes j)^{-}d(h) = \overline{d(h)^*} = \overline{d(h^*)} ;$$

but, if $\mu \in \mathscr{M}(G)$, then for all Borel sets $E \subseteqq G\times G$ we have

$$\overline{d(\mu)}(E) = \overline{\int \chi_E(\mathfrak{x}, \mathfrak{x})d\mu(\mathfrak{x})} = \int \chi_E(\mathfrak{x}, \mathfrak{x})d\bar{\mu}(\mathfrak{x}) = d(\bar{\mu})(E) ,$$

so we get

$$(j \otimes j)^{-} d(h) = d(\bar{h}^*) = d(j(h)) .$$

Hence $(j \otimes j)^{-} d = dj$.

Let ε be the one-dimensional representation of G given by $\varepsilon: \mathfrak{x} \in G \to 1 \in \boldsymbol{C}$. Then ε extends to a $*$-representation $\varepsilon: A \to \boldsymbol{C}$, i.e., ε is a nonzero $*$-homomorphism. It is clear that ε satisfy (3.10). If $g \in L^1(G \times G)$, one can show easily that $(\varepsilon \otimes I)g$, $(I \otimes \varepsilon)g \in L^1(G)$ and, for a.e. $\mathfrak{x} \in G$,

$$(\varepsilon \otimes I)g(\mathfrak{x}) = \int g(\mathfrak{s}, \mathfrak{x}) d\mathfrak{s} \quad \text{and} \quad (I \otimes \varepsilon)g(\mathfrak{x}) = \int g(\mathfrak{x}, \mathfrak{s}) d\mathfrak{s} .$$

From this it follows (by the argument used in Lemmas 3.4 and 3.6) that $(\varepsilon \otimes I)^{-}(\nu)$, $(I \otimes \varepsilon)^{-}(\nu) \in \mathscr{M}(G)$ for all $\nu \in \mathscr{M}(G \times G)$ and, for all Borel sets $B \subseteq G$,

$$(\varepsilon \otimes I)^{-}(\nu)(B) = \int_{G \times G} \chi_B(\mathfrak{t}) d\nu(\mathfrak{s}, \mathfrak{t}) ,$$

$$(I \otimes \varepsilon)^{-}(\nu)(B) = \int_{G \times G} \chi_B(\mathfrak{s}) d\nu(\mathfrak{s}, \mathfrak{t}) .$$

Thus, if $h \in L^1(G)$ and $B \subseteq G$ is a Borel set,

$$(\varepsilon \otimes I)^{-} d(h)(B) = \int_{\chi_B} (\mathfrak{t}) h(\mathfrak{t}) d\mathfrak{t}) ,$$

$$(I \otimes \varepsilon)^{-} d(h)(B) = \int_{\chi_B} (\mathfrak{s}) h(\mathfrak{s}) d\mathfrak{s} ,$$

so $(\varepsilon \otimes I)^{-} d(h) = h = (I \otimes \varepsilon)^{-} d(h)$ for all $h \in L^1(G)$. Hence $(\varepsilon \otimes I)^{-} d = I = (I \otimes \varepsilon)^{-} d$ and ε is a coidentity for $\{A, d\}$. □

DEFINITION 3.10. Let G be a locally compact group and let $\{A, d, j, \varepsilon\}$ be as in Theorem 3.9. We say that $\{A, d, j, \varepsilon\}$ is the *Hopf-C^*-algebra associated with G.*

LEMMA 3.11. *Let G be a locally compact group with associated Hopf-C^*-algebra $\{A, d, j, \varepsilon\}$. If we identify* (*as in* [6]) *A^* with the Fourier-Stieltjes algebra $B(G)$ of G, then the multiplication induced in A^* by d* (cf. *Proposition* 2.4) *coincide with pointwise multiplication in $B(G)$.*

Proof. Let $\psi_u, \psi_v \in A^*$ be associated with $u, v \in B(G)$ respectively, i.e.,

$$\langle h, \psi_u \rangle = \int h(\mathfrak{s}) u(\mathfrak{s}) d\mathfrak{s} \quad \text{and} \quad \langle h, \psi_v \rangle = \int h(\mathfrak{s}) v(\mathfrak{s}) d\mathfrak{s}$$

for all $h \in L^1(G)$. We have to show that

$$\langle h, \psi_u \psi_v \rangle = \int h(\mathfrak{s})u(\mathfrak{s})v(\mathfrak{s})d\mathfrak{s} \tag{3.11}$$

for all $h \in L^1(G)$. Since $u, v \in B(G)$, there exist $\pi_1, \pi_2 \in \Sigma(G)$, $\xi_1, \eta_1 \in \mathscr{H}_1 = \mathscr{H}_{\pi_1}$, $\xi_2, \eta_2 \in \mathscr{H}_2 = \mathscr{H}_{\pi_2}$ such that

$$u(\mathfrak{x}) = (\pi_1(\mathfrak{x})\xi_1 | \eta_1)_{\mathscr{H}_1} \quad \text{and} \quad v(\mathfrak{x}) = (\pi_2(\mathfrak{x})\xi_2 | \eta_2)_{\mathscr{H}_2}$$

for all $\mathfrak{x} \in G$ (cf. [6], Definition 2.2]). Define

$$\pi\colon (\mathfrak{x}, \mathfrak{y}) \in G \times G \longrightarrow \pi_1(\mathfrak{x}) \otimes \pi_2(\mathfrak{y}) \in \mathscr{B}(\mathscr{H}_1 \otimes \mathscr{H}_2) .$$

Then $\pi \in \Sigma(G \times G)$. Define

$$u \otimes v\colon (\mathfrak{x}, \mathfrak{y}) \in G \times G \longmapsto u(\mathfrak{x})v(\mathfrak{y}) \in \boldsymbol{C} .$$

Then:

$$\begin{aligned}(u \otimes v)(\mathfrak{x}, \mathfrak{y}) &= u(\mathfrak{x})v(\mathfrak{y}) = (\pi_1(\mathfrak{x})\xi_1|\eta_1)_{\mathscr{H}_1}(\pi_2(\mathfrak{y})\xi_2|\eta_2)_{\mathscr{H}_2} \\ &= (\pi_1(\mathfrak{x})\xi_1 \otimes \pi_2(\mathfrak{y})\xi_2|\eta_1 \otimes \eta_2)_{\mathscr{H}_1 \otimes \mathscr{H}_2} = (\pi(\mathfrak{x}, \mathfrak{y})\xi|\eta)_{\mathscr{H}_1 \otimes \mathscr{H}_2} ,\end{aligned}$$

where $\xi = \xi_1 \otimes \xi_2$, $\eta = \eta_1 \otimes \eta_2$. Hence $u \otimes v \in B(G \times G)$. Now, if $g \in L^1(G) \odot L^1(G)$, a simple computation shows that

$$\langle g, \psi_u \otimes \psi_v \rangle = \langle g, \psi_{u \otimes v} \rangle .$$

Since $L^1(G) \odot L^1(G)$ is dense in $C^*(G \times G)$, we get $\psi_u \otimes \psi_v = \psi_{u \otimes v}$. Hence, by [6; (2.6) Remark 3°], if $\nu \in \mathscr{M}(G \times G)$,

$$\langle \nu, \psi_u \otimes \psi_v \rangle = \int_{G \times G} u(\mathfrak{s})v(\mathfrak{t})d\nu(\mathfrak{s}, \mathfrak{t}) . \tag{3.12}$$

Now, let $h \in L^1(G)$. Then $d(h) \in \mathscr{M}(G \times G)$ so, by (3.12),

$$\langle h, \psi_u \psi_v \rangle = \langle d(h), \psi_u \otimes \psi_v \rangle = \int u(\mathfrak{s})v(\mathfrak{s})h(\mathfrak{s})d\mathfrak{s} .$$

Hence (3.11) holds and lemma follows. □

THEOREM 3.12. *Let G be a locally compact group with associated Hopf-C^*-algebra $\{A, d, j, \varepsilon\}$. Then G is isomorphic and homeomorphic to the intrinsic group of $\{A, d\}$.*

Proof. By Lemma 3.11, we can identify the commutative Banach algebras $B(G)$ and $A^* = (C^*(G))^*$. Then, by [11, Theorem 1], $G \cong \sigma(A^*) \cap A_u^{**}$; but the right hand side is just the intrinsic group of $\{A, d\}$, so the theorem follows. □

REMARK. We could have proved the above theorem directly. Under the natural inclusion $G \hookrightarrow (C^*(G))^*$, it is clear that $G \subseteqq \sigma(A^*) \cap A_u^{**}$. However, to show that this inclusion is onto, we would have to repeat the argument used by Walter (cf. [11, p. 28]). In view of this we decided to apply Walter's theorem.

As a corollary of Theorem 3.12 and Proposition 2.9, we get:

THEOREM 3.13. *If G_1 and G_2 are two locally compact groups whose associated Hoph-C^*-algebras are isomorphic, then they are homeomorphic and isomorphic.*

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