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THE STRONG APPROXIMATION THEOREM AND LOCALLY BOUNDED TOPOLOGIES ON $F(X)$

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To within equivalence, the only valuations on the field $F(X)$ of rational functions over F that are improper on F are the valuations v_p, where p is a prime polynomial of $F[X]$, and the valuation v_∞, defined by the prime polynomial X^{-1} of $F[X^{-1}]$. It is classic that if F is a finite field, the set $\mathscr{P}'$ defined by, $\mathscr{P}'=\{p: p$ is a prime polynomial over $F\} \cup \{\infty\}$, has the Strong Approximation Property, that is, for any finite subset G of $\mathscr{P}'$, any $q \in \mathscr{P}' \backslash G$, any family $(a_g)_{g \in G}$ of elements of $F(X)$ indexed by G, and any $M > 0$, there exists a nonzero element h in $F(X)$ such that $v_p(h - a_p) > M$ for all p in G and $v_p(h) \geqq 0$ for all p in $\mathscr{P}' \backslash (G \cup \{q\})$. We shall first prove that $\mathscr{P}'$ satisfies this condition when F is infinite as well. We then apply this result to obtain a characterization of all locally bounded topologies on $F(X)$ for which the subfield F is bounded.

1. The strong approximation theorem. Here, $\mathscr{P}$ is the set of prime polynomials in $F[X]$ and $\mathscr{P}'$ is the set $\mathscr{P} \cup \{\infty\}$.

THEOREM 1 (*The strong approximation theorem*). *For any finite subset G of $\mathscr{P}'$, any $q \in \mathscr{P}' \backslash G$, any family $(a_g)_{g \in G}$ of elements of $F(X)$ indexed by G, and any positive number M, there exists a nonzero h in $F(X)$ such that $v_p(h - a_p) \geqq M$ for all $p \in G$ and $v_d(h) \geqq 0$ for all $d \in \mathscr{P}' \backslash (G \cup \{q\})$.*

Proof. Let $S = \mathscr{P}' \backslash \{q\}$. By [5, Theorem 2.2, p. 322], it suffices to show that for distinct elements r and s in S and $M > 0$, there exists an h in $F(X)$ such that $v_r(h - 1) > M$, $v_s(h) > M$ and $v_d(h) \geqq 0$ for all $d \in S \backslash \{r, s\}$.

Case 1. $\infty \notin S$. Then r and s are distinct prime polynomials and so there exist polynomials f and g in $F[X]$ such that $f r^{M+1} + gs^{M+1} = 1$. Define h by, $h = gs^{M+1}$. Then $h - 1 = -fr^{M+1}$ and so $v_r(h - 1) \geqq M + 1 > M$. Furthermore, $v_s(h) \geqq M + 1 > M$. As h is a polynomial in $F[X]$, $v_d(h) \geqq 0$ for all $d \in \mathscr{P}$ and so in particular $v_d(h) \geqq 0$ for all $d \in S \backslash \{r, s\}$.

Case 2. $r = \infty$. Then s and q are distinct prime polynomials in $F[X]$. As v_∞ and v_s are independent valuations on $F(X)$, there exist polynomials f and g such that $v_\infty(f/g - 1) > M$ and $v_s(f/g) > M$

[1, Theorem 1, p. 134]. Choose a positive integer t such that $t \deg q > (M+1)\deg s + \deg f + M$. By the division algorithm, there exist polynomials w and z in $F[X]$ such that $q^t = ws^{M+1}g + z$ where $\deg z < (M+1)\deg s + \deg g$. So $fq^t = fws^{M+1}g + fz$ and hence $f/g = fws^{M+1}/q^t + fz/q^t g$. Let h be defined by $h = fws^{M+1}/q^t$. Then $v_s(h) \geqq M + 1 > M$ and for all prime polynomials p which are distinct from q, $v_p(h) \geqq 0$. So it suffices to show that $v_\infty(h-1) > M$.

Observe that $v_\infty(f/g - h) = v_\infty(fz/gq^t) = \deg g + t\deg q - \deg f - \deg z > \deg g + (M+1)\deg s + \deg f + M - \deg f - (M+1)\deg s - \deg g = M$. Therefore $v_\infty(h-1) = v_\infty(h - f/g + f/g - 1) \geqq \min\{v_\infty(h-f/g), v_\infty(f/g - 1)\} > M$.

Case 3. $s = \infty$. Then r and q are distinct prime polynomials. Let f be a polynomial such that $v_r(f-1) > M$. Choose a positive integer t such that $t\deg q > (M+1)\deg r + M$. By the division algorithm, there exist polynomials w and z in $F[X]$ such that $q^t f = wr^{M+1} + z$ where $\deg z < (M+1)\deg r$. Then $f = wr^{M+1}/q^t + z/q^t$. Let h be defined by $h = z/q^t$. Then $v_r(f-h) = v_r(wr^{M+1}/q^t) \geqq M + 1 > M$ and so $v_r(h-1) = v_r(h - f + f - 1) \geqq \min\{v_r(h-f), v_r(f-1)\} > M$. Furthermore,

$$v_\infty(h) = t\deg q - \deg z > (M+1)\deg r + M - (M+1)\deg r = M\,.$$

Finally for $d \in \mathscr{P}\backslash\{q\}$, $v_d(h) \geqq 0$.

Case 4. $\infty \in S\backslash\{r, s\}$. Then r, s and q are distinct prime polynomials in $F[X]$. By Case 1, there exists a polynomial f in $F[X]$ such that $v_r(f-1) > M$ and $v_s(f) > M$. Choose t so large such that $t\deg q > (M+1)(\deg r + \deg s)$ and let w and z be polynomials in $F[X]$ such that $fq^t = wr^{M+1}s^{M+1} + z$ where $\deg z < (M+1)(\deg r + \deg s)$. Then $f = wr^{M+1}s^{M+1}/q^t + z/q^t$. Define h by $h = z/q^t$. Then $v_r(f-h) = v_r(wr^{M+1}s^{M+1}/q^t) \geqq M + 1 > M$ and similarly $v_s(f-h) > M$. Hence $v_r(h-1) > M$ and $v_s(h) > M$. Furthermore for all polynomials p in $\mathscr{P}\backslash\{q\}$, $v_p(h) \geqq 0$. So it suffices to show that $v_\infty(h) \geqq 0$. As $v_\infty(h) = t\deg q - \deg z > (M+1)(\deg r + \deg s) - (M+1)(\deg r + \deg s) = 0$, $v_\infty(h) \geqq 0$.

2. Locally bounded topologies on $F(X)$. Let R be a ring and let $\mathscr{T}$ be a ring topology on R (that is, $\mathscr{T}$ is a topology on R making $(x, y) \to x - y$ and $(x, y) \to xy$ continuous from $R \times R$ to R). A subset S of R is *bounded* for $\mathscr{T}$ if given any neighborhood V of 0, there exists a neighborhood U of 0 such that $SU \subseteqq V$ and $US \subseteqq V$. $\mathscr{T}$ is a *locally bounded topology on* R if there is a bounded neighborhood of 0 for $\mathscr{T}$.

A *norm* $\|\cdot\cdot\|$ on a field K is a function from K to the nonnegative reals satisfying $\|x\| = 0$ if and only if $x = 0$, $\|x - y\| \leqq \|x\| + \|y\|$ and $\|xy\| \leqq \|x\|\|y\|$ for all x, y in K. Observe that a subset of K is bounded in norm if and only if it is bounded for the topology defined by the norm; in particular the topology defined by a norm is a locally bounded topology.

A subset I of a field K is an *almost order of* K if (1) $0, 1 \in I$, (2) $-I \subseteqq I$, (3) $II \subseteqq I$, (4) there exists a nonzero element h in I such that $h(I + I) \subseteqq I$, and (5) for each $x \in K^*$, there exist y, z in I^* such that $x = yz^{-1}$.

LEMMA 1 [6, *Theorems* 5 *and* 6]. *If $\mathscr{T}$ is a nondiscrete, locally bounded ring topology on a field K, then there is an almost order I of K that is a bounded neighborhood of zero. Conversely, if I is an almost order of K, then there exists a unique nondiscrete, locally bounded ring topology $\mathscr{T}$ on K for which I is a bounded neighborhood of zero. Furthermore, the topology $\mathscr{T}$ defined by I is Hausdorff if and only if $I \neq K$.*

In [7] we investigated locally bounded topologies on the quotient fields of certain Dedekind domains. We recall the results of that paper.

Let K be the quotient field of a Dedekind domain R that is not a field, $\mathscr{P}$ the set of nonzero prime ideals of R and $\mathscr{P}_\infty$ a set $\{|\cdot\cdot|_1, \cdots, |\cdot\cdot|_n\}$ of n mutually inequivalent proper absolute values on K such that for each $i \in [1, n]$ and each $p \in \mathscr{P}$, the topology $\mathscr{T}_i$ defined by $|\cdot\cdot|_i$ is distinct from the topology $\mathscr{T}_p$ defined by the valuation v_p. Let $\mathscr{P}'$ be defined by $\mathscr{P}' = \mathscr{P} \cup \mathscr{P}_\infty$. For each subset S of $\mathscr{P}'$, we define $O(S)$ by $O(S) = \{x \in K: v_p(x) \geqq 0$ for all $p \in S \cap \mathscr{P}$ and $|x|_i \leqq 1$ for all $|\cdot\cdot|_i \in S \cap \mathscr{P}_\infty\}$.

We placed the following conditions on K, R and $\mathscr{P}'$:

I. Class Number Condition (CC). The class number of K over R is finite.

II. Approximation Condition (AC). For any finite subset G of $\mathscr{P}'$, any $\gamma \in \mathscr{P}' \backslash G$, any family $(a_g)_{g \in G}$ of elements of K indexed by G, and any positive numbers M and e, there exists a nonzero element h in K such that $v_p(h - a_p) \geqq M$ for all $p \in G \cap \mathscr{P}$, $|h - a_{|\cdot\cdot|_k}|_2 \leqq e$ for all $|\cdot\cdot|_2 \in G \cap \mathscr{P}_\infty$ and $h \in O(\mathscr{P}' \backslash (G \cup \{\gamma\}))$.

III. Discreteness Condition (DC). The only ring topology on K for which $O(\mathscr{P}')$ is a neighborhood of zero is the discrete topology.

IV. Euclidean Condition (EC). There exist positive numbers $\beta_1, \cdots, \beta_n$ such that if $a, b \in R$ with $b \neq 0$, then there exist q, r in R satisfying $a = bq + r$, $|r|_i \leqq |b|_i \beta_i$ for all i in $[1, n]$.

LEMMA 2 [7, Lemma 2]. *If S is a nonempty, proper subset of $\mathscr{P}'$, then $O(S)$ is an almost order for a unique, Hausdorff, nondiscrete, locally bounded ring topology $\mathscr{T}_S$ on K.*

LEMMA 3 [7, *Theorem* 3, *Statement* 3]. *Let $\mathscr{T}$ be a Hausdorff, nondiscrete, locally bounded ring topology on K with the following property.*

V. Boundedness Condition (BC). For any $M > 0$, the set $O(\mathscr{P}) \cap \{y \in K: |y|_i \leqq M$ for all $|\cdot\cdot|_i \in \mathscr{P}_\infty\}$ is a bounded set for $\mathscr{T}$.

If $\mathscr{P}_\infty$ has exactly one element, then $\mathscr{T} = \mathscr{T}_S$ for some nonempty, proper subset S of $\mathscr{P}'$.

THEOREM 2. *Let F be a field and X an indeterminate over F. Let $\mathscr{P}$ be the set of all prime polynomials over F, v_∞ the valuation on $F(X)$ defined by $v_\infty(f/g) = \deg g - \deg f$ and let $\mathscr{P}_\infty = \{|\cdot\cdot|_\infty\}$ where $|y|_\infty = 2^{-v_\infty(y)}$ for all y in $F(X)$. Then $F(X)$, $F[X]$ and $\mathscr{P}' = \mathscr{P} \cup \mathscr{P}_\infty$ satisfy* (CC), (AC), (DC) *and* (EC). *Moreover, if $\mathscr{T}$ is a Hausdorff, nondiscrete, locally bounded ring topology on $F(X)$ for which the subfield F is bounded, then $\mathscr{T}$ satisfies* (BC) *and hence $\mathscr{T} = \mathscr{T}_S$ for some nonempty, proper subset S of $\mathscr{P}'$.*

Proof. As $F[X]$ is a principal ideal domain, (CC) holds. By Theorem 1, (AC) holds. Furthermore, (DC) holds. Indeed, as $O(\mathscr{P}') = F$, if $\mathscr{T}$ is a ring topology on $F(X)$ for which $O(\mathscr{P}')$ is a neighborhood of zero, then the set $F \cap FX = \{0\}$ is a neighborhood of zero for $\mathscr{T}$. Thus $\mathscr{T}$ is discrete. By the division algorithm, (EC) holds with $\beta_1 = 1$. So it suffices to prove that (BC) holds when $\mathscr{T}$ is a locally bounded topology on $F(X)$ for which the subfield F is bounded.

Notice that for $M > 0$, $O(\mathscr{P}) \cap \{y \in F(X): |y|_\infty \leqq M\} = \{y \in F[X]: \deg y \leqq N\}$ where $N = lnM/ln2$. Consequently, if $\mathscr{T}$ is a locally bounded topology on $F(X)$ for which the subfield F is bounded, then $\mathscr{T}$ satisfies (BC) and therefore by Lemma 3, $\mathscr{T} = \mathscr{T}_S$ for some nonempty, proper subset S of $\mathscr{P}'$.

COROLLARY [7, *Corollary* 4]. *If F is a finite field and $\mathscr{T}$ is a Hausdorff, nondiscrete, locally bounded topology on $F(X)$, then there exists a nonempty, proper subset S of $\mathscr{P}'$ such that $\mathscr{T} = \mathscr{T}_S$.*

The following theorem is a generalization of Theorem 2 of [3].

THEOREM 3. *Let $\mathcal{T}$ be a Hausdorff, nondiscrete, locally bounded ring topology on $F(X)$ for which the subfield F is bounded. The following statements are equivalent.*

1° *$\mathcal{T}$ is a field topology.*

2° *$\mathcal{T}$ is the supremum of finitely many valuation topologies $\mathcal{T}_p$ where $p \in \mathcal{P}'$.*

3° *There exists a nonzero element a in $F(X)$ such that $\lim_{n\to\infty} a^n = 0$.*

4° *$\mathcal{T}$ is defined by a norm.*

Proof. Let S be a nonempty, proper subset of $\mathcal{P}'$ such that $\mathcal{T} = \mathcal{T}_S$.

To show that 1° implies 2°, it suffices to show that S is finite. As $\mathcal{T}$ is a field topology and $O(S) + 1$ is a neighborhood of 1 in $\mathcal{T}$, there exists a y in $O(S)\backslash\{0\}$ such that $(yO(S) + 1)^{-1} \subseteq O(S) + 1$. If S is infinite, pick $p \in S \cap \mathcal{P}$ such that $v_p(y) = 0$. By Theorem 1, there exists a z in $F(X)$ such that $v_p(z + y^{-1}) > 0$ and $z \in O(S\backslash\{p\})$. Then $v_p(z) = v_p(z + y^{-1} - y^{-1}) \geqq \min\{v_p(z + y^{-1}), v_p(y^{-1})\} \geqq 0$ and so $z \in O(S)$. Hence $yz + 1 \in yO(S) + 1$ and $v_p(yz + 1) = v_p(y(z + y^{-1})) = v_p(y) + v_p(y + z^{-1}) > 0$. Therefore, $v_p((yz + 1)^{-1}) < 0$. But $(yz + 1)^{-1} \in O(S) + 1$ and $v_p(w) \geqq 0$ for all $w \in O(S) + 1$. Contradiction! Therefore S is finite.

To prove that 2° implies 3°, we note that if S is any nonempty, finite subset of $\mathcal{P}'$ and a is any nonzero element of $F(X)$ such that $|a|_\infty < 1$ when $|\cdot\cdot|_\infty \in S$ and $v_p(a) > 0$ for all p in $S \cap \mathcal{P}$, then $\lim_{n\to\infty} a^n = 0$ in $\mathcal{T}_S$. The existence of such an element is guaranteed by Theorem 1. The statement 3° implies 4° is a special case of Cohn's theorem [4, Theorem 6.1]. Finally the proof that 4° implies 1° is the same as that for normed algebras found on p. 77 of [2].

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