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MULTIPLICITY THEORY FOR BOOLEAN ALGEBRAS OF L^p-PROJECTIONS

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As an application of the integral module representation we investigate the multiplicity function in the sense of Bade for Boolean algebras of L^p-projections. In the case of finite multiplicity and the restriction of the multiplicity function to invariant subspaces we obtain results which have no analogue in the case of arbitrary Boolean algebras of projections.

0. **Introduction.** Let X be a fixed real Banach space and p a fixed number, $1 \leqq p < \infty$. An L^p-projection is a continuous projection $E: X \to X$ such that $\|x\|^p = \|Ex\|^p + \|x - Ex\|^p$ for all $x \in X$. L^p-projections have been investigated in [3], [4], [5], [8].

We consider a fixed complete Boolean algebra, say $\mathscr{B}$, of L^p-projections on X (for definitions, in particular that of the multiplicity function, see § 1 below). First, we give a short survey of the integral module representation which was introduced in the thesis of the second-named author ([8]). We use this representation to obtain results concerning the relationship between pairs of closed subspaces invariant w.r.t. $\mathscr{B}$ - the $\mathscr{B}$-cycles. The main result in this connection is the following: If M is a $\mathscr{B}$-cycle and x an element of X, then there is a $z \in X$ such that $(M + S(x))^- = M \oplus S(z)$ whereby $S(x)$ denotes the smallest $\mathscr{B}$-cycle containing x. It follows that in the case of finite multiplicity n there are vectors $x_1, \cdots, x_n$ with $X = S(x_1) \oplus \cdots \oplus S(x_n)$. This is in marked contrast to the case of arbitrary Boolean algebras of projections, where there is a Banach space X containing vectors x_1 and x_2 such that $X = (S(x_1) + S(x_2))^-$ but in which it is impossible to find vectors y_1, y_2 with $X = S(y_1) + S(y_2)$ (see [6]).

The case of finite multiplicity will be treated in some detail in § 4. It turns out that in this case the space X may be arbitrarily well approximated by finite families of Bochner spaces $L^p(V: \mu)$ with finite-dimensional V.

The last section is devoted to the restriction of the multiplicity to invariant subspaces, i.e. cycles. There seem to be no results in this direction in the general case (cf [7], p. 2289). We shall show that, for L^p-projections, the multiplicity function behaves as one would expect, i.e., the multiplicity of a cycle is less than or equal to that of the whole space. Partial results concerning the multiplicity of adjoints in dual spaces have been excluded in order to avoid the technicalities which are involved (e.g., the dual integral module representation).

1. Preliminaries.

Boolean algebras of L^p-projections. The concept of an L^p-projection was defined in the introduction. The range space of such a projection is called an L^p-summand. For the basic properties of L^p-projections and L^p-summands, the reader is referred to [4]. Note that $X = X_1 \oplus_p X_2 \oplus_p \cdots \oplus_p X_n$ is a short way of indicating that the X_i are L^p-summands with $X_i \cap X_j = \{0\} (i \neq j)$ and $X = X_1 + X_2 + \cdots + X_n$ — it follows that $\|x_1 + \cdots + x_n\|^p = \|x_1\|^p + \cdots + \|x_n\|^p$ if $x_i \in X_i$ for all i.

It is known that for $p \neq 2$ every pair of L^p-projections commute [3]. It follows that the set of all L^p-projections is in this case a complete Boolean algebra of projections in the sense of Bade [1] (i.e., an increasing net converges to its supremum, in the strong operator topology). We shall consider the more general case in which $\mathscr{B}$ is a fixed complete Boolean algebra of L^p-projections for some p, whereby the case $p = 2$ is not excluded and we do not assume that this algebra contains all the L^p-projections.

A $\mathscr{B}$-summand is an L^p-summand whose corresponding L^p-projection lies in $\mathscr{B}$. For the general properties of these summands and of Boolean algebras of projections we refer the reader to [4] and [1]. We need the following facts and definitions:

(a) There is a compact Hausdorff extremally disconnected space Ω such that the elements of $\mathscr{B}$ correspond to the clopen subsets of Ω (Ω is just the Stonean space of $\mathscr{B}$). By E_D we mean the element of $\mathscr{B}$ which corresponds to the clopen set D.

(b) The Banach algebra generated by $\mathscr{B}$, $(\operatorname{lin} \mathscr{B})^-$, is called the *Cuningham algebra* of $\mathscr{B}$ and is written $\mathscr{C}_{\mathscr{B}}$. It is isometrically isomorphic to $C(\Omega)$, the Banach algebra of continuous functions on Ω is such a way that E_D corresponds to χ_D for each clopen set D in Ω. For $T \in \mathscr{C}_{\mathscr{B}}$ we write $\hat{T}$ for the associated function in $C(\Omega)$.

(c) For every Borel measurable function on Ω there is a (uniquely determined) continuous numerical valued function on Ω such that these functions coincide on the complement of a set of first category. Each Borel set in Ω is the symmetric difference of a clopen set and a set of first category.

(d) For each $x \in X$ we define $\rho_x(D) := \|E_D x\|^p$ for all clopen D in Ω. ρ_x can be extended to a regular Borel measure on Ω which we also write ρ_x.

(e) If D is a clopen subset of Ω we say that D satisfies the *countable chain condition* (CCC) if each strictly increasing chain of clopen sets in D is at most countable (equivalently: each family of disjoint clopen subsets of D is at most countable). In this case we shall also say that the corresponding projection E satisfies the CCC.

It is easy to see that supp ρ_x satisfies the CCC since ρ_x is finite. It follows that Ω is the supremum of a family of clopen subsets each of which satisfies the CCC. We can thus restrict ourselves to the case where Ω and I themselves have this property.

(f) For $Y \subset X$ there is a smallest projection E such that $Ey = y$ for all $y \in Y$ (the carrier projection of Y). Note that if $F \wedge E = 0$ then $FY = \{0\}$. If $Y = \{x\}$, we shall write E_x for this projection.

(g) We note that with the notation of (d), (e), (f) the following conditions are equivalent:

(i) Ω satisfies the CCC.

(ii) there is an $x \in X$ such that supp $\rho_x = \Omega$.

(iii) there is an $x \in X$ such that $E_x = I$.

Following [8] we call a closed subspace of X which is invariant w.r.t. all projections in $\mathscr{B}$ a $\mathscr{B}$-cycle. It is easily verified that the intersection and the closed linear hull of arbitrary collections of $\mathscr{B}$-cycles are again $\mathscr{B}$-cycles. It follows that there is for each $x \in X$ a smallest $\mathscr{B}$-cycle containing x. This cycle is the closed linear hull of $\{Ex \mid E \in \mathscr{B}\}$ and is written $S(x)$.

Multiplicity theory. The cycles of the form $S(x)$ behave in many respects as one-dimensional subspaces (cf. f. ex. [4], Satz D1). The aim of multiplicity theory is to investigate a "dimension" with respect to the projection algebra, using the $S(x)$'s as building blocks. If the projection algebra is the trivial one, consisting of 0 and I, multiplicity theory is simply dimension theory. If, as in our case, the projection algebra consists only of L^p-projections, multiplicity measures, in some sense, how close X behaves as an abstract L^p-space. In § 4, for example, where we discuss the case of finite multiplicity, we shall see that X behaves almost as a space $L^p(V; \mu)$, which is the natural generalization of the L^p-spaces. Following Bade ([2]) we define:

DEFINITION 1.1.

(i) $\mathscr{C} := \{E \mid E \in \mathscr{B},\ E$ satisfies the CCC$\}$.

(ii) For $E \in \mathscr{C}$, $m(E)$ (the *multiplicity* of E) is defined by $m(E) := \inf\{\text{Card } I \mid$ there exists a family $(x_i)_{i \in I}$ in EX such that $(\text{lin}(\bigcup_{i \in I} S(x_i))^-) = EX\}$.

(iii) $m(\cdot)$ is extended to the whole of $\mathscr{B}$ by means of $m(E) := \sup\{m(F) \mid F \leqq E,\ F \in \mathscr{C}\}$ for $E \in \mathscr{B}$.

(iv) For $E \in \mathscr{B}$ the multiplicity of E is said to be *uniform*, if $m(F) = m(E)$ for all $F \in \mathscr{B}$, $0 \neq F \leqq E$.

Clearly, from the definition, $m(E) = m(E|_{EX})$ whereby the second multiplicity is calculated in the Boolean algebra $\mathscr{B} \circ E$. Also, by (iii), the multiplicity is completely determined by those E which satisfy the CCC. It follows that any procedure for calculating the multiplicity of the identity on spaces where it satisfies the CCC can be used to calculate the multiplicity of an arbitrary projection on an arbitrary space. We shall sometimes use this fact to simplify our proofs.

For a discussion of the properties of the multiplicity function we refer the reader to [2]. In particular the following properties hold:

PROPOSITION 1.2.

(i) $m(0) = 0$.

(ii) *m is monotone and* $m(\sup_{j \in J} E_j) = \sup_{j \in J} m(E_j)$ *for every family* $(E_j)_{j \in J}$ *in* $\mathscr{B}$.

(iii) *There is a disjoint partition of $\mathscr{B}$ into a family (E_c), indexed by a set of cardinals, such that each E_c has uniform multiplicity c (by "disjoint partition" we mean that the E_c are pairwise disjoint and $\sup E_c = I$). Note that this partition corresponds to a disjoint family (Ω_c) of clopen subsets of Ω with* $(\bigcup \Omega_c)^- = \Omega$.

Suppose that M is a $\mathscr{B}$-cycle. By restriction, $\mathscr{B}$ induces a complete Boolean algebra of L^p-projections on M. If we denote the multiplicity function on the restricted algebra by $\tilde{m}$, it is natural to expect that $\tilde{m}(E|_M) \leqq m(E)$ for $E \in \mathscr{B}$. We shall see in § 5 that this is indeed true.

Since $S(x) = \{Tx \mid T \in \mathscr{C}_{\mathscr{B}}\}^-$, the multiplicity of E corresponds to the dimension of EX as a normed $\mathscr{C}_{\mathscr{B}}$-module in the case where E satisfies the CCC. If E does not satisfy the CCC then the dimension of EX is generally larger than the multiplicity of E.

2. The integral module representation. In this chapter we shall describe without proofs the representation of X as a space of vector-valued functions on Ω such that the elements of $\mathscr{B}$ have the form of multiplication by characteristic functions which was introduced in [8]. We shall only treat a special case which is general enough for our purpose, namely that where Ω satisfies the countable chain condition. For the general case and a complete description of the construction, see [8], [5].

As before, let $\mathscr{B}$ be a complete Boolean algebra of L^p-projections on the real Banach space X. We assume that Ω satisfies the countable chain condition. Thus there is an element x in X such that $Ex \neq 0$ for all $E \neq 0$ in $\mathscr{B}$, wlog $||x|| = 1$ (see introduction). Let y be some element of X. Since $\operatorname{supp} \rho_x = \Omega$, the measure ρ_y is absolutely continuous w.r.t. ρ_x. Thus there is an $L^1(\rho_x)$-function f^y on Ω such that $\rho_y = f^y \rho_x$. As every ρ_x-measurable function is almost everywhere equal to some unique continuous function on Ω, we can assume that f^y itself is continuous. Note that $f^y \geqq 0$ and is finite almost everywhere.

We denote by $C^p(\Omega, \rho_x)$ the space of continuous numerical p-integrable functions on the measure space (Ω, ρ_x) with the obvious structures of addition and multiplication by $C(\Omega)$-functions. $||g||_p$ denotes the L^p-norm of an element in $C^p(\Omega, \rho_x)$. Now, if we set $[y]: = (f^y)^{1/p}$, the mapping $y \mapsto [y]$ maps X into $C^p(\Omega, \rho_x)$. This mapping has the following properties:

LEMMA 2.1.
(i) $||y|| = ||[y]||_p$ *for all* y *in* X.
(ii) $[Ty] = |T|\,[y]$ *for* T *in* $\mathscr{C}_{\mathscr{B}}$ *and* y *in* X.
(iii) $[y + z] \leqq [y] + [z]$ *for* y, z *in* X.

These are the defining properties of a *norm resolution.*

By 2.1 (ii) and (iii) the mapping $y \mapsto [y](k)$ is, for each k in Ω, a semi-norm on $Y_k: = \{y \mid y \text{ in } X, [y](k) < \infty\}$, a linear subspace of X. Let X_k be the associated Banach space, i.e., the completion of $Y_k/\{y \mid [y](k) = 0\}$ in the norm $||\cdot||_k$ induced by $y \mapsto [y](k)$. This construction provides us with a family of Banach spaces indexed by the points of Ω. The *p-direct integral* of this family over Ω with respect to ρ_x, $\int_{\Omega}^{p} X_k d\rho_x$, is the set of all mappings $f: \Omega \to \bigcup_{\Omega} \cup \{\infty\}$ such that $f(k)$ lies in $X_k \cup \{\infty\}$ for each k and for which the scalar-valued function $k \mapsto ||f(k)||_k (||\infty||_k: = \infty)$ lies in $C^p(\Omega, \rho_x)$. A p-direct integral is not, in general, closed under addition so that it cannot be given a natural vector space structure. However a subset Z of the p-direct integral for which the following properties hold:

DEFINITION 2.2.

(i) For all f, g in Z there is an h in Z with $f(k) + g(k) = h(k)$ wherever $f(k)$ and $g(k)$ lie in X_k.

(ii) For all f in Z and g in $C(\Omega)$ there is an h in Z with $g(k) \cdot f(k) = h(k)$ wherever $f(k)$ lies in X_k

can be given a natural $C(\Omega)$-module structure since continuity implies that the h of 2.2 is unique.

If, further:

(iii) For each k, $X_k = \{f(k) \mid f \in Z, f(k) \neq \infty\}^-$.

(iv) Z is complete in the norm $N(f) := \|(\|f(\cdot)\|.)\|_p$

we call Z an *integral module*.

If we now map X into $\int_\Omega^p X_k d\rho_x$ by means of $y \mapsto \langle y \rangle$ whereby

$$\begin{aligned}\langle y\rangle(k) &:= \text{the equivalence class of } y \text{ in } X_k \text{ if } [y](k) < \infty \\ &:= \infty \qquad \text{if } [y](k) = \infty\end{aligned}$$

all the structures of X are transferred to the range space so that 2.2 (i), (ii) and (iv) are automatically satisfied and (iii) follows from the definition of the X_k. We thus have the following representation theorem:

THEOREM 2.3. *Let X, $\mathscr{B}$, Ω and x be as above.*

*There is a family of Banach spaces X_k, indexed by the points of Ω such that X is isometrically isomorphic as a $C(\Omega)$-module to an integral module in $\int_\Omega^p X_k d\rho_k$. ($X$ is a $C(\Omega)$-*module *by virtue of $C(\Omega) \cong \mathscr{C}_x$).*

The most important advantage of the integral module representation is of course that the operators in $\mathscr{C}_x$ now correspond to the pointwise multiplication by continuous functions. In particular a projection in $\mathscr{B}$ has the action of restriction to the corresponding set in Ω.

Since we are interested in $\mathscr{B}$-cycles in X we need to know how these look in the integral module representation. As is to be expected they turn out to have a particularly simple form.

PROPOSITION 2.4. *Let $\langle\cdot\rangle: X \to \int_\Omega^p X_k d\rho_x$ be the representation of* 2.3.

If M is a $\mathscr{B}$-cycle in X then:

(i) *$M_k := \{\langle y\rangle(k) \mid \langle y\rangle(k) \neq \infty, y \in M\}$ is a closed subspace of X_k for each k.*

(ii) *If y is an element of X then y is in M if and only if $\langle y\rangle(k) \in M_k \cup \{\infty\}$ far all k.*

(iii) *For each natural number n the set $\{k \mid \dim M_k = n\}$ is clopen in Ω. In particular the set* $\operatorname{supp} M := \{k \mid M_k \neq \{0\}\}^-$ *is clopen and corresponds to the carrier projection of M.*

Proof.

(i) That M_k is a subspace is clear, since $\langle ay+bz\rangle(k) = a\langle y\rangle(k) + b\langle z\rangle(k)$ for a, b in $\boldsymbol{R}$, y, z in M. It remains to show that M_k is closed. Suppose that y^k is in X_k with $\langle y_n\rangle(k) \to y^k$ for some sequence $\{y_n\}$ in M. We suppose that in addition $\|\langle y_{n+1}\rangle(k) - \langle y_n\rangle(k)\|_k = [y_{n+1} - y_n](k) \leqq 1/2^{n+1}$ for each n. Set $z_1 = y_1$. Since $[y_2-y_1](k) \leqq 1/4$ there is a clopen set D_1 containing k on which $[y_2 - y_1] \leqq 1/2$. Set $z_2 = y_1 + E_{D_1}(y_2 - y_1)$. Then $[z_2 - z_1] \leqq 1/2$ and $\langle z_2\rangle(k) = \langle y_2\rangle(k)$. By induction we obtain a sequence $\{z_n\}$ in M with $[z_{n+1} - z_n] \leqq 1/2^n$ and $\langle z_n\rangle(k) = \langle y_n\rangle(k)$ for each n. Thus $\{z_n(l)\}$ converges uniformly on Ω, in particular $\{z_n\}$ converges in M to z with $\langle z\rangle(l) = \lim \langle z_n\rangle(l)$ for each 1. Since $\langle z_n\rangle(k) = \langle y_n\rangle(k)$ for all n we have $\langle z\rangle(k) = \lim \langle y_n\rangle(k) = y^k$. Thus M_k is closed.

(ii) Suppose that y is in X and that $\langle y\rangle(k) \in M_k \cup \{\infty\}$ for each k. Let $\varepsilon > 0$ and E_D be such that $\|E_D y - y\| < \varepsilon$ and $\langle y\rangle(k) \in M_k$ for k in D. Then there is for each k in D a z_k in M with $\langle z_k\rangle(k) = \langle y\rangle(k)$. Let $D_k \subseteqq D$ be a clopen set containing k on which $[z_k - y] < \varepsilon$. Then D is covered by finitely many D_k's, say $D_1, \cdots, D_n$ corresponding to $k_1, \cdots, k_n$. Wlog we may assume that the D_i's are disjoint. Put $z = E_{D_1}z_{k_1} + \cdots + E_{D_n}z_{k_n} \in M$. Then

$$\|z - E_D y\| \leqq \left(\int_D [z - y]^p d\rho_x\right)^{1/p} = \left(\sum_1^n \int_{D_i} [z_{k_i} - y]^p d_{\rho x}\right)^{1/p}$$
$$\leqq (\varepsilon^p)^{1/p} = \varepsilon .$$

Thus $\|z - y\| \leqq 2\varepsilon$. Since ε was arbitrary and M is closed we have $y \in M$.

The 'only if' part follows immediately from the definition of M_k.

(iii) The continuity of $[y - z]$ for y, z in M implies that if $\|\langle y\rangle(k) - \langle z\rangle(k)\|_k > \varepsilon$ for some k then $\|\langle y\rangle(l) - \langle z\rangle(l)\|_l > \varepsilon$ for all l in some clopen neighbourhood of k. Using this and a standard argument invoking the compactness of the unit ball in a finite-dimensional space we obtain: If $y, z_1, \cdots, z_m$ are in M and $d(\langle y\rangle(k), \operatorname{lin}\{\langle z_1\rangle(k), \cdots \langle z_m\rangle(k)\}) > \varepsilon$ for some k then $d(\langle y\rangle(l), \operatorname{lin}\{\langle z_1\rangle(l), \cdots, \langle z_m\rangle(l)\}) > \varepsilon$ for l in some clopen neighbourhood of k. It follows that if $\langle z_1\rangle(k), \cdots, \langle z_m\rangle(k)$ are linearly independent for some k then $\langle z_1\rangle(l), \cdots, \langle z_m\rangle(l)$ are linearly independent in some neighbourhood of k. Thus for each n the set $\{k \mid \dim M_k \geqq n\}$ is open. Let k_0 be a point in the closure of this set and suppose that $\dim M_{k_0} = m < n$. Let $z_1, \cdots, z_m$ in M be m elements such that $\langle z_1\rangle(k_0), \cdots, \langle z_m\rangle(k_0)$ is a basis for M_{k_0}. Let $A = \{z \mid z \in M, [z] \leqq 3, d(\langle z\rangle(k), \operatorname{lin}\{\langle z_1\rangle(k), \cdots, \langle z_m\rangle(k)\}) \geqq [z](k)/2 \geqq 1/2$ wherever $[z](k) \neq 0\}$. We order the elements of A by means of $z \leqq z'$ if and only if $\langle z\rangle(k) = \langle z'\rangle(k)$ wherever $\langle z\rangle(k) \neq 0$. If $\{z_\alpha\}_{\alpha \in A}$ is a totally ordered chain in A then it converges to an element z_0 in M which also satisfies the defining conditions of A. Clearly z_0 is an upper bound for the chain so that we can apply Zorn's Lemma to obtain a maximal element in A, say z. Since $\langle z_1\rangle(k_0), \cdots, \langle z_m\rangle(k_0)$ span M_{k_0}, $\langle z\rangle(k_0)=0$. But $[z](k) \geqq 1$ where $\langle z\rangle(k) \neq 0$ so, by the continuity of $[z](k)$, we have that $\langle z\rangle(k) = 0$ on some clopen neighborhood D of k_0. Let k be a point in D with $\dim M_k \geqq n$. Then there is a z' in M with $[z'](k) = 2$ and $d(\langle z'\rangle(k), \operatorname{lin}\{\langle z_1\rangle(k), \cdots, \langle z_m\rangle(k)\}) > 3/2 > [z'](k)/2$. By continuity there is a clopen set $D_0 \subseteq D$ containing k such that $E_{D_0}z'$ lies in A. But then $z + E_{D_0}z'$ lies in A and is strictly larger than z. Since this is a contradiction it follows that $\dim M_{k_0} \geqq n$. Thus $\{k \mid \dim M_k \geqq n\}$ is clopen for all n. Since $\{k \mid \dim M_k = n\} = \{k \mid \dim M_k \geqq n\} \backslash \{k \mid \dim M_k \geqq n+1\}$ this set is also clopen for each n. $\operatorname{supp} M = \{k \mid \dim M_k \geqq 1\}$ and is therefore clopen. That it corresponds to the carrier projection of M is trivial.

By applying 2.4 (i) to X itself we immediately obtain:

COROLLARY 2.5. *For all* $k \in \Omega$, *and* y^k *in* X_k *there is a* $y \in X$ *with* $\langle y\rangle(k) = y^k$, i.e., *the completion in the definition of* X_k *is superfluous.*

The Zorn's Lemma argument of 2.4 (iii) occurs over and over again when working with integral modules. In order to save repeating it each time it is formulated in the following lemma for which we need a preparatory definition.

DEFINITION 2.6. Let X, $\mathscr{B}$ be as above and n a natural number.

$$QS(X^n, \mathscr{B}) := \{(x_1, \cdots, x_n; F) \mid x_i \in X, F \in \mathscr{B} \text{ and } E_{x_i} = F \text{ for all } i\} .$$

THE EXISTENCE LEMMA 2.7. *Let* A *be a subset of* $QS(X^n, \mathscr{B})$ *for which the elements of* X *which occur are bounded in norm and let* $E \in \mathscr{B}$ *satisfy the countable chain condition. If*

(i) *For each* F in $\mathscr{B}$, $0 \lneqq F \leqq E$ *there is an* $a := (a_1, \cdots, a_n; F_a)$ *in* A *with* $0 \lneqq F_a \leqq F$.

(ii) *Whenever* $\{F_\alpha\} \nearrow F \leqq E$ *and* a *is an element of* $QS(X^n, \mathscr{B})$ *such that* $F_\alpha a := (F_\alpha a_1, \cdots, F_\alpha a_n; F_\alpha F_a)$ *is in* A *for all* α *then* Fa *is also in* A.

(iii) *Whenever* a *and* a' *are in* A *and* $F_a F_{a'} = 0$ *then* $a + a' := (a_1 + a_1', \cdots, a_n + a_n'; F_a + F_{a'})$ *is in* A.

Then there is an element a *in* A *with* $F_a = E$.

Proof. We order the elements of A by means of $a \leqq a'$ if and only if $F_a \leqq F_{a'}$ and $a_i = F_a a_i'$ for all i. Let A_E be the set of all

a in A with $F_a \leqq E$. A strictly increasing totally ordered chain in A_E can be at most countable since E satisfies the countable chain condition, say $a^1, \cdots, a^m, \cdots$. By the boundedness in norm of the elements occurring in A, $a_i^1, a_i^2, \cdots, a_i^m, \cdots$, is a Cauchy sequence for each i and converges in X to an element a_i. Clearly $a: = (a_1, \cdots, a_n; \sup F_{a^m})$ is in $QS(X^n, \mathscr{B})$ and $F_{a^m}a = a^m$ for each natural number m. By (ii) a is in A and is an upper bound for the chain $a^1, \cdots, a^m, \cdots$. Since $F_{a^m} \leqq E$ for each m, a is in A_E. By Zorn's Lemma A_E contains a maximal element. By (i) and (iii) the projection in this element must be E.

Note that it is not essential for the validity of the existence lemma that $\mathscr{B}$ consists of L^p-projections (although that is of course the case in the present paper). It suffices that the norm boundedness implies that every sequence in which each element is an extension of the previous one is Cauchy. In practice we shall only use the cases $n = 1$ and $n = 2$ and shall normally only verify (i), leaving it to the reader to check that (ii) and (iii) are trivially satisfied. If the elements of A are characterized by some property then verifying (i) corresponds to showing that the set of projections for which it holds is co-final in every ultrafilter containing E. The existence lemma then allows us to conclude that the property holds for E itself.

3. Distance functions. If y is an element of a Banach space Y and M a subspace, the distance of y from M, $d(y, M)$ is $\inf_{m \in M} \|y + m\|$. This well-known distance function suggests the following definition of the distance between two subspaces of Y:

DEFINITION 3.1. The distance between two subspaces M_1, M_2 of a Banach space Y, $d(M_1, M_2)$ is $\inf \{\|m_1 + m_2\| \mid m_1 \in M_1, m_2 \in M_2, \max \{\|m_1\|, \|m_2\|\} = 1\}$, whereby we define the infimum of the empty set to be unity.

Clearly, if $d(M_1, M_2) = d$, we have $\|m_1 + m_2\| \geqq d\|m_1\|$ and $\geqq d\|m_2\|$ for $m_1 \in M_1$, $m_2 \in M_2$. This distance function takes values in $[0, 1]$.

PROPOSITION 3.2. *Let M_1, M_2 be closed subspaces of Y.*

(i) *If $M_1 \cap M_2 \neq \{0\}$ then $d(M_1, M_2) = 0$.*

(ii) *$d(M_1, M_2) > 0$ if and only if $M_1 \cap M_2 = \{0\}$ and $M_1 + M_2$ is closed.*

Proof.

(i) Trivial.

(ii) The 'only if' part is trivial and the 'if' part follows from the closed graph theorem.

We are, of course, interested in the case where M_1 and M_2 are $\mathscr{B}$-cycles in a Banach space X. ($\mathscr{B}$ as in the foregoing section.) We have seen in the last section how M_1 and M_2 define families $(M_1)_k$ and $(M_2)_k$ of subspaces of the spaces X_k in a given integral module representation. The next proposition relates $d(M_1, M_2)$ to the distance between these subspaces.

PROPOSITION 3.3. *Let M, M_1 and M_2 be $\mathscr{B}$-cycles in a Banach space X and $y \in X$.*

(i) $d_k(M_1, M_2) := d((M_1)_k, (M_2)_k)$ *is continuous as a function of* k.

(ii) $d(M_1, M_2) = \inf_\Omega d_k(M_1, M_2)$.

(iii) $d_k(S(y), M)$ *vanishes on a nonvoid clopen subset of Ω if and only if* $S(y) \cap M \neq \{0\}$.

Note. 3.3 (iii) does not necessarily hold for two arbitrary $\mathscr{B}$-cycles.

Proof.

(i) Suppose for some k in Ω, $d_k(M_1, M_2) < a \in \boldsymbol{R}$. Then there is also some $\varepsilon > 0$ such that $d_k(M_1, M_2) < a(1-\varepsilon)$. By the definition of the distance function there are elements $m \in (M_1)_k$, $m' \in (M_2)_k$ with $\max\{\|m\|_k, \|m'\|_k\} = 1$ and $\|m + m'\|_k < a(1-\varepsilon)$. Let $m_1 \in M_1$, $m_2 \in M_2$ be such that $\langle m_1\rangle(k) = m$, $\langle m_2\rangle(k) = m'$. By continuity there is a clopen set D containing k with $\|\langle m_1\rangle(l)\|_l > \|m\|_k - \varepsilon$, $\|\langle m_2\rangle(l)\|_l > \|m'\|_k - \varepsilon$ and $\|\langle m_1\rangle(l) + \langle m_2\rangle(l)\|_l < a(1-\varepsilon)$ for all l in D. But then $a(1-\varepsilon) > \|\langle m_1\rangle(l) + \langle m_2\rangle(l)\|_l \geqq d_l(M_1, M_2)\cdot\mu_1 > (1-\varepsilon)d_l(M_1, M_2)$ whereby $\mu_1 = \max\{\|\langle m_1\rangle(l)\|_l, \|\langle m_2\rangle(l)\|_l\} > 1-\varepsilon$. It follows that $d_l(M_1, M_2) < a$ for l in D.

On the other hand suppose that every neighborhood of k contains a point l where $d_l(M_1, M_2) < a$. By the preceding part of the proof there is a clopen set D_l containing l and elements $m_1 \in M_1$, $m_2 \in M_2$ with $\|\langle m_1\rangle(l') + \langle m_2\rangle(l')\|_{l'} < a \max\{\|\langle m_1\rangle(l')\|_{l'}, \|\langle m_2\rangle(l')\|_{l'}\}$ and $\max\{\|\langle m_1\rangle(l')\|_{l'}, \|\langle m_2\rangle(l')\|_{l'}\} > 1/2$ for l' in D_l. Let D be the clopen set $\{l \mid d_l(M_1, M_2) < a\}^-$. We apply the existence lemma to $A := \{(m_1, m_2; E_C) \mid m_1 \in M_1, m_2 \in M_2, (m_1, m_2; E_C) \in QS(X^2 \mathscr{B})\ \|\langle m_1\rangle(l) + \langle m_2\rangle(l)\|_l \leqq a \max\{\|\langle m_1\rangle(l)\|_l, \|\langle m_2\rangle(l\|)_l\}$ and $\max\{\|\langle m_1\rangle(l)\|_l, \|\langle m_2\rangle(l)\|_l\} \geqq 1/2$ for l in $C\}$ and E_D. The conditions (ii) and (iii) are trivially satisfied and (i) we havejust shown. Thus there is a pair $\tilde{m}_1$, $\tilde{m}_2$ such that these inequalities hold everywhere in D, in particular also at k. But then $a \max\{\|\langle\tilde{m}_1\rangle(k)\|_k, \|\langle\tilde{m}_2\rangle(k)\|_k\} \geqq \|\langle\tilde{m}_1\rangle(k) + \langle\tilde{m}_2\rangle(k)\|_k \geqq d_k(M_1, M_2)\max\{\|\langle\tilde{m}_1\rangle(k)\|_k, \|\langle\tilde{m}_2\rangle(k)\|_k\}$. It follows that $d_k(M_1, M_2) \leqq a$. Thus the component-wise distance function is both upper and lower semicontinuous and therefore continuous.

(ii) Let $d:=\inf_\Omega d_k(M_1, M_2)$. If $m_1 \in M_1$, $m_2 \in M_2$ and $\max\{\|m_1\|, \|m_2\|\} = 1$ then

$$\begin{aligned}\|m_1+m_2\|^p &= \int_\Omega ([m_1+m_2](k))^p d\rho_x = \int_\Omega (\|\langle m_1\rangle(k)+\langle m_2\rangle(k)\|_k)^p d\rho_x \\ &\geqq \int_\Omega (d_k(M_1, M_2))^p (\max\{\|\langle m_1\rangle(k)\|_k, \|\langle m_2\rangle(k)\|_k\})^p d\rho_x \\ &\geqq d^p \int_\Omega (\max\{\|\langle m_1\rangle(k)\|_k, \|\langle m_2\rangle(k)\|_k\})^p d\rho_x \geqq d^p\end{aligned}$$

so that $d(M_1, M_2) \geqq d$.

On the other hand since $d_k(M_1, M_2)$ is continuous and Ω compact, there is a point k in Ω such that $d_k(M_1, M_2) = d$. Let $1 > \varepsilon > 0$. As in (i) we can find elements $m_1 \in M_1$, $m_2 \in M_2$ so that $\max\{\|\langle m_l\rangle(l)\|_1, \|\langle m_2\rangle(l)\|_l\}$ (= say $\|\langle m_1\rangle(l)\|_l) > 1-\varepsilon$ and $\|\langle m_1\rangle(l)+\langle m_2\rangle(l)\|_l < d(1+\varepsilon)$ for l in some clopen set D containg k. But then:

$$\begin{aligned}\frac{d(1+\varepsilon)}{(1-\varepsilon)} \max\{\|E_D m_1\|, \|E_D m_2\|\} &\geqq \frac{d(1+\varepsilon)}{(1-\varepsilon)} \|E_D m_1\| \\ &= \frac{d(1+\varepsilon)}{(1-\varepsilon)} \left(\int_D \|\langle m_1\rangle(l)\|_l^p d\rho_x\right)^{1/p} > \left(\int_D (d(1+\varepsilon))^p d\rho_x\right)^{1/p} \\ &> \left(\int_D \|\langle m_1\rangle(l)+\langle m_2\rangle(l)\|_l^p d\rho_x\right)^{1/p} = \|E_D(m_1+m_2)\| \\ &\geqq d(M_1, M_2) \max\{\|E_D m_1\|, \|E_D m_2\|\}.\end{aligned}$$

So $d(M_1, M_2) < d(1+\varepsilon)/(1-\varepsilon)$ (note $\|E_D m_1\| \neq 0$). Since ε was arbitrary subject to $1 > \varepsilon > 0$, $d(M_1, M_2) = d$.

(iii) It is straightforward to show that the spaces $(S(y))_k$ are one-dimensional at most and equal to $\operatorname{lin}(\langle y\rangle(k))$ where $\langle y\rangle(k)$ is finite and nonzero. Thus, if $d_k(S(y), M) = 0$ on a clopen set $D \neq \phi$ then, by 3.2 (ii) $(S(y))_k \cap M_k \neq \{0\}$ and so $\langle y\rangle(k) \in M_k \cup \{\infty\}$ on D. By 2.5 (ii) $E_D y \in M$. Since the distance between $\{0\}$ and any other subspace is 1, $E_D y \neq 0$. Thus $S(y) \cap M \neq \{0\}$.

On the other hand, if $S(y) \cap M \neq \{0\}$ let z be an element in the intersection with $z \neq 0$ and $1 \leqq \|\langle z\rangle(k)\|_k \leqq 2$ on $\operatorname{supp} z$. Then $\langle z\rangle(k)$ is a nonzero element in $(S(y))_k \cap M_k$ for all k in $\operatorname{supp} z$. By 3.2 (i) $d_k(S(y), M) = 0$ on $\operatorname{supp} z$.

4. The case of finite multiplicity. In this section we consider the case where $m(I) < \infty$. It turns out that in this situation the spaces behave as a sort of continuous analogue to the Bochner spaces $L^p(V; \mu)$ with finite-dimensional V. In particular, X can be split up in a very simple manner in contrast to the situation for arbitrary Boolean algebras of projections with finite multiplicity (cf. the counterexample in [6]).

LEMMA 4.1. *Let $\mathscr{B}$ be a complete Boolean algebra of L^p-projections on X, $M \subset X$ a $\mathscr{B}$-cycle. For each $x \in X$, there is a $z \in X$ with*

$$(M + S(x))^- = M + S(z) .$$

Proof. Let $D: = \{k \mid d_k(M, S(x)) \neq 0\}^-$, a clopen set in Ω. We apply the existence lemma to E_D and

$$A: = \{(z, F) \mid (z, F) \in QS(X, \mathscr{B}) , \quad d(M, S(z)) \geqq 1/4 , \\ F(M + S(z)) = F((M + S(x))^-)\} .$$

That (ii) and (iii) are satisfied is easily verified. Now suppose that $0 \lneqq F = E_C \leqq E_D$. Then there is a point $k \in C$ with $d_k(M, S(x)) \neq 0$. Then $M_k \subsetneqq M_k + \text{lin}(\langle x\rangle(k))$ so that there is a point $y_k \in M_k + \text{lin}(\langle x\rangle(k))$ with $\|y_k\| = 1$ and $d(M_k, \text{lin}(y_k)) > 1/3$. Let $m \in M$ and $a \in \boldsymbol{R}$ such that $y_k = \langle m + ax\rangle(k)$ (note that $a \neq 0$). Then there is, by the continuity of $d.(M, S(m + ax))$ and $[m + ax](\cdot)$ a clopen set $C_0 \subset C$ containing k such that $d_l(M, S(m + ax)) > 1/4$ and $[m + ax](l) > 1/2$ for $l \in C_0$. Let $z_0 = E_{C_0}(m + ax)$. Clearly $(z_0, E_{C_0}) \in QS(X, \mathscr{B})$ and $d(M, S(z_0)) \geqq 1/4$. But $E_{C_0}x = (z_0 - E_{C_0}m)/a \in M + S(z_0)$. It follows that $E_{C_0}(M + S(z_0)) = E_{C_0}((M + S(x))^-)$. Thus (i) also holds. By the existence lemma there is a $(z, E_D) \in A$, i.e., an element $z \in X$ with $E_D(M + S(z)) = E_D((M + S(x))^-)$.

But $(M + S(x))^- = E_D((M + S(X))^-) + M = E_D(M + S(z)) + M = M + S(z)$ since $d_k(M, S(x)) = 0$ outside D implies $(I - E_D)(S(x)) \subset M$.

THEOREM 4.2. *Let $\mathscr{B}$ be a complete Boolean algebra of L^p-projections on the real Banach space X.*

(i) *If Ω satisfies the* CCC *and I has finite multiplicity, say n, then there exist $x_1, \cdots, x_n$ in X such that*

$$X = S(x_1) \oplus \cdots \oplus S(x_n) .$$

For any $x \in X$ with $Ex \neq 0$ for all $E \in \mathscr{B}$, $E \neq 0$, the Banach spaces X_k in the integral module representation of X w.r.t. *x are finite-dimensional, and $n = \max_{k \in \Omega}(\dim X_k)$.*

(ii) *Conversely, if there is an $x \in X$ with $E_x = I$ such that the X_k in the corresponding intgral module representation are finite-dimensional with $n: = \max_{k \in \Omega}(\dim X_k)$ then $m(I)$ is finite and equal to n.*

Proof.

(i) The first part of the proposition follows from 4.1 by induction on n.

Let $y \mapsto \langle y \rangle$ be the integral module representation of X w.r.t. x. Each $y \in X$ can be written as $y_1 + \cdots + y_n$ with $y_i \in S(x_i)$ for each i. Thus $\langle y \rangle = \langle y_1 \rangle + \cdots + \langle y_n \rangle$ so that $X_k = [S(x_1)]_k + \cdots + [S(x_n)]_k$ for k in Ω. It follows that $\max_{k \in \Omega} (\dim X_k) \leqq n$.

However suppose that $\max_{k \in \Omega} (\dim X_k) < n$. This means that for each $k \in \Omega$ there is an x_i with $\langle x_i \rangle(k) = 0$. Let $D_i := \{k \mid \langle x_i \rangle(k) = 0\}$ for $i = 1, \cdots, n$. Then $\bigcup_{i=1}^n D_i = \Omega$ so that $\{k \mid \langle x_n \rangle(k) \neq 0\} \subset \bigcup_{i=1}^{n-1} D_i$. Since this set is open it is contained in $\bigcup_{i=1}^{n-1} C_i$ whereby $C_i := (D_i)^0$ for $i = 1, \cdots, n-1$ (note that Ω is extremally disconnected). Let $y_i := x_i + E_{C_i} x_n$ for these i. Then $(I - E_{C_i}) y_i = x_i$ and $E_{C_i} y_i = E_{C_i} x_n$. It follows that $S(y_1) + \cdots S(y_{n-1}) \supset S(x_1) + \cdots + S(x_n) = X$. This is in contradiction to the fact that X has multiplicity n.

(ii) Suppose that $\dim X_k = m$ for some $k \in \Omega$. Let $\hat{x}_1, \cdots, \hat{x}_m$ be a basis of X_k, $x_1, \cdots, x_m$ elements in X with $\langle x_i \rangle(k) = \hat{x}_i$ for each i and $M := [S(x_1) + \cdots S(x_m)]^-$. Then $M_k = X_k$ and by the continuity of the dimension $M_{k'} = X_{k'}$ for k' in some clopen neighbourhood D of k. Then $E_D X = E_D M$, and E_D has finite multiplicity less than or equal to m. Thus there is a neighbourhood of each point k such that the corresponding projection has multiplicity not greater than n. It follows from 1.2 (ii) that $m(I) \leqq n$ and then the equality follows from (i).

Note. It follows from 4.2 that $m(E_D) = \max_{k \in D} (\dim X_k)$ for each clopen D in Ω and thus that uniform multiplicity n of I implies $\dim X_k = n$ for all k in Ω.

We now investigate the restriction of the multiplicity function to cycles in the case where I has finite multiplicity.

THEOREM 4.3. *Let X be a Banach space, $\mathscr{B}$ a complete Boolean algebra of L^p-projections on X. Suppose that I has finite multiplicity n, that Ω satisfies the* CCC *and that M is a $\mathscr{B}$-cycle. We denote the multiplicity in $\mathscr{B}|_M$ by $\tilde{m}$.*

(i) *$\tilde{m}(E|_M) \leqq m(E)$ for all $E \in \mathscr{B}$.*

(ii) *$M = X$ if and only if $m(E) = \tilde{m}(E|_M)$ for all $E \in \mathscr{B}$. In particular, if I and $I|_M$ have uniform multiplicity and $E_M = I$, $M = X$ if and only if $m(I) = \tilde{m}(I|_M)$.*

(iii) *If $I|_M$ has finite uniform multiplicity $\tilde{k}$, $E_M = I$ and $x_1, \cdots, x_{\tilde{k}}$ are elements of M such that $M = S(x_1) \oplus \cdots \oplus S(x_{\tilde{k}})$, then there are $x_{\tilde{k}+1}, \cdots, x_n$ such that $X = S(x_1) \oplus \cdots \oplus S(x_n)$.*

Proof.

(i) The same proof as 4.2 (i) gives $\tilde{m}(E|_M) = \max_{k \in D} (\dim M_k) \leqq \max_{k \in D} (\dim X_k) = m(E)$ (whereby D is the clopen subset of Ω associated with E).

(ii) $M = X$ if and only if $M_k = X_k$ for all $k \in \Omega$. Since $\dim X_k$ is finite, it suffices that $\dim M_k = \dim X_k$ for each k and, since Ω is totally disconnected and $k \mapsto \dim M_k$, $k \mapsto \dim X_k$ are continuous, that $\sup_{k \in D}(\dim M_k) = \sup_{k \in D}(\dim X_k)$ for all clopen D. This is clearly equivalent to $\tilde{m}(E|_M) = m(E)$ for each $E \in \mathscr{B}$.

(iii) Let $D_{\tilde{k}+1} = \{k \mid \dim X_k \geqq \tilde{k} + 1\}$. Apply the existence lemma to $E_{D_{\tilde{k}+1}}$ and the set $\{(y, E_y) \mid d(S(y), M) \geqq 1/2\}$ to obtain $x_{\tilde{k}+1}$ with $E_{x_{\tilde{k}+1}} = E_{D_{\tilde{k}+1}}$ and $d(S(x_{\tilde{k}+1}), M) \geqq 1/2$. Let $D_{\tilde{k}+2} = \{k \mid \dim X_k \geqq \tilde{k}+2\}$ and apply the existence lemma to $E_{D_{\tilde{k}+2}}$ and $\{(y, E_y) \mid d(S(y), M \oplus S(x_{\tilde{k}+1})) \geqq 1/2\}$ to obtain $X_{\tilde{k}+2}$. Continue until x_n. Since $\dim M_k = \tilde{k}$ for all k and $\operatorname{supp}\langle x_{\tilde{k}+r}\rangle = D_{\tilde{k}+r}$ for $1 \leqq r \leqq n - \tilde{k}$ we have $\dim[S(x_1) \oplus \cdots \oplus S(x_n)]_k = \dim X_k$ for all k and thus $X = S(x_1) \oplus \cdots \oplus S(x_n)$.

We now show that in the case of finite multiplicity, the space X may be arbitrarily well approximated by sums of Bochner spaces $L^p(V; \mu)$.

Let $(S, \sum, \mu)$ be a finite measure space and V a finite-dimensional Banach space with $n := \dim V$. By $\mathscr{B}_\Sigma$ we mean the Boolean algebra on $L^p(V; \mu)$ which consists of the characteristic projections E_D, $f \mapsto f\chi_D$ for $D \in \Sigma$. $\mathscr{B}_\Sigma$ is obviously a σ-complete algebra of L^p-projections. Since (S, Σ, μ) is finite, $I \in \mathscr{B}_\Sigma$ satisfies the CCC and σ-completeness is equivalent to completeness.

LEMMA 4.4. *$I \in \mathscr{B}_\Sigma$ has finite uniform multiplicity and $m(I) = \dim V$.*

Proof. Let $e_1, \cdots, e_n$ be a basis of V, $e_1, \cdots, e_n \in L^p(V; \mu)$ the corresponding constant functions. For $D \in \Sigma$ with $\mu(D) > 0$ it is easy to see that $E_D L^p(V; \mu) (\cong L^p(V; \mu|_D)) = S(\chi_D e_1) + \cdots + S(\chi_D e_n)$ so that $m(E_D) \leqq n$ for all E_D.

On the other hand, suppose that for some $E_D \neq 0$, $S(f_1) \oplus \cdots \oplus S(f_k) = E_D L^p(V; \mu)$ with $k < n$.

Then either (a) the set $\{\omega \in D \mid f(\omega) \notin \operatorname{lin}\{f_1(\omega), \cdots, f_k(\omega)\}\}$ has zero measure for all $f \in E_D L^p(V; \mu)$ or (b) there is an $f \in E_D L^p(V; \mu)$ and a measurable set $C \subset D$, $\mu(C) > 0$ such that $f(\omega) \notin \operatorname{lin}\{f_1(\omega), \cdots, f_k(\omega)\}$ for $\omega \in C$.
(Note that both (a) and (b) are independent of the choice of representatives.)

If (a) holds, let K be the unit ball of V and $v_1, \cdots, v_r$ a finite family of elements in K with $K \subset \bigcup_{i=1}^r B(v_i, 1/4)$. The set $G := \bigcup_{i=1}^r \{\omega \mid \omega \in D,\ v_i(\omega) (= v_i) \notin \operatorname{lin}\{f_1(\omega), \cdots, f_k(\omega)\}\}$ has by (a) zero measure. Thus there is a point $\omega \notin G$. But then $v_i \in \operatorname{lin}\{f_1(\omega), \cdots, f_k(\omega)\}$ for all i and thus $d(v, \operatorname{lin}\{f_1(\omega), \cdots, f_k(\omega)\}) \leqq 1/4$ for all $v \in K$. This of course contradicts the fact that $k < \dim V$.

However if (b) holds, then there is an $\varepsilon > 0$ and $C_\varepsilon \subset C$, $\mu(C_\varepsilon) > 0$ with $d(f(\omega), \operatorname{lin}\{f_1(\omega), \cdots, f_k(\omega)\}) \geqq \varepsilon$ for $\omega \in C_\varepsilon$. But then $\|\chi_\varepsilon f - \sum_{i=1}^k g_i f_i\|^p \geqq \varepsilon^p \mu(C_\varepsilon)$ for all $g_1, \cdots, g_k$ in the Cunningham algebra of $\mathscr{B}_\Sigma$ which contradicts the fact that $E_D L^p(V; \mu) = S(f_1) \oplus \cdots \oplus S(f_k)$.

Thus $m(E_D) = n$ for all $E_D \in \mathscr{B}_\Sigma$, $E_D \neq 0$.

LEMMA 4.5. *Let X, $\mathscr{B}$, Ω etc. satisfy the conditions of Theorem 4.2.*

(i) *For each k in Ω and each $\varepsilon > 0$ there is a clopen neighbourhood D of k such that $\delta(X_k, X_l) \leqq \varepsilon$ for l in D.*

Here $\delta(X_k, X_l)$ denotes the Banach-Mazur distance of X_k from X_l i.e., $\delta(X_k, X_l) := \inf\{\log \|\varphi\| \cdot \|\varphi^{-\wedge}\| \mid \varphi: X_k \to X_l$ *isomorphism*$\}$ *with* $\inf \emptyset := \infty$.

(ii) *For each k in Ω and each $\varepsilon > 0$ there is a clopen neighbourhood D of k such that $\delta(L^p(X_k, (D, \rho_x|_D)), E_D X) \leqq \varepsilon$.*

Proof.

(i) Suppose ε and k are given and choose an $1 > \varepsilon' > 0$ such that $\log(1+\varepsilon')/(1-\varepsilon') < \varepsilon$. Since X and $\mathscr{B}$ satisfy the conditions of Theorem 4.2, X_k has finite dimension say n. Then there is a clopen neighbourhood D_n of k such that X_l has dimension n for l in D_n (2.4 (iii)). Let $y_1^*, \cdots, y_n^*$ be a base of X_k with $\|y_i^*\|_k = 1$ for all i and let $r > 0$ be a number such that $\|\sum_1^n \lambda_i y_i^*\|_k \geqq r \max|\lambda_i|$ for all real numbers $\lambda_1, \cdots, \lambda_n$. (It is in fact possible to choose a base for which r can be put equal to 1.)

Let $y_1, \cdots, y_n$ be elements of X such that $\langle y_i \rangle(k) = y_i^*$ for each i. Since the unit ball of X_k is totally bounded it can be covered by finitely many balls of radius $r\varepsilon'/4n$, say those with centers $z_1, \cdots, z_m$. Finally, if $\langle y_i \rangle(l) \neq \infty$, let $\varphi_l: X_k \to X_l$ be the linear map which maps y_i^* into $\langle y_i \rangle(l)$ for each i. Due to the continuity of the norm resolution $\|\varphi_l(z)\|_l$ is continuous as a function of l for each z in X_k. Thus there is a clopen set D containing k and contained in D_n such that:

(a) $\|z_i\|_k - \|\varphi_l(z_i)\|_l < \varepsilon'/4 \quad i = 1, \cdots, m$,

(b) $1 - \varepsilon' < \|\langle y_i \rangle(l)\|_l < 1 + \varepsilon' \quad i = 1, \cdots, n$,

for all l in D.

But then for each $z \in X_k$, $\|z\|_k = 1$ there is a z_i with $\|z - z_i\|_k < r\varepsilon'/4n z - z_i = \sum_1^n \lambda_j y_j^*$ for suitable λ_j's in $\boldsymbol{R}$ and from $\|z - z_i\|_k < r\varepsilon'/4n$ it follows that $\lambda_j < \varepsilon'/4n$ for all j. Thus

$$\big|\|\varphi_l(z)\|_l - \|z\|_k\big| \leqq \big|\|\varphi_l(z)\|_l - \|\varphi_l(z_i)\|_l\big| + \big|\|\varphi_l(z_i)\|_l - \|z_i\|_k\big| + \big|\|z_i\|_k - \|z\|_k\big|$$

$$< ||\varphi_l(z-z_i)||_l + ||z-z_i||_k + (\varepsilon'/4) < ||\sum_A^n \lambda_j \langle y_j \rangle (l)||_l + (r\varepsilon'/4n) + (\varepsilon'/4)$$
$$< n(\varepsilon'/4n)(1+\varepsilon') + (r\varepsilon'/4n) + (\varepsilon'/4) < \varepsilon' .$$

In particular, since $\varepsilon' < 1$ each φ_l is an isomorphism and

$$||\varphi_l|| \leqq 1 + \varepsilon' , \quad ||\varphi_l^{-1}|| \leqq 1/(1-\varepsilon') .$$

But then

$$\delta(X_k, X_l) \leqq \log ||\varphi_l|| \cdot ||\varphi_l^{-1}|| \leqq \log \frac{1+\varepsilon'}{1-\varepsilon'} < \varepsilon .$$

(ii) Suppose ε and k are given and that ε' is a positive number which we shall determine later. As in (i) we can determine a clopen set D containing k such that $\delta(X_k, X_l) \leqq \varepsilon'$ for l in D. With the notation of (i) let $\bar{y}_i$ be the elements of $L^p(X_k, (D, \rho_x|_D))$ which have the constant value y_i^* on D. Consider the map φ which maps every element of the form $\sum_A^n f_i y_i$ in $E_D X$ into $\sum_A^n f_i \bar{y}_i$ in $L^p(X_k, (D, \rho_x|_D))$ where the f_i's are continuous functions on Ω which vanish outside D. Then φ is clearly well-defined and linear and its domain and range are dense subsets of $E_D X$ and $L^p(X_k, (D, \rho_x|_D))$ respectively. Since

$$\Big|\Big| \sum_A^n f_i y_i \Big|\Big|^p = \int_D \Big|\Big| \sum_A^n f_i(l) \langle y_i \rangle (l) \Big|\Big|_l^p d\rho_x$$

and also

$$\Big|\Big| \sum_A^n f_i \bar{y}_i \Big|\Big|^p = \int_D \Big|\Big| \sum_A^n f_i(l) y_i^* \Big|\Big|_k^p d\rho_x$$

we have

$$\Big| \Big|\Big| \sum_A^n f_i y_i \Big|\Big|^p - \Big|\Big| \sum_A^n f_i \bar{y}_i \Big|\Big|^p \Big| \leqq \int_D \Big| \Big|\Big| \sum_A^n f_i(l) \langle y_i \rangle (l) \Big|\Big|_l^p - \Big|\Big| \sum_A^n f_i(l) y_i^* \Big|\Big|_k^p \Big| d\rho_x .$$

But

$$\sum_A^n f_i(l) \langle y_i \rangle (l) = \varphi_l \Big(\sum_A^n f_i(l) y_i^* \Big) .$$

It follows that

$$\Big| \Big|\Big| \sum_A^n f_i(l) \langle y_i \rangle (l) \Big|\Big|_l - \Big|\Big| \sum_A^n f_i(l) y_i^* \Big|\Big|_k \Big| \leqq \varepsilon' \Big|\Big| \sum_A^n f_i(l) y_i^* \Big|\Big|_k$$

and thus that

$$\left| \left\| \sum_A^n f_i(l)\langle y_i\rangle(l) \right\|_l^p - \left\| \sum_A^n f_i(l) y_i^* \right\|_k^p \right| \leqq p\varepsilon'(1+\varepsilon')^{p-1} \cdot \left\| \sum_A^n f_i(l) y_i^* \right\|_k^p .$$

Thus

$$\left| \left\| \sum_A^n f_i y_i \right\|^p - \left\| \sum_A^n f_i \bar{y}_i \right\|^p \right| \leqq p\varepsilon'(1+\varepsilon')^{p-1} \left\| \sum_A^n f_i \bar{y}_i \right\|^p$$

and

$$\left| \left\| \sum_A^n f_i y_i \right\| - \left\| \sum_A^n f_i \bar{y}_i \right\| \right| \leqq \frac{\varepsilon'(1+\varepsilon')^{p-1}}{(1-\varepsilon'(1+\varepsilon')^{p-1})} \left\| \sum_A^n f_i \bar{y}_i \right\| .$$

For sufficiently small ε' we thus have that φ is $1-1$ and

$$\log \|\varphi\| \, \|\varphi^{-1}\| \leqq \varepsilon .$$

We can thus extend φ to an isomorphism between $E_D X$ and $L^p(X_k, (D, \rho_x|_D))$ for which this inequality also holds.

We are now in a position to prove the promised result, that it can be arbitrarily well approximated by sums of Bochner spaces. The reader will note that we actually prove rather more than is contained in the statement of the theorem, we have however chosen this formulation for the sake of clarity.

THEOREM 4.6. *Let X be a Banach space and a complete Boolean algebra of L^p-projections on X, satisfying the* CCC, *with $m(I)=n$, finite. Then, for each $\varepsilon>0$, there is a finite family of finite measure spaces $(S_j, \sum_j, \mu_j)_{1\leqq j\leqq r}$ and a family $(V_j)_{1\leqq j\leqq r}$ of Banach spaces of dimension less than or equal to n, such that, for $Y:= L^p(V_1, \mu_1) \oplus_p L^p(V_2, \mu_2) \oplus_p, \cdots, \oplus_p L^p(V_r, \mu_r)$ $\delta(Y, X) \leqq \varepsilon$.*

Proof. By Lemma 4.5 we can find for each point k a clopen neighbourhood D such that $(L^p(X_k, (D, \rho_x|_D)), E_D X) \leqq \varepsilon$. Since Ω is compact there is a finite set $k_1, \cdots, k_r$ such that the corresponding D_j's cover Ω, Wlog we may assume that the D_j's are disjoint. Set $(S_j, \sum_j, \mu_j) = (D_j$, Borel sets on $D_j, \rho_w|_{D_j})$ and $V_j = X_{k_j}$ for each j. $\dim V_j = \dim X_{k_j} \leqq n$. Since $X = E_{D_1}X \oplus_p E_{D_2}X \oplus_p, \cdots, \oplus_p E_{D_r}X$ it follows immediately that $\delta(Y, X) \leqq \varepsilon$ by composing the individual isomorphisms.

5. The restriction of the multiplicity function. Let M be a fixed $\mathscr{B}$-cycle in X. $\mathscr{B}$ induces a complete Boolean algebra of L^p-projections $\tilde{\mathscr{B}} := \mathscr{B}|_M$ on M, and it is natural to ask whether it is possible to compare the respective multiplicity functions. If we denote by $\tilde{m}$ the "restricted" multiplicity function, one would expect that $\tilde{m}(E|_M) \leqq m(E)$ for $E \in \mathscr{B}$. It turns out that this is true for our situation, but there seems to be no result in this direction for the general case of Boolean algebras of projections. To avoid notational complications we will restrict ourselves to the case where $\mathscr{B}$ satisfies the CCC (note that this is then also true for $\tilde{\mathscr{B}}$ since the representation space of $\tilde{\mathscr{B}}$ is a subspace of Ω).

The following problem crops up: Is it possible given a fixed family of cycles generated by elements of X to find the same number of elements in M such that the cycles generated by these elements approximate the same elements in M? The case of finite multiplicity is already settled (cf. § 4) whereas the general case needs a certain amount of technical preliminaries.

LEMMA 5.1.

(1) *Let x be an element of X such that $E_x = I$, $S(x) \cap M = 0$, $\|x\| = 1$. Then there is an $m \in M$ such that $[x-m](k) \leqq 3d_k(S(x), M)$ for $k \in \Omega$. If m' is any element of M such that $\|E_D x - m'\| \leqq \varepsilon$ (where $\varepsilon \geqq 0$, $D \subset \Omega$ clopen), then we have $\|E_D m - m'\| \leqq 4\varepsilon$.*

(2) *For every $x \in X$ there is an $m \in M$ such that $\|Ex - m'\| \leqq \varepsilon (E \in \mathscr{B}, \varepsilon \geqq 0, m' \in M)$ implies $\|Em - m'\| \leqq 4\varepsilon$.*

Note. In (1) $d_k(S(x), M)$ is taken in the integral module representation with respect to this x.

Proof.

(1) We shall work in the integral module representation of X with respect to x. Let Ω' be the open dense set $\{k \mid d_k(S(x), M) > 0\}$. For $k \in \Omega'$, select an m_k in M with $[m_k - x](k) \leqq 2d_k(S(x), M)$. By continuity there is a clopen neighbourhood D_k of k on which we have $[m_k - x](l) \leqq 3d_l(S(x), M)$ for l in D_k. By applying the existence lemma we obtain an element m in M for which $[m - x](k) \leqq 3d_k(S(x), M)$ for all k in $(\Omega)^- = \Omega$. $[E_D x - m'] \leqq \varepsilon$ implies $\varepsilon^p \geqq \int_\Omega [E_D x - m']^p(k) d\rho_x \geqq \int_D [x - m']^p(k) d\rho_x$. On the other hand $[x - m'](k) \geqq d_k(S(x), M)[x](k) = d_k(S(x), M)$ (all $k \in \Omega$) so that

$$\begin{aligned} \|E_D m - E_D x\|^p &= \int_\Omega \chi_D [m - x]^p(k) d\rho_x = \int_D [m - x]^p(k) d\rho_x \\ &\leqq \int_D 3^p (d_k(S(x), M))^p d\rho_x \\ &\leqq 3^p \varepsilon^p \int_D [x - m]^p(k) d\rho_x \leqq 3^p \varepsilon^p . \end{aligned}$$

But this implies $\|E_D m - m'\| \leqq \|E_D m - E_D x\| + \|E_D x - m'\| \leqq 4\varepsilon$.

(2) If E_x means the carrier projection of x, E_M the projection in $\mathscr{B}$ with $S(E_M x) \cap M = 0$, $S((I - E_M)x) \subset M$ (cf. [2], 4.7) we have, with $E_0 := E_x \circ E_M$, $S(E_0 x) \cap E_0 M = 0$, $S((I - E_0)x) \subset M$, $E_{E_0 x} = E_0$. We apply (1) to the Banach space $E_0 X$, the Boolean algebra $\mathscr{B}|_{E_0 X}$, the $\mathscr{B}|_{E_0 X}$-cycle $E_0 M$, and the distinguished element $E_0 x / \|E_0 x\|$ (if $E_0 x = 0$, we have $S(x) \subset M$ so that we may choose $x = m$). We get an $m_0 \in M$ for which the conclusion of (1) is valid. Define $m := m_0 \|E_0 x\| + (I - E_0)x \in M$. Suppose $m' \in M$, $\varepsilon \geqq 0$, $E \in \mathscr{B}$ are given with $\|Ex - m'\| \leqq \varepsilon$. It follows that $\varepsilon^p \geqq \|EE_0 x - E_0 m'\|^p + \|(I - E_0)Ex - (I - E_0)m'\|^p$. By (1), we have $\|Em_0 - E_0(m'/\|E_0 x\|)\| \leqq 4 \|E(E_0 x/\|E_0 x\|) - E_0(m'/\|E_0 x\|)\|$, i.e., $\|E(\|E_0 x\| m_0) - E_0 m'\| \leqq 4 \|EE_0 x - E_0 m'\|$.
It follows that

$$
\begin{aligned}
\|Em - m'\|^p &= \|E_0 Em - E_0 m'\|^p + \|(I - E_0)Em - (I - E_0)m'\|^p \\
&= \|E(\|E_0 x\| m_0) - E_0 m'\|^p + \|(I - E_0)Ex - (I - E_0)m'\|^p \\
&\leqq 4^p \|EE_0 x - E_0 m'\|^p + \|(I - E_0)(Ex - m)\|^p \\
&\leqq 4^p [\|E_0(Ex - m')\|^p + \|(I - E_0)(Ex - m')\|^p] \\
&\leqq 4^p \|Ex - m'\|^p \leqq 4^p \varepsilon^p .
\end{aligned}
$$

LEMMA 5.2. *Let $(x_i)_{i \in I}$ be an infinite family of elements in X. There is a family $(m_j)_{j \in J}$ in M such that* $\mathrm{Cz}\, I = \mathrm{Cz}\, J$, $(\mathrm{lin} \bigcup_{i \in I} S(x_i))^- \cap M \subset (\mathrm{lin} \bigcup_{j \in J} S(m_j))^-$.

Proof. Let J be the set of all $(i_1, \cdots, i_r; a_1, \cdots, a_r)$ where $i_1, \cdots, i_r$ is a finite subset of I and $a_1, \cdots, a_r$ are rational numbers. It is clear that $\mathrm{Cz}\, I = \mathrm{Cz}\, J$. For $j = (i_1, \cdots, i_r; a_1, \cdots, a_r)$, let m_j be that element in M which is associated with $x = a_1 x_{i_1} + \cdots + a_r x_{i_r}$ as in Lemma 5.1 (2).

Let m' be an element of $(\mathrm{lin} \bigcup_{i \in I} S(x_i))^- \cap M$, $\varepsilon > 0$. It is possible, by the definition of $S(x_i)$, to find elements $x_{i_1}, \cdots, x_{i_r}$, operators $\sum_{\lambda=1}^{n_\rho} a_\lambda^\rho E_\lambda^\rho (\rho = 1, \cdots, r)$ such that the a_λ^ρ are rational and $\|\sum_{\rho=1}^{r} \sum_{\lambda=1}^{n_\rho} a_\lambda^\rho E_\lambda^\rho x_{i_\rho} - m'\| \leqq \varepsilon$. An easy computation shows that we may write $\sum_{\rho=1}^{r} \sum_{\lambda=1}^{n_\rho} a_\lambda^\rho E_\lambda^\rho x_{i_\rho}$ as $\sum_{\lambda=1}^{l} \hat{E}_\lambda (b_1^\lambda x_{i_1} + \cdots + b_r^\lambda x_{i_r})$ whereby the family $(\hat{E}_\lambda)_{\lambda=1,\dots,l}$ is a disjoint partition of I and the b_ρ^λ are rational numbers. With $j_\lambda := (i_1, \cdots, i_r; b_1^\lambda, \cdots, b_r^\lambda)$, $m_\lambda := m_{j_\lambda}$, $\varepsilon_\lambda := \|\hat{E}_\lambda (b_1^\lambda x_{i_1} + \cdots + b_r^\lambda x_{i_r}) - \hat{E}_\lambda m'\|$ it follows that $\|\hat{E}_\lambda m_\lambda - \hat{E}_\lambda m'\| \leqq 4\varepsilon$ and therefore that $\|\sum_{\lambda=1}^{l} \hat{E}_\lambda m_\lambda - m'\|^p = \sum_{\lambda=1}^{l} \|\hat{E}_\lambda m_\lambda - \hat{E}_\lambda m'\|^p \leqq 4^p \sum_{\lambda=1}^{l} \varepsilon_\lambda^p = 4^p \sum_{\lambda=1}^{l} \hat{E}_\lambda (b_1^\lambda x_{i_1} + \cdots + b_r^\lambda x_{i_r}) - m'\|^p \leqq 4^p \varepsilon^p$. Thus we have $m' \in (\mathrm{lin} \bigcup_{j \in J} S(m_j))^-$.

THEOREM 5.3. $\tilde{m}(E|_M) \leqq m(E)$ *for all* $E \in \mathscr{B}$.

Proof. By the usual techniques we can restrict ourselves to the case when $E = I$ and Ω satisfies the CCC. But then 5.3 is a consequence of 4.2 and 5.2 in the case of $m(E)$ finite and infinite, respectively.

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