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ON CURVATURE OPERATORS OF BOUNDED RANK

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A curvature operator, that is, a linear map $R: \Lambda^2 V \to \Lambda^2 V$, has bounded rank $2r$ if it maps simple bivectors into bivectors of rank $\leqq 2r$. It is shown here that this condition is equivalent to the following:

$$\sum R(x_{i_1} \wedge y_1) \wedge \cdots \wedge R(x_{i_{r+1}} \wedge y_{r+1}) = 0$$

for all $x_1, \cdots, x_{r+1}, y_1, \cdots, y_{r+1}$ in V, with the sum taken over all permutations $(i_1, \cdots, i_{r+1})$ of $(1, 2, 3, \cdots, r+1)$. An application to Riemannian geometry is given.

1. Introduction.

The Riemann curvature tensor has been studied in many different algebraic contexts. In particular, it can be formulated as a linear map $R: \Lambda^2 V \to \Lambda^2 V$, called the curvature operator, where V is a real n-dimensional vector space and $\Lambda^2 V$ is its associated space of bivectors.

The concept of bivector rank is reviewed in § 2. Our main result appears as Theorem 3.4 in § 3. The application to Riemannian geometry is given in § 4. The reader is referred to [1] and [2] for background material in exterior algebra.

The author wishes to thank Professor Marvin Marcus for supplying an elegant proof for Theorem 3.4.

2. The rank of a bivector.

The bivector space $\Lambda^2 V$ is isomorphic to the space $o(V)$ of linear maps $V \to V$ which are skew-symmetric with respect to any fixed inner product on V. Namely, choose a basis $e_1, \cdots, e_n$ of V. Then for arbitrary $\alpha \in \Lambda^2 V$ we have $\alpha = \Sigma a^{ij} e_i \wedge e_j$, where the sum is taken either over $1 \leqq i < j \leqq n$, or over $i, j = 1, \cdots, n$ with the understanding that $a^{ji} = -a^{ij}$ (and the a^{ij} are divided by 2). The linear map $A: V \to V$ defined by $Ae_i = \Sigma a^{ij} e_j$ is skew-symmetric with respect to any inner product for which the basis $e_1, \cdots, e_n$ is orthonormal. It is easy to check that if a different basis is chosen, the range of A still stays the same; hence, $U_\alpha = A(V)$ is a uniquely defined subspace of V associated to α. The rank of α is simply the rank of such a corresponding linear map $A \in o(V)$, i.e., rank $(\alpha) = \dim U_\alpha$.

Note rank $(\alpha) = 0$ means $\alpha = 0$. Bivectors of minimal nonzero rank, that is, of rank 2, are called simple or decomposable.

We shall need some equivalent definitions of the rank of α, expressed in the context of $\Lambda^2 V$ rather than $o(V)$. These facts are summarized as follows.

PROPOSITION 2.1. *Let $\alpha \in \Lambda^2 V$, $\alpha \neq 0$.*

(a) *Rank* $(\alpha) = 2r$ *if and only if there exist independent vectors* $x_1, \cdots, x_{2r}$ *such that*

$$\alpha = x_1 \wedge x_2 + \cdots + x_{2r-1} \wedge x_{2r} .$$

(b) *Rank* $(\alpha) = 2r$ *if and only if* $\alpha^r \neq 0$ *and* $\alpha^{r+1} = 0$.

(c) *The rank of* α *is the smallest dimension of any subspace* $U \subset V$ *such that* $\Lambda^2 U$ *contains* α.

(d) *The rank of* α *is twice the smallest number of terms in any expression of* α *as a sum of simple bivectors.*

Proof.

(a) Write $\alpha = \Sigma a^{ij} e_i \wedge e_j$, with the sum taken over $1 \leqq i < j \leqq n$. Since $\alpha \neq 0$ by hypothesis, some a^{ij} must be nonzero; hence the basis vectors e_i can be relabeled to obtain $a^{12} \neq 0$. Set

$$x_1 = a^{12} e_i - \sum_{3 \leqq j} a^{2j} e_j , \quad x_2 = e_2 + \sum_{3 \leqq i} \frac{a^{1i}}{a^{12}} e_i .$$

Then the expression $\alpha = \sum a^{ij} e_i \wedge e_j$ can be rewritten as

$$\begin{aligned} \alpha &= x_1 \wedge x_2 + \sum_{3 \leqq i < j} a^{ij} e_i \wedge e_j - \sum_{3 \leqq i, j} \frac{a^{1i} a^{2j}}{a^{12}} e_i \wedge e_j \\ &= x_1 \wedge x_2 + \sum_{3 \leqq i < j} \frac{1}{a^{12}} (a^{12} a^{ij} - a^{1i} a^{2j} + a^{1j} a^{2i}) e_i \wedge e_j \\ &= x_1 \wedge x_2 + \sum_{3 \leqq i < j} \frac{1}{a^{12}} (\alpha \wedge \alpha)^{12ij} e_i \wedge e_j \\ &= x_1 \wedge x_2 + \alpha_1 . \end{aligned}$$

Note that $x_1, x_2, e_3, \cdots, e_n$ are linearly independent and that $\alpha_1 \in \Lambda^2\{e_3, \cdots, e_n\}$ (brackets $\{\cdots\}$ denote span).

Now an induction can be performed. If $\alpha_1 = 0$, we are done. If $\alpha_1 \neq 0$, relabel the e_i for $3 \leqq i$ to make $a_1^{34} \neq 0$. The above procedure is then repreated on α_1 to get

$$\begin{aligned} \alpha_1 &= x_3 \wedge x_4 + \sum_{5 \leqq i < j} \frac{1}{a^{34}} (\alpha_1 \wedge \alpha_1)^{34ij} e_i \wedge e_j \\ &= x_3 \wedge x_4 + \alpha_2 . \end{aligned}$$

Thus $\alpha = x_1 \wedge x_2 + x_3 \wedge x_4 + \alpha_2$, with $x_1, \cdots, x_4, e_5, \cdots, e_n$ linearly independent, and $\alpha_2 \in \Lambda^2\{e_5, \cdots, e_n\}$. Eventually, one of the α_k's is zero, since we run out of e_i's to operate on. Hence $\alpha = x_1 \wedge x_2 + \cdots + x_{2r-1} \wedge x_{2r}$, for some $2r$. Since the vectors $x_1, \cdots, x_{2r}$ are independent, $2r \leqq n$.

Note that $\alpha \in \Lambda^2\{x_1, \cdots, x_{2r}\}$. Moreover, if we extend $x_1, \cdots, x_{2r}$

to a basis of V, then in this basis the coordinates of α are given by $a^{12} = a^{34} = a^{56} = \cdots = 1$, $a^{21} = a^{43} = a^{65} = \cdots = -1$, all other $a^{ij} = 0$. Hence for this basis the vectors Ae_i are given by $Ae_{2k-1} = x_{2k}$, $Ae_{2k} = -x_{2k-1}$. It follows that $U_\alpha = \{x_2, -x_1, x_4, -x_3, \cdots, x_{2r}, -x_{2r-1}\} = \{x_1, \cdots, x_{2r}\}$, and therefore $\operatorname{rank}(\alpha) = 2r$. This proves (a).

(b) The power α^r stands for the exterior product $\alpha \wedge \cdots \wedge \alpha$ where α occurs r times. Let us substitute the "canonical" expansion given in part (a), $\alpha = x_1 \wedge x_2 + \cdots + x_{2r-1} \wedge x_{2r}$, into this product; notice that it has exactly r summands. Since $x \wedge x = 0$, the nonzero terms of the product α^r are obtained by choosing a different summand $x_{2i-1} \wedge x_{2i}$ from each α and multiplying these together. Since the exterior product of bivectors is commutative, each of these terms equals $x_1 \wedge x_2 \wedge \cdots \wedge x_{2r-1} \wedge x_{2r}$. Now there are $r!$ of these terms, since a typical term can be built up in $r!$ different ways. Therefore $\alpha^r = r!\,(x_1 \wedge \cdots \wedge x_{2r})$. Since the x_i are independent, we see that $\alpha^r \neq 0$; and also $\alpha^{r+1} = 0$, since each term of the product $\alpha^r \wedge \alpha$ contains a factor of form $x \wedge x$.

On the other hand, suppose $\alpha^s \neq 0$, $\alpha^{s+1} = 0$, and let $\operatorname{rank} \alpha = 2r$. Then the above argument gives $\alpha^r \neq 0$, $\alpha^{r+1} = 0$. If $s < r$, then $s + 1 \leqq r$, so $\alpha^{s+1} = 0$ contradicts $\alpha^r \neq 0$; and if $r < s$, then $r + 1 \leqq s$, so $\alpha^{r+1} = 0$ contradicts $\alpha^s = 0$. Therefore only $s = r$ is possible. This proves (b).

(c) Let s be the smallest dimension of any subspace $U \subset V$ such that $\alpha \in \Lambda^2 U$. Let $e_1, \cdots, e_k$ be a basis of U_α such that $e_1, \cdots, e_n$ is a basis of V. Hence each $Ae_i = \sum a^{ij} e_j$ is a linear combination of $e_1, \cdots, e_k$ only, so that no nonzero term with e_j, $j > k$, appears in these sums. Since $a^{ji} = -a^{ij}$, this means that the coefficients a^{ij} which involve $i, j > k$ must all vanish. Therefore the expression $\alpha = \sum a^{ij} e_i \wedge e_j$ reduces to a sum over $i, j = 1, \cdots, k$, whence $\alpha \in \Lambda^2 U_\alpha$. This implies $s \leqq k$.

Conversely, by definition of s, there is a basis $b_1, \cdots, b_2, b_{s+1}, \cdots, b_n$ of V such that $\alpha = \sum x^{ij} b_i \wedge b_j$, summed over $1 \leqq i < j \leqq s$. Taking this as a sum over all $i, j = 1, \cdots, n$, we see that $x^{ij} = 0$ for $i, j > s$. Hence for this basis we have $Ae_i = \sum x^{ij} b_j$, summed over $1 \leqq j \leqq s$, which implies that $U_\alpha \subset \{b_1, \cdots, b_s\}$. Therefore $k \leqq s$, and thus $k = s$. This proves (c).

(d) By (a), a simple bivector is of form $x_1 \wedge x_2$. The required statement follows directly from (a) and (c).

COROLLARY 2.2.

(a) *A bivector α is simple if and only if $\alpha \wedge \alpha = 0$.*

(b) *If $\alpha = \sum a^{ij} y_i \wedge y_j$, $i, j \leqq p$, then $\operatorname{rank}(\alpha) \leqq p$; and if the y_i are linearly dependent, then $\operatorname{rank}(\alpha) < 0$.*

(c) *If $\alpha = y_1 \wedge y_2 + \cdots + y_{2r-1} \wedge y_{2r}$, then $\operatorname{rank}(\alpha)$ is $= 2r$ or* *is* $<2r$ *as the* y_i *are linearly independent or dependent.*

Proof. Part (a) is clear. For (b), note $\alpha \in \Lambda^2 U_y$ where $U_y = \{y_1, \cdots, y_p\}$. Now rank $(\alpha) \leqq \dim U_y$ by Proposition 2.1 (c). If the y_i are dependent, $\dim U_y < p$. Hence rank $(\alpha) < p$. For (c) note that if y_i are dependent, then rank $(\alpha) < 2r$ by (b). On the other hand, if the y_i are independent, then rank $(\alpha) = 2r$ by Proposition 2.1 (a).

3. Curvature operators of bounded rank. The space $\Lambda^2 V$ is a disjoint union of the subsets of bivectors of the different possible ranks $2, 4, \cdots, 2[n/2]$. We wish to consider how a curvature operator $R: \Lambda^2 V \to \Lambda^2 V$ maps the simple bivectors.

The image of a simple bivector is a bivector having a certain rank. At worst, this rank is $2[n/2] = n - 1$ or n (as n is odd or even), but it could be a smaller number. Let us say that a curvature operator R has *bounded rank* $2r$ if the image of each simple bivector has rank $\leqq 2r$. This means that the range $R(\Lambda^2 V)$ is contained in the union of the sets of vectors of ranks $2, 4, \cdots, 2r$. Our purpose here is to give a characterization for curvature operators R of bounded rank $2r$.

Curvature operators of bounded rank 2 are those that map simple bivectors into simple bivectors, or in other words, preserve decomposability; they were studied in [3] and [4]. We first state some results concerning this special case.

PROPOSITION 3.1. *If a curvature operator R has bounded rank 2, then it maps bivectors of rank $2r$ into bivectors of rank $\leqq 2r$, for all r.*

Proof. Consider a bivector α of rank $2r$. By Proposition 2.1 (a) it can be written as $\alpha = x_1 \wedge x_2 + \cdots + x_{2r-1} \wedge x_{2r}$. Since R is linear, $R\alpha = R(x_1 \wedge x_2) + \cdots + R(x_{2r-1} \wedge x_{2r})$. But each of these terms is a simple bivector; hence $R\alpha = y_1 \wedge y_2 + \cdots + y_{2r-1} \wedge y_{2r}$ for suitable $y_1, \cdots, y_{2r} \in V$. Now Corollary 2.2 implies that rank $(R\alpha) \leqq 2r$.

THEOREM 3.2. [4, Prop. 3.1]. *A curvature operator R has bounded rank 2 if and only if $R(x_1 \wedge x_2) \wedge R(x_3 \wedge x_4) + R(x_1 \wedge x_3) \wedge R(x_2 \wedge x_4) = 0$ for all $x_1, x_2, x_3, x_4 \in V$.*

THEOREM 3.3. [3, Thm. 1]. *Let V have an inner product, suppose the curvature operator R is symmetric in the induced inner product on $\Lambda^2 V$ and is nonsingular, and let $n \geqq 5$. Then R has* *bounded rank* 2 *if and only if* $R = \pm\Lambda^2 L$ *for some linear map* L: $V \to V$.

Now we return to the general case and state our main theorem, which is a generalization of Theorem 3.2. Let S_r denote the symmetric group on r objects.

THEOREM 3.4. *A curvature operator R has bounded rank* $2r$ *if and only if*

$$(1) \qquad \sum_{\sigma \in S_{r+1}} R(x_{\sigma(1)} \wedge y_1) \wedge \cdots \wedge R(x_{\sigma(r+1)} \wedge y_{r+1}) = 0 ,$$

for all $x_1, \cdots, x_{r+1}, y_1, \cdots, y_{r+1} \in V$.

Proof. [Marvin Marcus]. By definition, R has bounded rank $2r$ if and only if $R(x \wedge y)$ has rank $\leqq 2r$ for every $x, y \in V$. By Proposition 2.1 (b), this occurs if and only if $(R(x \wedge y))^{r+1} = 0$ for all $x, y \in V$. But this in turn occurs if and only if

$$(2) \qquad \left[R\left(\sum_1^{r+1} \lambda_i x_i\right) \wedge \left(\sum_1^{r+1} \mu_j y_j\right)\right]^{r+1} = 0$$

for all $x_1, \cdots, x_{r+1}, y_1, \cdots, y_{r+1} \in V$ and all real $\lambda_1, \cdots, \lambda_{r+1}, \mu_1, \cdots, \mu_{r+1}$.

The left side of (1) can be considered as an $\Lambda^{2(r+1)}V$-valued polynomial in the indeterminates $\lambda_1, \cdots, \lambda_{r+1}, \mu_1, \cdots, \mu_{r+1}$. Upon expanding and collecting terms, we find that the coefficient of $\lambda_1 \cdots \lambda_{r+1} \mu_1 \cdots \lambda_{r+1}$ is precisely the left side of equation (1). But if a polynomial is identically zero, then all its coefficients must vanish. Therefore $R(x \wedge y)^{r+1} = 0$ for all $x, y \in V$ implies (1).

On the other hand, if (1) holds for all $x_1, \cdots, x_{r+1}, y_1, \cdots, y_{r+1}$, then we can put $x_1 = \cdots = x_{r+1} = x$ and $y_1 = \cdots = y_{r+1} = y$, to get $R(x \wedge y)^{r+1} = 0$ for all $x, y \in V$.

Theorem 3.4 can be restated in terms of a basis $e_1, \cdots, e_n$ of V. Let $R(e_i \wedge e_j) = R_{ij}$. Then

$$R(x \wedge y) = \sum_{i,j} x^i y^j R(e_i \wedge e_j) = \sum_{i,j} x^i y^j R_{ij} ,$$

since both R and the exterior product are linear in their arguments. Note that the R_{ij} are the columns of the matrix of R in terms of basis $e_i \wedge e_j$, $i < j$, of $\Lambda^2 V$.

THEOREM 3.5. *A curvature operator R has bounded rank* $2r$ *if and only if*

$$\sum_{\sigma \in S_{r+1}} R_{i_{\sigma(1)} j_1} \wedge \cdots \wedge R_{i_{\sigma(r+1)} j_{r+1}} = 0 ,$$

for all $1 \leqq i_\nu, j_\nu \leqq n$.

Proof.

$$\sum_{\sigma \in S_{r+1}} R(x_{\sigma(1)} \wedge y_1) \wedge \cdots \wedge R(x_{\sigma(r+1)} \wedge y_{r+1})$$
$$= \sum (R_{i_{\sigma(1)}j_1} \wedge \cdots \wedge R_{i_{\sigma(r+1)}j_{r+1}}) x_{\sigma(1)}^{i_{\sigma(1)}} \cdots x_{\sigma(r+1)}^{i_{\sigma(r+1)}} y_1^{j_1} \cdots y_{r+1}^{j_{r+1}} .$$

Now $x_{\sigma(r+1)}^{i_{\sigma(r+1)}} \cdots x_{\sigma(1)}^{i_{\sigma(1)}} = x_1^{i_1} \cdots x_{r+1}^{i_{r+1}}$. Hence this sum can be rewritten as

$$\sum_{\substack{i_1, \cdots, i_r \\ j_1 \cdots j_r}} (R_{i_{\sigma(1)}j_1} \wedge \cdots \wedge R_{i_{\sigma(r+1)}j_{r+1}}) x_1^{i_1} \cdots x_{r+1}^{i_{r+1}} y_1^{j_1} \cdots y_{r+1}^{j_{r+1}} .$$

Now this sum is zero for all $x_\nu^{i_\nu}$, $y_\nu^{j_\nu}$ if and only if the coefficients $\sum_\sigma R_{i_{\sigma(1)}j_1} \wedge \cdots \wedge R_{i_{\sigma(r+1)}j_{r+1}}$ are identically zero.

COROLLARY 3.6. *A curvature operator R has an image bivector of rank $> 2r$ if and only if there exist integers $1 \leqq i_1, \cdots, i_{r+1}$, $j_1, \cdots, j_{r+1} \leqq n$ such that*

$$\sum_{\sigma S_{r+1}} R_{i_{\sigma(1)}j_1} \wedge \cdots \wedge R_{i_{\sigma(r+1)}j_{r+1}} \neq 0 .$$

4. An application. Let M^n be an n-dimensional Riemannian manifold and let V denote the tangent space at any point p of M^n. If M^n admits local isometric embedding of a neighborhood of p into Euclidean space E^{n+r}, then the curvature operator R at p decomposes into a sum $R = \Lambda^2 L_1 + \cdots + \Lambda^2 L_r$, where the maps L_i: $V \to V$ are the second fundamental form operators. Hence $R(x \wedge y) = L_1(x) \wedge L_1(y) + \cdots + L_r(x) \wedge L_r(y)$ for each $x, y \in V$, which implies that each $R(x \wedge y)$ has rank $\leqq 2r$ (by Proposition 2.1 (d)). Hence we get the following results, which are relevant for $r \leqq [n/2]$.

LEMMA 4.1. *If the neighborhood of a point in a Riemannian manifold M^n admits isometric embedding into E^{n+r}, then the curvature operator at that point has bounded rank $2r$.*

THEOREM 4.2. *Let M^n be a Riemannian manifold, and set $R_{ij} = 1/2 \sum_{k,l} R_{ij}^{kl} e_k \wedge e_l$, where R_{ij}^{kl} is the curvature tensor and $e_1, \cdots, e_n$ is a basis of the tangent space at a point of M^n. If there exists a point in M^n where*

$$\sum_{\sigma \in S_{r+1}} R_{i_{\sigma(1)}j_1} \wedge \cdots \wedge R_{i_{\sigma(r+1)}j_{r+1}} \neq 0$$

for some integers $1 \leqq i_1, \cdots, i_{r+1}, j_1, \cdots, j_{r+1} \leqq n$, then M^n cannot be isometrically immersed in E^{n+r}.

References

1. M. Marcus, *Finite-dimensional Multilinear Algebra*, Vols. I and II, Marcel Dekker, New York, 1974 and 1975.
2. W. Slebodzinski, *Exterior forms and their application*, Warszawa, Polish Sci. Pub., 1970.
3. J. Vilms, *Local isometric imbedding of Riemannian n-manifolds into Euclidean (n+1)-space*, J. Differential Geometry, **12** (1977), 197-202.
4. ———, *Factorization of curvature operators*, Trans. Amer. Math. Soc., to appear.

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