PACIFIC JOURNAL OF MATHEMATICS
Vol. 87, No. 2, 1980

ALMOST-PERIODIC FUNCTIONS WITH UNBOUNDED INTEGRAL

RUSSELL A. JOHNSON

Let B be an almost-periodic (a.p.) function with mean value zero. Let $G(t)=\int_0^t B(s)ds$. The well-known theorem of Bohr states that $G(t)$ is uniformly bounded iff $G(t)$ is a.p. This theorem may be reformulated in the following way. Let Ω be the hull of B, and let $(\Omega, \boldsymbol{R})$ be the flow on Ω defined by translation. Since B is a.p., Ω is a compact abelian topological group. There is a continuous $b: \Omega \rightarrow \boldsymbol{R}$ and an $\omega_0 \in \Omega$ such that $b(\omega_0 \cdot t)=B(t)$. I.e., b "extends B to Ω". Then Bohr's theorem is equivalent to the following: $G(t)$ is bounded iff there is a continuous $r: \Omega \rightarrow \boldsymbol{R}$ such that $r(\omega \cdot t)-r(\omega)=\int_0^t b(\omega \cdot s)ds (\omega \in \Omega, t \in \boldsymbol{R})$.

In this paper, we consider the case when $G(t)$ is unbounded. Two results are obtained. The first is a generalization of Bohr's theorem: let μ be (normalized) Haar measure on Ω, and let $g_\omega(t)=\int_0^t b(\omega \cdot s)ds (\omega \in \Omega, t \in R)$; then $\overline{\lim}_{n\to\infty} 1/2n\gamma\{t \in [-n, n] \mid g_\omega(t) \in I\}>0$ for some compact $I \subset \boldsymbol{R}$ and some $\omega \in \Omega$ iff there exists a μ-measurable $r: \Omega \rightarrow \boldsymbol{R}$ such that $r(\omega \cdot t)-r(\omega)=\int_0^t b(\omega \cdot s)ds (\omega \in \Omega, t \in \boldsymbol{R})$. Here γ is Lebesgue measure on $\boldsymbol{R}$. Thus, r exists if some $g_\omega(t)$ is not too badly unbounded. This theorem is stated for the class of "minimal" functions (see below), which includes the a.p. ones.

Now, an example in ([10]) shows that there exist a.p. functions b with $g_w(t)$ unbounded which admit a discontinuous, μ-measurable r as above. It is natural to ask whether r *always* exists. Our second result (§ 4) states that this is false; residually many functions $b \in C(\Omega)$ with mean value zero admit no μ-measurable r. This is, at first glance, a bit disappointing. However, combining our two theorems, we can at least draw this conclusion: even a "measure-theoretic" Bohr's theorem applies to only a small (though non-vacuous) set of a.p. functions.

The proof of the first result may be of interest. We make use of techniques and results from ergodic theory, lifting theory ([9]), and the theory of linear skew-product flows ([14], [15]). Of special importance is a close examination of a disintegration ([3], [9]) of a certain ergodic measure. Said examination involves a deep theorem of Furstenberg concerning such disintegrations ([7], Theorem 4.1).

His theorem is stated for *integer* flows. Since our interest is in *real* flows, we extend his theorem to this case (in fact, to the case of an *arbitrary* phase group; we also make other generalizations. See 2.2, 2.3, 2.4). The extension is performed by mimicking Furstenberg's proof.

1. Preliminaries. In 1.1–1.7, X is a locally compact Hausdorff space unless there is a statement to the contrary.

DEFINITIONS 1.1. Let $M(X)$ be the set of nonnegative (Radon) measures on X ([1], Chpt. III, § 1, n° 3, Def. 2). We will always give $M(X)$ the topology of pointwise convergence (i.e., $\mu_n \to \mu$ iff $\mu_n(f) \to \mu(f)$ for each continuous $f\colon X \to C$ with compact support). Let $M_1(X) = \{\mu \in M(X) \mid \|\mu\| \equiv \mu(X) = 1\}$. If $\mu \in M(X)$, we use

$$\int_X f(x) d\mu(x)\ , \quad \int_X f d\mu\ ,$$

or $\mu(f)$ to denote an integral with respect to μ. Let Supp μ be the support of μ.

DEFINITIONS, REMARKS 1.2. Let $\mu \in M(X)$, and let π map X to a topological space Y. Say π is *μ-Lusin-measurable* if, for each compact $K \subset X$ and $\varepsilon > 0$, there is a compact $K_1 \subset K$ such that $\mu(K \sim K_1) < \varepsilon$ and $\pi \mid K_1$ is continuous. If Y is separable metric, then π is μ-Lusin-measurable iff $\pi^{-1}(B)$ is μ-measurable for every closed ball $B \subset Y$. See ([1], Chpt. IV, § 5, Prop. 1 of n°1 and Thm. 4 of n°5).

DEFINITIONS, REMARKS 1.3. Let Y be locally compact Hausdorff, let $\mu \in M(X)$, and let $\pi\colon X \to Y$ be μ-Lusin-measurable. Say π is *μ-proper* if, for every compact $C \subset Y$, $\pi^{-1}(C)$ is essentially μ-integrable (i.e., $\sup_K \mu(\pi^{-1}(C) \cap K) < \infty$, where $K \subset X$ is compact). If π is μ-proper, one can define an image measure $\nu = \pi(\mu)$ ([2], § 6, n°1, Def. 1). If X is compact and π is μ-Lusin-measurable, then π is necessarily μ-proper. If X and Y are compact, then $\nu = \pi(\mu)$ has the following property: $f \in L^1(Y, \nu)$ iff $f \circ \pi \in L^1(X, \mu)$, and

$$\int_Y f d\nu = \int_X f \circ \pi d\mu\ .$$

See ([2], § 6, n°2, Thm. 1).

DEFINITION 1.4. Let X and Y be compact Hausdorff, $\mu \in M(X)$, $\pi\colon X \to Y$ a μ-proper map, and $\nu = \pi(\mu)$. A map $\lambda\colon Y \to M(X)\colon y \to \lambda_y$ is a *disintegration of μ with respect to ν* (or *with respect to π*) if:

(a) Supp $\lambda_y \subset \pi^{-1}(y)(y \in Y)$;

(b) $||\lambda_y|| = 1$ $(y \in Y)$;

(c) λ is ν-adequate ([2], § 3, n°1, Def. 1); see also 1.5(a);

(d) if $f: X \to \boldsymbol{R}$ is continuous, then $y \to \lambda_y(f)$ is ν-integrable, and $\int_X f(x)d\mu(x) = \int_Y \lambda_y(f)d\nu(y)$.

REMARKS 1.5. (a) If λ is ν-Lusin-measurable, it is ν-adequate ([2], § 3, n°1, Prop. 2).

(b) One can define the notion of disintegration if X and Y are locally compact; slight modifications are needed in 1.4(d). See [3], § 3, n°1, Thm. 1).

THEOREM 1.6. *Let X and Y be compact metric, $\pi: X \to Y$ a μ-proper map, $\nu = \pi(\mu)$.*

(a) *There exists a disintegration λ of μ with respect to ν.*

(b) *If λ': $Y \to M(X)$ is another map satisfying* (a), (c), *and* (d) *of* 1.4, *then* $\lambda' = \lambda$ $\nu -$ a.e.

(c) *If $f \in L^1(X, \mu)$, then $y \to \lambda_y(f)$ is defined* $\nu -$ a.e., *is ν-integrable, and* $\mu(f) = \int_Y \lambda_y(f)d\nu(y)$.

Parts (a) and (b) of 1.6 follow from a more general theorem, in which X and Y are locally compact second countable ([3], § 3, n°1, Thm. 1). Part 1.6(c) follows from 1.6(a) and ([2], § 3, n°3, Thm. 1).

DEFINITION 1.7. Let $\mu \in M(X)$, and let $M^\infty(X, \mu) = \{f: X \to \boldsymbol{R} | f$ is bounded and μ-measurable$\}$. A map $\rho: M^\infty(X, \mu) \to M^\infty(X, \mu)$ is a *lifting* of $M^\infty(X, \mu)$ if (i) it is linear, (ii) $\rho(f) = f$ locally $\mu -$ a.e.; (iii) if $f_1 = f_2$ locally $\mu-$a.e., then $\rho(f_1) = \rho(f_2)$ *everywhere*; (iv) $f \geqq 0 \Rightarrow \rho(f) \geqq 0$; (v) $\rho(f_1 \cdot f_2) = \rho(f_1) \cdot \rho(f_2)$. If, in addition, (vi) $\rho(f) = f$ for every continuous $f \in M^\infty(X, \mu)$, then ρ is a *strong lifting* of $M^\infty(X, \mu)$. See ([9], Chpt. III, Def. 1).

THEOREM 1.8 ([8]). *Let X be a locally compact topological group with left Haar measure μ. There exists a strong lifting ρ of $M^\infty(X, \mu)$ which commutes with left translations (thus, let $(T_x f)(\bar{x}) = f(x \cdot \bar{x})(f \in M^\infty(X, \mu); x, \bar{x} \in X)$; one has $\rho(T_x f) = T_x(\rho(f))$).*

DEFINITIONS 1.9. A (right) transformation group (or *flow*) is a triple (X, T, Φ), where X is a topological space, T is a topological group, and $\Phi: X \times T \to X: (x, t) \to x \cdot t$ is a continuous map such that: (i) $x \cdot \mathrm{idy} = x(x \in X; \mathrm{idy} = \text{identity in } T)$; (ii) $(x \cdot t) \cdot s = x \cdot (t \cdot s)(x \in X; t, s \in T)$. We will always suppress Φ, writing just (X, T) when referring to a flow. If $t \in T$ and $A \subset X$, define $A \cdot t = \{x \cdot t \mid x \in A\}$. If $t \in T$ and $f: X \to Y$, define $(t \cdot f)(x) = f(x \cdot t)(x \in X)$. If X is compact Hausdorff and $\mu \in M(X)$, define $(\mu \cdot t)(f) = \mu(t \cdot f)(t \in T, f \in C(X))$. Equivalently, one could define $(\mu \cdot t)(A) = \mu(A \cdot t^{-1})$ for each μ-measurable $A \subset X$.

DEFINITIONS, REMARKS 1.10. Let (X, T) be a flow with X compact Hausdorff. Let $\mu \in M_1(X)$. A set $A \subset X$ is *T-invariant* if $\mu(A \cdot t \Delta A) = 0$ for each $t \in T$. It is *strictly T-invariant* if $A \cdot t = A (t \in T)$. The element μ of $M_1(X)$ is *T-invariant* if $\mu \cdot t = \mu (t \in T)$. It is *T-ergodic* if, in addition, $\mu(A) = 0$ or $\mu(A) = 1$ for every T-invariant set A. If (X, T) has only one invariant measure μ, then μ is ergodic ([13]).

DEFINITIONS 1.11. Let $B: \boldsymbol{R} \to \boldsymbol{R}^n$ be a uniformly bounded, uniformly continuous map. Let $C(\boldsymbol{R}, \boldsymbol{R}^n)$ be the space of continuous maps from $\boldsymbol{R}$ to $\boldsymbol{R}^n$, with the compact-open topology. For each $\tau \in R$, define $f_\tau(t) = f(t+\tau)(f \in C(\boldsymbol{R}, \boldsymbol{R}^n), t \in \boldsymbol{R})$, and let $\Omega = \text{cls}\{B_\tau \mid \tau \in \boldsymbol{R}\} \subset C(\boldsymbol{R}, \boldsymbol{R}^n)$. Then Ω is compact metric ([12]), and the translation $(f, \tau) \to f_\tau$ induces a flow $(\Omega, \boldsymbol{R})$. The space Ω is the *hull* of B. Let ω_0 represent the element B of Ω. Define $b: \Omega \to \boldsymbol{R}^n$: $b(\omega) = \omega(0)$. Then $b(\omega_0 \cdot t) = B_t(0) = B(t)$. Thus b "extends B to Ω". If $B(t)$ is almost periodic, then ([5]) Ω is a compact abelian topological group, with dense subgroup $\boldsymbol{R}$; the flow $(\Omega, \boldsymbol{R})$ is defined by the group operation ($\omega \cdot t$ is the product of $\omega \in \Omega$ and $t \in \boldsymbol{R} \subset \Omega$). The unique invariant measure for $(\Omega, \boldsymbol{R})$ is normalized Haar measure. If $(\Omega, \boldsymbol{R})$ is minimal (i.e., the only nonempty closed invariant subset of Ω is Ω itself), we say that B is *minimal*. If B is a.p., then B is minimal.

2. Furstenberg's theorem.

In this section, we generalize Furstenberg's theorem. We have tried to compromise between, on the one hand, ignoring the fact that Furstenberg's proof is readily available, and, on the other, giving no details at all and simply giving references to that proof.

NOTATION 2.1. For the most part, we adopt the notation of ([7]). However, a disintegration of a measure μ will be written $\omega \to \mu_\omega$, rather than $\omega \to \mu(\omega)$. Compare also with 1.4, where we let λ denote a disintegration. If B is a set, we let $|B|$ be its cardinality. Let Ω denote a compact metric space. If (Ω, T) is a flow, we sometimes write ωt for $\omega \cdot t(\omega \in \Omega, t \in T)$. *In* §2, *T is an arbitrary topological group.*

THEOREM 2.2. *Let (Ω, T) be a flow with μ_0 a T-ergodic measure on Ω. Let K be the unit circle, let $\Sigma = \Omega \times K$, and let $\pi: \Sigma \to \Omega$: $(\omega, \zeta) \to \omega$ be the projection. Let (Σ, T) be a flow satisfying $\pi((\omega, \zeta)\cdot t) = [\pi(\omega, \zeta)]\cdot t = \omega\cdot t((\omega, \zeta) \in \Sigma)$. (Equivalently, suppose $(\omega, \zeta)\cdot t = (\omega t, h_t(\omega, \zeta))$ for continuous functions $h_t: \Sigma \to K$). Let μ be a measure ergodic with respect to (Σ, T) such that $\pi(\mu) = \mu_0$, and let $\omega \to \mu_\omega: \Omega \to M_1(\Sigma)$ be a disintegration of μ with respect to $\mu_0(1.6)$. If μ is not the only ergodic measure on (Σ, T) such that $\pi(\mu) = \mu_0$, then there is an integer n such that $|\text{Supp}\,\mu_\omega| = n\ \mu_0 -$ a.e.*

Proof. We divide the proof into steps.

(1) Let $t \in T$, and define $\tilde{\mu}_\omega \in M_1(\Sigma)$ by $\tilde{\mu}_\omega(f) = \mu_{\omega t}(t^{-1}f)(f \in C(\Sigma))$. Clearly $\omega \to \bar{\mu}_\omega$ satisfies 1.4(a), (b), (c). Since

$$\int_\Omega \tilde{\mu}_\omega(f)d\mu_0(\omega) = \int_\Omega \mu_{\omega t}(t^{-1}f)d\mu_0(\omega) = (\text{by } 1.3)$$
$$\int_\Omega \mu_\omega(t^{-1}f)d\mu_0(\omega) = \mu(t^{-1}f) = \mu(f)\ , \quad \omega \longrightarrow \tilde{\mu}_\omega$$

satisfies 1.4(d). By 1.6(b), $\tilde{\mu}_\omega = \mu_\omega\ \nu -$a.e.; i.e., $= (\mu_\omega)\cdot t\ \mu_0$-a.e. for each fixed t.

(2) For each integer n, let $B_n = \{\omega \in \Omega \mid |\text{Supp}\,\mu_\omega| \leqq n\}$. We claim B_n is μ_0-measurable. For, let Γ be a compact set such that $\omega \to \mu_\omega$ is continuous on Γ. It suffices to show that $B \cap \Gamma$ is closed. Let $\omega_l \to \omega$, $\omega_l \in B_n \cap \Gamma$. Suppose μ_{ω_l} is supported on points

$$(\omega_l, \zeta_1(\omega_l)), \cdots, (\omega_l, \zeta_k(\omega_l))(k \leqq n)\ ;$$

letting δ_ρ denote the Dirac measure at ρ, we write

$$\mu_{\omega_l} = \sum_{i=1}^n \alpha_i(\omega_l)\delta_{(\omega_l, \zeta_i(\omega_l))}\ ,$$

where $0 = \alpha_{k+1}(\omega_l) = \cdots = \alpha_n(\omega_l)$ if $k < n$. Choosing a subsequence, we assume $\alpha_i(\omega_l) \to \bar{\alpha}_i$, $\zeta_i(\omega_l) \to \bar{\zeta}_i (1 \leqq i \leqq n)$. If $f \in C(\Sigma)$, then $\mu_\omega(f) = \lim_{l\to\infty} \mu_{\omega_l}(f) = \sum_{i=1}^n \bar{\alpha}_i f(\omega, \bar{\zeta}_i)$.

(a) Suppose $\Delta = \text{Supp}\,\mu_\omega$ is infinite. Since K is compact, μ_ω assigns positive measure to open subsets of Δ. Let V be an open set in Δ whose closure does not contain $\bar{\zeta}_1, \cdots, \bar{\zeta}_n$. Let $0 \leqq f \in C(\Sigma)$ be equal to 1 on V, and equal to zero at $(\omega, \bar{\zeta}_i)(1 \leqq i \leqq n)$. We obtain a contradiction; hence $|\text{Supp}\,\mu_\omega| < \infty$.

(b) Suppose $|\text{Supp}\,\mu_\omega| < \infty$, with $\mu_\omega = \sum_{i=1}^r \alpha_i\delta_{(\omega, \zeta_i)}$. Let $\bar{\alpha}_i, \bar{\zeta}_i$ be as above. Then

(*) each $\zeta_i \in \{\bar{\zeta}_j \mid \bar{\alpha}_j \neq 0, 1 \leqq j \leqq n\}$, if $\alpha_i \neq 0$;

(**) each $\bar{\zeta}_i \in \{\zeta_j \mid \alpha_j \neq 0, 1 \leqq j \leqq r\}$, if $\bar{\alpha}_i \neq 0$. For suppose (*) is false. Choose $0 \leqq f \in C(\Sigma)$ such that $f(\zeta_i) = 1$ and $f(\bar{\zeta}_j) = 0$ if

$\bar{\alpha}_j \neq 0$; one obtains a contradiction. Similarly for (**). Now, by (*), we must have $|\mathrm{Supp}\,\mu_\omega| \leqq n$; hence $\mu_\omega \in B_n$.

We have shown B_n is μ_0-measurable. As a corollary to the proof, $C_n = \{\omega \in \Omega \mid |\mathrm{Supp}\,\mu_\omega| > n\}$ is μ_0-measurable, since $C_n \cap \Gamma$ is open in Γ.

From (1) and (2), we obtain two conclusions.

(3) If $D = \{\omega \in \Omega \mid |\mathrm{Supp}\,\mu_\omega| < \infty\}$, then $D = \bigcup_{n=1}^{\infty} B_n$ is μ_0-measurable. Also, D is T-invariant; hence $\mu_0(D) = 0$ or 1.

(4) Note $D = \bigcup_{n=1}^{\infty} D_n$, where $D_n = B_n \cap C_{n-1}$ is μ_0-measurable and T-invariant. Suppose $\nu(D) = 1$. Then, for some n, $D_n = \{\omega \in \Omega \mid |\mathrm{Supp}\,\mu_\omega| = n\}$ has μ_0-measure 1.

From now through (15), assume for contradiction that $|\mathrm{Supp}\,\mu_\omega| = \infty$ μ_0 – a.e. Let μ' be another ergodic measure on Σ, with disintegration $\omega \to \mu'_\omega$, such that $\pi(\mu') = \mu_0$.

(5) As on p. 593 of ([7]), one can show that μ_ω is nonatomic (i.e., no point has nonzero measure) μ_0 – a.e. As on p. 594 of ([7]), one can show that μ'_ω is also nonatomic μ_0 – a.e.

Let $\lambda = 1/2(\mu + \mu')$. Then $\omega \to \lambda_\omega = 1/2(\mu_\omega + \mu'_\omega)$ is a disintegration of λ with respect to μ_0, and λ_ω is μ_0–a.e. nonatomic. Fix $\zeta_0 \in K$, let $\{\zeta_0, \zeta\} \subset K$ denote the interval from ζ_0 to ζ (counterclockwise), and let K' be the unit circle in the complex plane. Let $\pi' \colon \Omega \times K' \to \Omega$ denote the projection $(\omega, \zeta) \to \omega$.

Define

$$(6) \qquad \psi \colon \Sigma \longrightarrow \Omega \times K' \colon (\omega, \zeta) \longrightarrow (\omega, e^{2\pi i \lambda_\omega(\{\zeta_0, \zeta\})}) ,$$

and for each t,

$$(7) \qquad g_t \colon \Omega \longrightarrow K' \colon \omega \longrightarrow e^{2\pi i \lambda_{\omega t}(\{\zeta_0, h_t(\omega, \zeta_0)\})} .$$

For each $t \in T$, define $\tilde{t} \colon \Omega \times K' \to \Omega \times K' \colon (\omega, \zeta) \to (\omega \cdot t, g_t(\omega)\zeta)$. Denote the image of (ω, ζ) under $\tilde{t}$ by $(\omega, \zeta) \cdot \tilde{t}$. As on p. 594 of ([7]), one has

(8) $\psi((\omega, \zeta) \cdot t) = [\psi(\omega, \zeta)] \cdot \tilde{t}$ for $(\omega, \zeta) \in \pi^{-1}(B)$, where $B \subset \Omega$ has μ_0-measure 1 (B depends on t).

(9) We show that ψ is η-Lusin-measurable for any measure η on Σ such that $\pi(\eta) = \mu_0$[1] (in particular, for μ, μ', λ). If $\zeta_0 \neq \zeta \in K$ and $m > 1$, construct continuous functions $\tilde{g}_{m,\zeta} \colon K \to \boldsymbol{R}$ such that (i) $\lim_{m\to\infty} \tilde{g}_{m,\zeta}(\bar{\zeta}) = \varphi_{\{\zeta_0,\zeta\}}(\bar{\zeta})$ (here φ denotes characteristic function); (ii) for fixed m, $\tilde{g}_{m,\zeta_n} \to \tilde{g}_{m,\zeta}$ uniformly if $\zeta_n \to \zeta$; (iii) $0 \leqq \tilde{g}_{m,\zeta}(\zeta') \leqq 1$ for all $m, \zeta, \bar{\zeta}$. Let $g_{m,\zeta}(\omega, \bar{\zeta}) \equiv \tilde{g}_{m,\zeta}(\bar{\zeta})$. Define $\tau_m \colon \Sigma \to \boldsymbol{R} \colon (\omega, \zeta) \to \lambda_\omega(g_{m,\zeta})$. Let $\Gamma \subset \Omega$ be a compact set on which $\omega \to \lambda_\omega$ is continuous. Let $(\omega_n, \zeta_n) \to (\omega, \zeta)$ in $\pi^{-1}(\Gamma)$. Then $|\lambda_{\omega_n}(g_{m,\zeta_n}) - \lambda_\omega(g_{m,\zeta})| \leqq |\lambda_{\omega_n}(g_{m,\zeta_n}) - \lambda_{\omega_n}(g_{m,\zeta})| + |\lambda_{\omega_n}(g_{m,\zeta}) - \lambda_\omega(g_{m,\zeta})| \leqq \| g_{m,\zeta_n} - g_{m,\zeta} \| + |\lambda_{\omega_n}(g_{m,\zeta}) -$

[1] Here and in (11) below, we assume η admits a disintegration $\omega \to \eta_\omega$ with η_ω nonatomic for all ω (hence $\eta(\Omega \times \{\zeta_0\}) = 0$).

$\lambda_\omega(g_{m,\zeta})| \to 0$ as $n \to \infty$. Hence τ_m is η-Lusin-measurable. Now, $\lim_{m\to\infty} \tau_m(\omega, \zeta) = \lambda_\omega(\{\zeta_0, \zeta\})$. Hence ψ is η-Lusin-measurable.

(10) In a similar fashion, each g_t is μ_0-Lusin-measurable. Hence each map $\tilde{t}$ is η'-Lusin-measurable for any measure η' on $\Omega \times K'$ which satisfies $\pi'(\eta') = \mu_0$.

(11) By 1.3 and (9), ψ is η-proper if $\pi(\eta) = \mu_0$. By 1.8 and (10), $\tilde{t}$ is η'-proper if $\pi'(\eta') = \mu_0$. Hence $\psi(\mu) \equiv \mu_*$ and $\psi(\mu') \equiv \mu'_*$ are Radon measures, and have unique disintegrations with respect to μ_0(1.6). As on p. 595 of ([7]), $\psi(\lambda) = \mu_0 \times m$, where m is normalized Lebesgue measure on K'. Moreover, by (8), (10), and ([2], § 6, n°3, Prop. 4(a)), one has $\mu_* \cdot \tilde{t} = \psi(\mu \cdot t) = \psi(\mu) = \mu_*$. Similarly $\mu'_* \cdot \tilde{t} = \mu'_*$, $(\mu_0 \times m) \cdot \tilde{t} = \mu_0 \times m (t \in T)$.

(12) Note $\mu_0 \times m = 1/2(\mu_* + \mu'_*)$. We show that the assumption $\mu_* \neq \mu'_*$ implies the existence of an $f \in L^1(\Omega \times K', \mu_0 \times m)$ such that (i) for each t, $f((\omega, \zeta)\cdot\tilde{t}) = f(\omega, \zeta)$ $\mu_0 \times m$ – a.e.; (ii) f is not equal to a constant $\mu_0 \times m$ – a.e. (The existence of f does not follow from standard ergodic theory, since the flow $(\omega, \zeta, t) \to (\omega, \zeta)\cdot\tilde{t}$ has not been proved measurable. However, we need only imitate a standard proof.) Note that $\mu_* \ll \mu_0 \times m$. Hence, if $E \subset \Omega \times K'$ is μ_*-measurable, then $\mu_*(E) = \int_E f d(\mu_0 \times m)$ for a unique $f \in L^1(\mu_0 \times m)$. Now, $\mu_*(E) =$ (1.3 and (11))$\mu_*(E \cdot \tilde{t}^{-1}) = \int_{E\cdot\tilde{t}^{-1}} f d(\mu_0 \times m) =$ (1.3 and (11)) $\int_E f((\omega, \zeta)\cdot\tilde{t}) d(\mu_0 \times m)(\omega, \zeta)$. Hence $f((\omega, \zeta)\cdot\tilde{t} = f(\omega, \zeta)$ $\mu_0 \times m$ – a.e. for each $\tilde{t}$, and (i) is proved. If $f =$ const. $\mu_0 \times m$ – a.e., then const. $= 1$(let $E = \Omega \times K'$). But then $\mu_* = \mu_0 \times m$, contradicting $\mu_* \neq \mu'_*$. So (ii) holds, also.

(13) Using Fubini's theorem, expand f in a partial Fourier series: $f \sim \sum_{m-\infty}^{\infty} a_m(\omega)\zeta^m$. Fix $t \in T$. By (i) in (12) and uniqueness of Fourier coefficients, $a_m(\omega\cdot t)g_t^m(\omega) = a_m(\omega)\mu_0$ – a.e. $(-\infty < m < \infty)$. Since μ_0 is ergodic, f is not a function of ω alone (otherwise (ii) of (12) is violated). Hence there exists $k \neq 0$ with $a_k(\omega) \neq 0$. Arguing as in Lemma 2.1 of ([7]), we see that, for each $t \in T$, $g_t^k(\omega) = R(\omega\cdot t)/R(\omega)\mu_0$ – a.e., where $|R(\omega)| = 1$ μ_0 – a.e. (in fact, $R(\omega) = \overline{a_k(\omega)}/|a_k(\omega)|$).

(14) As on p. 595 of ([7]), let $J \subset K'$ be any interval, and let $\Lambda'(J) = \{(\omega, \zeta) \in \Omega \times K' \mid R(\omega)^{-1}\zeta^k \in J\}$. Then (p. 595)$\mu_0 \times m(\Lambda'(J)) = m(J)$. By (11) and (13), and arguing as on p. 595, one has $\mu_0 \times m(\Lambda'(J)\cdot\tilde{t}\Delta\Lambda'(J)) = 0$ for each $t \in T$. By (8) and (11), $\lambda(\Lambda(J)\cdot t\Delta\Lambda(J)) = 0$ if $\Lambda(J) = \psi^{-1}(\Lambda'(J))(t \in T)$. Also, $\lambda(\Lambda(J)) = m(J)$. So λ has invariant sets of all measures. Argue as on p. 595 again to obtain a contradiction to the assumption $\mu_* \neq \mu'_*$. We conclude $\mu_* = \mu'_*$.

(15) Note $\psi|_{\{\omega\}\times K}$ is continuous for μ_0 – a.a.ω. Hence $\mu_{*,\omega} \equiv \psi(\mu_\omega)$ and $\mu'_{*,\omega} \equiv \psi(\mu'_\omega)$ are defined μ_0 – a.e., and can be shown to be disintegrations of μ_*, μ'_* with respect to μ_0. By 1.6(b), $\mu_{*,\omega} =$

μ_0 — a.e. As on p. 595 of ([7]), it follows that $\mu_\omega = \mu'_\omega$ μ_0 — a.e., and hence that $\mu = \mu'$.

We have contradicted our assumption that $|\operatorname{Supp}\mu_\omega| = \infty \mu_0$ — a.e. By (3), $|\operatorname{Supp}\mu_\omega| < \infty$ μ_0 — a.e., and by (4), $|\operatorname{Supp}\mu_\omega| = n$ μ_0 — a.e. for some n.

THEOREM 2.3. *1.2 remains true if Ω is compact Hausdorff.*

Proof. The proof is not a repetition of that just given, since 1.6 does not now apply. Even if it did, a map $\omega \to \mu_\omega$ which satisfies 1.4(a), (b), (c), (d) need not be μ_0 — Lusin measurable since $M_1(\Sigma)$ is not metrizable. (We used μ_0 — Lusin-measurable of $\omega \to \mu_\omega$ heavily.) However, note that K acts *freely* ([3]) on $\Omega \times K$ by group multiplication. By ([11], Theorem 1.9), *every* measure η on Σ (T-invariant or not), has a $\pi(\eta)$-Lusin-measurable disintegration $\omega \to \eta_\omega$ with respect to $\pi(\eta)$; moreover, $\omega \to \eta_\omega$ is unique in the sense that, if $\omega \to \bar{\eta}_\omega$ is another $\pi(\eta)$-Lusin-measurable disintegration, then $\bar{\eta}_\omega = \eta_\omega$ $\pi(\eta)$ — a.e.

Now go through the proof of 2.2, using μ_0-Lusin-measurable disintegrations $\omega \to \mu_\omega$, $\omega \to \mu'_\omega$. Nothing changes in steps (1)-(10), except that sequences are replaced by nets in various places. In (11), "unique disintegrations" is replaced by "unique μ_0-Lusin-measurable disintegrations". All is the same in steps (12)-(14). In (15), however, we hit a snag. It is not clear that the maps $\omega \to \mu_{*,\omega}$ and $\omega \to \mu'_{*,\omega}$ are μ_0-Lusin-measurable; hence we cannot apply uniqueness to conclude that $\mu_{*,\omega} = \mu'_{*,\omega}$ μ_0 — a.e. We escape as follows. Define $\mu_{*,\omega} = \psi(\mu_\omega)$, $\mu'_{*,\omega} = \psi(\mu'_w)$. Recall K' is the unit circle. Let π_2: $\Omega \times K' \to K'$ be the projection. Define elements $\alpha_\omega, \alpha'_w \in M_1(K')$ by $\alpha_\omega(h) = \mu_{*,\omega}(h \circ \pi_2)$, $\alpha'_\omega(h) = \mu'_{*,\omega}(h \circ \pi_2)$. Let $f \in L^1(\Omega, \mu_0)$. Note

$$\int_{\Omega \times K'} f(\omega) h \circ \pi_2(\omega, \zeta) d\mu_*(\omega, \zeta) = (1.3) \int_\Sigma (f \cdot h \circ \pi_2) \circ \psi d\mu = (1.6\text{c})$$
$$\int_\Sigma f(\omega) \mu_\omega(h \circ \pi_2 \circ \psi) d\mu_0(\omega) = (1.3) \int_\Omega f(\omega) \mu_{*,\omega}(h \circ \pi_2) d\mu_0(\omega)$$
$$= \int_\Omega f(\omega) \alpha_\omega(h) d\mu_0(\omega) .$$

Similarly,

$$\int f(\omega) h \circ \pi_2(\omega, \zeta) d\mu'_*(\omega, \zeta) = \int_\Omega f(\omega) \alpha'_\omega(h) d\mu_0(\omega) .$$

Recall $\mu_* = \mu'_*$, and define

$$S: L^1(\Omega, \mu_0) \to M(K'): S(f) \cdot h = \int_\Omega f(\omega) \mu_{*,\omega}(h \circ \pi_2) d\mu_0(\omega) .$$

The Dunford-Pettis theorem ([4], [9]) applies; there exists a unique (up to sets of μ_0-measure zero) map $\sigma: \Omega \to M(K'): \omega \to \sigma_\omega$ such that $S(f)\cdot h = \int_\Omega f(\omega)\sigma_\omega(h)d\mu_0(\omega)$ for all f, h. Hence $\alpha(\omega) = \sigma_\omega = \alpha'(\omega)$ μ_0—a.e.; since Supp $\mu_{*,\omega}$ and Supp $\mu'_{*,\omega}$ are subsets of $\{\omega\}\times K'$, $\mu_{*,\omega} = \mu'_{*,\omega}$. The rest of the proof of 2.2 is the same as before.

THEOREM 2.4. *2.3 remains true if K acts freely on Σ (with, of course, $\Omega = \Sigma/K$).*

We say *K acts freely on Σ* if (K, Σ) is a transformation group such that, if $\zeta\cdot\sigma = \sigma$ for some $\zeta \in K$ and $\sigma \in \Sigma$, then $\zeta = \text{idy}$ in K.

Proof. Using the technique of ([11], § 1), we construct a Borel isomorphism φ of $\Omega \times K$ onto Σ which (i) maps $\{\omega\} \times K$ homeomorphically onto $\pi^{-1}(\omega) \subset \Sigma$ for all $\omega \in \Omega$ ($\pi: \Sigma \to \Omega$ is the quotient map); (ii) is η-proper for *every* $\eta \in M(\Omega \times K)$. If $t \in T$, define $t_\varphi: \Sigma \to \Sigma$: $t_\varphi = \varphi \circ t \circ \varphi^{-1}$; one obtains a flow $(\Omega \times K, T_\varphi)$, where T_φ consists of Borel measurable maps which are η-proper for every $\eta \in M(\Omega \times K)$. We may apply all the steps of 2.2 (with the modifications of 2.3) to the flow $(\Sigma \times K, T_\varphi)$. (In step (10), some extra work must be done because T_φ does not consist of continuous maps, but the changes are straightforward.)

NOTATION 3.1. Let $B(t)$ be a minimal function (1.11), with $G(t) = \int_0^t B(s)ds$. Let Ω be the hull of B, and define $b \in C(\Omega)$ and $\omega_0 \in \Omega$ so that $b(\omega_0\cdot t) = B(t)(t \in \boldsymbol{R})$. If B is almost periodic (a.p.), let μ_0 be normalized Haar measure on Ω (see 1.11). By uniqueness of Haar measure and 1.10, μ_0 is $\boldsymbol{R}$-ergodic; it is the only ergodic measure on Ω.

3.2. Consider the set of two-dimensional ordinary differential equations $E(\omega)$: $\dot{x} = \begin{pmatrix} 0 & 0 \\ b(\omega\cdot t) & 0 \end{pmatrix} \times (x \in \boldsymbol{R}^2, \omega \in \Omega)$. (We read $E(\omega)$ as "the equation corresponding to ω".) The solutions to these ODEs generate a flow on $\Omega \times \boldsymbol{R}^2$, as follows: $(\bar{\omega}, \bar{x})\cdot t = (\bar{\omega}\cdot t, x(t))$, where $x(t)$ is the solution to $E(\bar{\omega})$ with initial condition $x(0) = \bar{x}$. The flow $(\Omega \times \boldsymbol{R}^2, \boldsymbol{R})$ is an example of a *linear skew-product flow* ([14], [15]). It is called "linear" because each map $N_{t,\omega}$: $\{\omega\} \times \boldsymbol{R}^2 \to \{\omega\cdot t\} \times \boldsymbol{R}^2$: $(\omega, x) \to (\omega, x)\cdot t$ is linear. Let P^1 = projective one-space = the set of lines through the origin in $\boldsymbol{R}^2$. By linearity, each map $N_{t,\omega}$ takes a line in $\{\omega\} \times \boldsymbol{R}^2$ to a line in $\{\omega\cdot t\} \times \boldsymbol{R}^2$; hence $(\Omega \times \boldsymbol{R}^2, \boldsymbol{R})$ induces a flow $(\Omega \times P^1, \boldsymbol{R})$. We let $\Sigma = \Omega \times P^1$, $\pi: \Sigma \to \Omega: (\omega, \zeta) \to \omega$. Note P^1 is homeomorphic to a circle.

3.3. We can describe $(\Sigma, \boldsymbol{R})$ more usefully. Let $S^1 \subset \boldsymbol{R}^2$ be the unit circle, with polar coordinate θ. We may visualize P^1 as that part of S^1 such that $-\pi/2 \leqq \theta \leqq \pi/2$, with $\theta = -\pi/2$ and $\theta = \pi/2$ identified. We will coordinatize P^1 with θ where $-\pi/2 < \theta \leqq \pi/2$ (note the strict inequality). The flow $(\Sigma, \boldsymbol{R})$ may now be given as follows: (i) if $(\omega, \theta) \in \Sigma$ with $-\pi/2 < \theta < \pi/2$, then $(\omega, \theta)\cdot t = (\omega\cdot t, \tan^{-1}\left(\theta + \int_0^t b(\omega\cdot s)ds\right)$; (ii) if $\theta = \pi/2$, then $(\omega, \pi/2)\cdot t = (\omega, \pi/2)(t \in \boldsymbol{R})$. One sees this by solving equations $E(\omega)$.

DEFINITIONS 3.4. Note that $\Sigma_0 = \{(\omega, \pi/2) \mid \omega \in \Omega\}$ is a compact invariant subset of Σ. The projection $\pi: \Sigma \to \Omega$ induces a homeomorphism $\pi_0 = \pi|_{\Sigma_0}$ of Σ_0 onto Ω which commutes with the flows. If μ_0 is an ergodic measure on Σ, then $\eta = \pi_0^{-1}(\mu_0)$ is a measure on Σ_0. If we view η as a measure on Σ in the obvious way, then η is supported on Σ_0 and ergodic with respect to $(\Sigma, \boldsymbol{R})$.

LEMMA 3.5. *Let $\Omega, \Sigma, \Sigma_0, \pi_0$ be as above (except that, in this lemma, Ω need not be metric). Suppose that every measure on Σ which is ergodic with respect to $(\Sigma, \boldsymbol{R})$ has the form $\pi_0^{-1}(\mu_0)$ for some ergodic μ_0 on Ω. Let $f \in C(\Sigma)$ satisfy $f|_{\Sigma_0} = 0$. Let $\nu_{t,\sigma}(g) = 1/t\int_0^t g(\sigma\cdot s)ds(\sigma \in \Sigma, t \in \boldsymbol{R}, g \in C(\Sigma))$. Then, given $\varepsilon > 0$, $\exists T$ such that $|t| \geqq T \Rightarrow |\nu_{t,\sigma}(f)| < \varepsilon$.*

Proof. Observe that $\eta(f) = 0$ for every ergodic η on Σ. Suppose for contradiction that f does not satisfy the conclusion of 3.5. Let t_n and ω_n be points such that $|t_n| > n$ and $|\nu_{t_n,\omega_n}(f)| \geqq \varepsilon$. Choose a subnet $(t_\alpha, \sigma_\alpha)$ of (t_n, ω_n) such that $\nu_{t_\alpha,\omega_\alpha}$ converges to some $\nu \in M_1(\Sigma)$. Then $\nu(f) \neq 0$. We may assume $t_\alpha \to +\infty$, $\omega_\alpha \to \omega$. But these two conditions imply that ν is invariant. Now, it is well-known that the set of invariant measures is the closed convex hull of the set of ergodic measures (in the topology of pointwise convergence). Hence $\nu(f) = 0$. This contradiction proves 3.5.

DEFINITION 3.6. If $I \subset \boldsymbol{R}$, $b \in C(\Omega)$, and $\omega \in \Omega$, let $A(n, I, \omega, b) = 1/2n\, \gamma\{t \in [-n, n] \mid g_\omega(t) \in I\}$, where $g_\omega(t) = \int_0^t b(\omega\cdot s)ds$ and γ is Lebesgue measure on $\boldsymbol{R}(\gamma[0, 1] = 1)$. When confusion cannot arise, we will write $A(n, I, \omega)$.

PROPOSITION 3.7. *Assume there is a compact set $I \subset \boldsymbol{R}$ such that $\overline{\lim}_{n\to\infty} A(n, I, \omega_0) > 0$ for some $\omega_0 \in \Omega$. Then there is an ergodic μ_0 on Ω and at least two ergodic measures η, μ on Σ such that $\pi(\eta) = \mu_0$.*

Proof. There is at least one ergodic measure on Ω([13]). If the conclusion of 3.7 is false, then, given an ergodic μ_0 on Ω, the measure $\eta = \pi_0^{-1}(\mu_0)$ (see 3.4) is the only ergodic measure on Σ satisfying $\pi(\eta) = \mu_0$. Let $I_1 = \tan^{-1} I \subset (-\pi/2, \pi/2) \subset P^1$, and let $\Sigma_1 = \Sigma \times I_1$. Let f be a continuous, nonnegative function which is 0 on Σ_0 and 1 on Σ_1. Let $E_n = \{t \in [-n, n] \mid g_{\omega_0}(t) \in I\}$. Then $f((\omega_0, 0)\cdot t) = 1$ if $t \in E_n$. Hence $\overline{\lim}_{n\to\infty} 1/2n\left[\int_0^n f(\omega_0, 0)\cdot s ds + \int_{-n}^0 f(\omega_0, 0)\cdot s ds\right] > 0$. This contradicts 3.5; 3.7 is proved.

Let μ_0 be an ergodic measure on Ω which satisfies the condition of 3.7. If $B(t)$ is a.p., then μ_0 is normalized Haar measure (3.1).

THEOREM 3.8. (a) *Suppose there exist $\omega_0 \in \Omega$ and a compact $I \subset \boldsymbol{R}$ such that $\overline{\lim}_{n\to\infty} A(n, I, \omega_0) > 0$. Then there is a μ_0-measurable function r on Ω such that $r(\omega\cdot t) - r(\omega) = \int_0^t b(\omega\cdot s)ds$ μ_0 –* a.e. *for each $t \in \boldsymbol{R}$.*

(b) *If $B(t)$ is* a.p., *then r may be chosen so that $r(\omega\cdot t) - r(\omega) = \int_0^t b(\omega\cdot s)ds$ for all $\omega \in \Omega$, $t \in \boldsymbol{R}$.*

Proof. (a) Using 3.6, we can find an ergodic μ_0 on Ω and ergodic measures $\eta = \pi_0^{-1}(\mu_0)$ and $\mu \neq \eta$ on Σ such that $\pi(\eta) = \pi(\mu) = \mu_0$. Let $\lambda: \Omega \to M_1(\Sigma): \omega \to \lambda_\omega$ be a disintegration of μ with respect to μ_0(1.6). Using uniqueness in 1.6 (1.6(b)), it is easy to see that $(*) \lambda_{\omega\cdot t} = (\lambda_\omega)\cdot t$ μ_0 – a.e. for each $t \in \boldsymbol{R}$.

By 2.2, there exists an integer n such that $|\operatorname{Supp} \lambda_\omega| = n$ on a set $B \subset \Omega$ of μ_0-measure 1. For $\omega \in B$, we write $\lambda_\omega = \sum_{i=1}^n \alpha_i(\omega)\delta_{(\omega, \theta_i(\omega))}$

$$(\delta = \text{Dirac measure}),\ \text{where } \theta_0(\omega) < \theta_2(\omega) < \cdots < \theta_n(\omega)\ .$$

Let $B_1 = \{\omega \in B \mid \theta_n(\omega) = \pi/2\}$ (recall θ has range $\pi/2 < \theta \leqq \pi/2$).

By (*) and invariance of Σ_0, B_1 is $\boldsymbol{R}$-invariant in the sense of 2.10.

We claim B_1 is μ_0-measurable. Let $\Gamma \subset B$ be a compact set such that $\lambda|_\Gamma$ is continuous. It suffices to show that $B_1 \cap \Gamma$ is closed. Let $\omega_l \in B_1 \cap \Gamma$, $\omega_l \to \omega \in B$. Choosing a subsequence, we may assume that $\alpha_i(\omega_l) \to \bar{\alpha}_i$, $\theta_i(\omega_l) \to \bar{\theta}_i$. It is easy to see that $\{\theta_1(\omega), \cdots, \theta_n(\omega)\} \subset \{\bar{\theta}_i \mid \bar{\alpha}_i \neq 0 (1 \leqq i \leqq n)\}$. Hence the two sets are equal, and no $\bar{\alpha}_i$ can be zero. Since $\bar{\theta} = \pi/2$, we must have $\theta_n(\omega) = \pi/2$. So B_1 is μ_0-measurable.

Since μ_0 is ergodic, $\mu_0(B_1) = 0$ or 1. It cannot be 1. For, suppose it is. The ergodic measures η and μ are mutually singular (considerably more is true; see, e.g., [13], pp. 496–508). Let D_1 and D_2 be Borel sets in Σ such that $1 = \eta(D_1) = \mu(D_2)$, $D_1 \cap D_2 = \phi$. Clearly $\eta(D_1 \cap \Sigma_0) = 1$. This implies that, for μ_0 – a.a. ω, one has $D_1 \cap \Sigma_0 \cap \pi^{-1}(\omega) = \phi$. Clearly $\eta(D_1 \cap \Sigma_0) = 1$. This implies that, for μ_0 – a.a. ω, one has $D_2 \cap \Sigma_0 \cap \pi^{-1}(\omega) = \phi$. Since $\mu_0(B_1) = 1$, we have $\lambda_\omega(D_2) < 1$ for μ_0 − a.a.ω. But $1 = \mu(D_2) = $ (1.6(c))

$$\int_\Omega \lambda_\omega(D_2) d\mu_0(\omega) < 1 .$$

The contradiction shows that $\mu_0(B_1) = 0$.

Let $B_2 = \Omega \sim B_1$; then $\mu_0(B_2) = 1$. Let $D_3 = \{(\omega, \theta) \in \Sigma \mid \omega \in B_2, \theta = \max_{1 \leq i \leq n} \theta_i(\omega)\}$. Then D_3 is $\boldsymbol{R}$-invariant ((*) and the fact that the flow on Σ preserves the θ-order). We claim D_3 is μ-measurable. Let $\Gamma \subset B_2$ be a compact set such that $\lambda|_\Gamma$ is continuous, and let $\Gamma_1 = \pi^{-1}(\Gamma)$. We show that $D_3 \cap \Gamma_1$ is closed. Let $(\omega_l, \theta_l) \in D_3 \cap \Gamma_1$, with $(\omega_l, \theta_l) \to (\omega, \theta)$. Then $\omega \in B_2$, and $\lambda_{\omega_l} \to \lambda_\omega$. Choosing a subsequence, we assume $\alpha_i(\omega_l) \to \bar{\alpha}_i$, $\theta_i(\omega_l) \to \bar{\theta}_i$. Now each θ_l is equal to $\theta_n(\omega_l)$. Hence $\theta = \bar{\theta}_n$. As before, $\{\theta_1(\omega), \cdots, \theta_n(\omega)\} = \{\bar{\theta}_1, \cdots, \bar{\theta}_n\} = \{\bar{\theta}_1, \cdots, \theta\}$. We claim that $\theta = \theta_n(\omega)$. Since the $\theta_i(\omega_l)$ were arranged in increasing order, it suffices to show that $\bar{\theta}_1 \neq -\pi/2$. But, if this were not the case, then $\theta_n(\omega)$ would be $\pi/2$. Since it is not ($\omega \in B_2$), we have $\theta = \theta_n(\omega)$, and hence $(\omega, \theta) \in D_3$.

Either $\mu(D_3) = 0$ or $\mu(D_3) = 1$. But $\mu(D_3) = \int_\Omega \lambda_\omega(D_3) d\mu_0(\omega)$, and $\lambda_\omega(D_3) > 0$ on B_2. Hence $\mu(D_3) = 1$. This implies λ_ω is supported on the point (ω, θ) if $\omega \in B_2$; i.e., $\lambda_\omega = \delta_{(\omega,\theta)}$.

We now define r. If $\omega \in B_2$, let (ω, θ) be the corresponding point in D_3, and let $r(\omega) = \tan\theta$. If $\omega \notin B_2$, define r arbitrarily. Since D_3 is $\boldsymbol{R}$-invariant, one has $r(\omega \cdot t) = \tan \tan^{-1}\left(\theta + \int_0^t b(\omega \cdot s) ds = r(\omega) + \int_0^t b(\omega \cdot s) ds\ \mu_0\right.$ − a.e. for each $t \in \boldsymbol{R}$. Also, it follows immediately from the proof of μ-measurability of D_3 that is μ_0-measurable. This completes the proof of (a).

(b) Let λ_ω be the disintegration of μ with respect to μ_0 of (a). We first arrange that $\lambda_{\omega \cdot t} = (\lambda_\omega) \cdot t$ for *all* $\omega \in \Omega$ and $t \in \boldsymbol{R}$. To do this, let ρ be a strong lifting of $M^\infty(\Omega, \mu_0)$ commuting with translations (1.8). As in ([9], Chpt. VI, Prop. 1), we may define a new disintegration λ' of μ with respect to μ_0 by the formula $\lambda'_\omega(f) = \rho(g)(\omega)$, where $g: \Omega \to \boldsymbol{R}: g(\bar{\omega}) = \lambda_{\bar{\omega}}(f) (f \in C(\Sigma))$. It is easily seen that $\lambda'_{\omega \cdot t} = (\lambda'_\omega) \cdot t$ for all ω, t.

Now go through the proof of (a) with λ' in place of λ. One finds that B_2 is *strictly* $\boldsymbol{R}$-invariant (in the sense of 1.10). If $\omega \in B_2$, define $r(\omega) = \tan\theta$; define r on $\Omega \sim B_2$ in any manner so that $r(\omega \cdot t) - r(\omega) = \int_0^t b(\omega \cdot s) ds$ holds. Then this equation holds for all ω, t. As in (a), r is μ_0-measurable. The proof of (b) is complete.

By restating the hypotheses of 3.8, we obtain a theorem whose converse is also true.

THEOREM 3.9. *The following are equivalent.*

(a) *There is an ergodic measure* μ_0 *on* Ω, *a set* $\Omega_1 \subset \Omega$ *with* $\mu_0(\Omega_1) = 1$ *and compact sets* $I_\omega \subset \boldsymbol{R}$ *such that* $\lim_{n\to\infty} A(n, I_\omega, \omega)$ *exists and is positive* ($\omega \in \Omega_1$).

(b) *There is an ergodic measure* μ_0 *on* Ω *and a* μ_0*-measurable function* r *such that* $r(\omega\cdot t) - r(\omega) = \int_0^t b(\omega\cdot s)ds$ μ_0—a.e. *for each* $t \in \boldsymbol{R}$. *If* $B(t)$ *is* a.p., *then* r *may be chosen so that equality holds for all* ω, t.

Proof. (a) $\Rightarrow$ (b): follows from 3.8.

(b) $\Rightarrow$ (a): Let J be any compact set such that $B = r^{-1}(J)$ has positive μ_0-measure. Let φ_B be the characteristic function of B. By the Birkhoff ergodic theorem ([13]), $1/t\int_0^t \varphi_B(\omega\cdot s)ds \to \mu_0(B)$ as $t \to \infty$ and as $t \to -\infty$, for μ_0 – a.a.ω. Fix such an ω. Note that $\omega\cdot s \in B$ iff $r(\omega\cdot s) \in J$. Let $I_\omega = \{s - r(\omega) \mid s \in J\}$. Then $\lim_{n\to\infty} A(n, I_\omega, \omega) = \mu_0(B) > 0$.

REMARKS 3.10. (a) Since (b) $\Rightarrow$ (a) in 3.9, we can conclude that the hypothesis of 3.8 implies 3.9(a). Thus the relative density hypothesis "extends from a point to almost all of the hull".

(b) Since J can be chosen to be an interval of arbitrarily small length, so can the sets I_ω.

(c) Theorems 3.8 and 3.9 say nothing about μ_0-integrability of r.

(d) Using the techniques of § 3, one can prove results analogous to 3.8, 3.9 for minimal *integer* flows (Ω, T) (Ω a compact metric space, $T: \Omega \to \Omega$ a homeomorphism). Let $b: \Omega \to \boldsymbol{R}$ be continuous. The analogues of 3.8, 3.9 are obtained by simply replacing $\int_0^t b(\omega \cdot s)ds$ by $g_\omega(m) = \sum_{k=0}^m b(\omega\cdot T^k)$ throughout (if m is negative, let $g_\omega(m) = \sum_{k=0}^{-m} b(\omega\cdot T^{-k})$), and by replacing $A(n, I, \omega, b)$ by $1/2n$ card $\{m \in [-n, n] \mid g_\omega(m) \in I\}$.

(e) Let (Ω, R) be a.p. minimal. Let $C_0(\Omega) = \{b \in C(\Omega) \mid b$ has mean value zero$\}$. There is a $b_0 \in C_0(\Omega)$, and a μ_0-measurable, discontinuous function $r_0: \Omega \to \boldsymbol{R}$ such that

$$r_0(\omega t) - r_0(\omega) = \int_0^t b_0(\omega\cdot s)ds(\omega\cdot\Omega,\ t \in \boldsymbol{R}) .$$

One can prove this by constructing r_0, using a method similar to that of ([7], p. 585). See also ([10]). We will not give details here. Now, in 4.3 below, it is shown that $V = \left\{v \in C_0(\Omega) \,\middle|\, \int_0^t v(\omega\cdot s)ds \text{ is bounded } (\omega \in \Omega)\right\}$ is dense in $C_0(\Omega)$. Then $b_0 + V$ is also dense in $C_0(\Omega)$. Hence the set of functions $b \in C_0(\Omega)$ with a α "μ_0-measurable, discontinuous antiderivative" r, is dense in $C_0(\Omega)$.

4. In this section, we show that "most" a.p. functions satisfy neither conditions of 3.9. To make this precise, we alter our point of view somewhat, and consider some almost periodic minimal set ([5])(Ω, $\boldsymbol{R}$). If Ω is metrizable, then Ω is the hull of some a.p. function $B(t)$. However, we will not assume Ω is metrizable. The result is then the following. Suppose (Ω, $\boldsymbol{R}$) is not a periodic flow (i.e., Ω is not the hull of a periodic function), and let $C_0(\Omega) = \{b \in C(\Omega) \mid b$ has mean value zero$\}$; then there is a residual subset C_1 of $C_0(\Omega)$ such that $b \in C_1 \Rightarrow \lim_{n\to\infty} 1/2n\, \gamma \left\{t \in [-n, n] \,\middle|\, \int_0^t b(\omega\cdot s)ds \in I\right\} = 0$ for all $\omega \in \Omega$ and all compact $I \subset \boldsymbol{R}$.

NATATION 4.1. Let (Ω, $\boldsymbol{R}$) be an a.p. minimal set. As in 3.6, let $A(n, I, \omega, b) = 1/2n\, \gamma \left\{t \in [-n, n] \,\middle|\, \int_0^t b(\omega\cdot s)ds \in I\right\}$ for $b \in C(\Omega)$ and compact $I \subset \boldsymbol{R}$. Recall that the mean value of $b \in C(\Omega)$ equals

$$\int_\Omega b(\omega)d\mu_0(\omega)(\mu_0 = \text{normalized Haar measure on } \Omega)\ .$$

Let $C_0(\Omega) = \{b \in C(\Omega) \mid b$ has mean value zero$\}$. Give $C(\Omega)$ the sup-norm topology.

LEMMA 4.2. *Suppose* (Ω, $\boldsymbol{R}$) *is not a periodic flow. Let* $0<\varepsilon<1$ *and compact* $I \subset \boldsymbol{R}$ *be given. Then there is a* $c \in C_0(\Omega)$ *with* $\|c\|=1$ *such that* $A(n, I, \omega, c) < \varepsilon$ *for all* ω *if* n *is sufficiently large.*

Proof. First pick $\omega_0 \in \Omega$ and $b \in C_0(\Omega)$. We may assume that $B(t) = b(\omega_0\cdot t)$ is not periodic in t. Expand $B(t)$ in a Bohr-Fourier series: $B(t) = \sum_{k=1}^{\infty} a_k e^{i\lambda_k t}$. We may assume $\lambda_k \neq 0$ for all k. Either (i) λ_m/λ_l is rational for all m and l, in which case $\underline{\lim}\, |\lambda_m| = 0$, or (ii) λ_m/λ_l is irrational for some m, l. Let Ω_1 be the hull of $B(t)$, and write $B_\tau(t) = B(t+\tau)(t, \tau \in \boldsymbol{R})$. The correspondence $\omega_0\cdot\tau \to B_\tau$: $\{\omega_0\cdot t \mid t \in \boldsymbol{R}\} \to \Omega_1$ is uniformly continuous, hence extends to a surjection τ_1: $\Omega \to \Omega_1$ which commutes with the flows.

Next, let K be the unit circle, and let $K_\infty = \prod_{k=1}^{\infty} K$. Define a flow ($K_\infty$, $\boldsymbol{R}$) as follows: $(e^{i\theta_k})_{k=1}^{\infty}\cdot t = (e^{i(\theta_k+\lambda_k t)})_{k=1}^{\infty}$. The correspondence $B_\tau \to (e^{i\lambda_k t})_{k=1}^{\infty}$: $\{B_\tau \mid \tau \in \boldsymbol{R}\} \to K_\infty$ is uniformly continuous, hence extends to a continuous map τ_2: $\Omega_1 \to K_\infty$. Let $\Omega_2 = $ Image (τ_2); then Ω_2 is compact invariant, and τ_2: $\Omega_1 \to K_\infty$ commutes with the flows.

Now consider case (i). Define c_m: $K_\infty \to \boldsymbol{R}$: $(e^{i\theta_k})_{k=1}^{\infty} \to \cos\theta_m$. Note that, if $p = (e^{i\theta_k})_{k=1}^{\infty}$, then $\int_0^t c_m(p\cdot s)ds = (1/\lambda_m)[\sin(\theta_m + \lambda_m t) - \sin\theta_m]$. The following is not hard to prove (we will not do so): if $\delta > 0$, and if $f_j(x, \delta) = (1/2j)\gamma\{y \in [-j, j] \mid |\sin(x + y) - \sin x| \leqq \delta\}(0 < j \in \boldsymbol{R}, x \in \boldsymbol{R})$, then given $\varepsilon > 0$, $\exists\delta > 0$ and J such that $j \geqq J \Rightarrow f_j(x, \delta) \leqq \varepsilon$ *uniformly* in x. Let I and ε be as in the statement of 4.2. Choose $M > 0$ so that $I \subset [-M, M]$. Choose $\delta > 0$ and J so that $j \geqq J \Rightarrow f_j(x, \delta) < \varepsilon$ for all $x \in \boldsymbol{R}$. There is a λ_m such that $|\lambda_m| \cdot M < \delta$. Let $n = j/|\lambda_m|$. Note that $(1/2n)\gamma\left\{t \in [-n, n] \,\middle|\, \int_0^t c_m(p \cdot s)ds \in [-M, M]\right\} = (1/2j)\gamma\{\tau \in [-j, j] \mid |\sin(\theta_m + \tau) - \sin\theta_m| \leqq \delta\} < \varepsilon$ if $j \geqq J$, for all $p \in K_\infty$. Let $C = c_m \circ \tau_2 \circ \tau_1$, and choose $N \geqq J/|\lambda_m|$. Then $n \in N \Rightarrow A(n, I, \omega, c) < \varepsilon$ for all $\omega \in \Omega$.

Finally, consider case (ii). Suppose λ_m/λ_l is irrational. The map $(e^{i\lambda_k t})_{k=1}^\infty \to (e^{i\lambda_m t}, e^{i\lambda_l t})$ of $\{(e^{i\lambda_k t})_{k=1}^\infty \mid t \in \boldsymbol{R}\}$ into the 2-torus $K \times K = K^2$ is uniformly continuous, hence extends to a continuous map τ_3 of Ω_2 onto K^2 which commutes with the flows (the flow on K^2 is of course the irrational twist defined by λ_m and λ_l). For integers τ and s, define $C_{\tau s}: K^2 \to \boldsymbol{R}: (e^{i\theta}, e^{i\varphi}) \to \cos(\tau\theta + s\varphi)$. We can choose τ and s so that $|\tau\lambda_m + s\lambda_l|$ is as small as we please. Therefore, we can apply an argument like that used in case (i) to show that, if $C = C_{\tau s} \circ \tau_3 \circ \tau_2 \circ \tau_1$, then (for appropriate τ and s) C satisfies 4.2.

LEMMA 4.3. *Let* $V = \left\{b \in C_0(\Omega) \,\middle|\, \int_0^t b(\omega \cdot s)ds \text{ is uniformly bounded } (\omega \in \Omega, t \in \boldsymbol{R})\right\}$. *Then V is dense in* $C_0(\Omega)$.

Proof. Let $\tilde{b} \in C_0(\Omega)$, $\omega_0 \in \Omega$, $\tilde{B}(t) = \tilde{b}(\omega_0 \cdot t)$, $\tilde{B}(t) = \sum_{k=0}^\infty a_k e^{i\lambda_k t}$.

Then $\tilde{B}(t)$ may be uniformly approximated by trigonometric polynomials without constant term whose frequencies are among the λ_ks([6]). Such a polynomial defines a function b on Ω such that $\int_0^t b(\omega \cdot s)ds$ is uniformly bounded as a function of $\omega \in \Omega$ and $t \in \boldsymbol{R}$. The lemma follows.

THEOREM 4.4. *Let $(\Omega, \boldsymbol{R})$ be a nonperiodic, almost periodic minimal set. Then there is a residual subset C_1 of $C_0(\Omega)$ such that* $\lim_{n\to\infty} A(n, I, \omega, b) = 0$ *for all $\omega \in \Omega$ and all compact $I \subset \boldsymbol{R}(b \in C_1)$.*

Proof. Let $Q(I, k, N) = \{b \in C_0(\Omega) \mid$ for some $\omega \in \Omega$ (depending on b), one has $A(n, I, \omega, b) \geqq 1/k$ for $n \geqq N\}$. By 3.8 and 3.3, $\bigcup_{I \subset R} \bigcup_{k=1}^\infty \bigcup_{N=1}^\infty Q(I, k, N) = \{b \in C_0(\Omega) \mid$ for some $\omega \in \Omega$ and some compact $I \subset \boldsymbol{R}$, $\overline{\lim}_{n\to\infty} A(n, I, \omega, b) > 0\}$.

Without loss of generality, we can restrict attention to sets I of the form $[-\alpha, \alpha]$, where α is an integer. It is easily seen that, if $I = [-\alpha, \alpha]$, then $\operatorname{cls} Q(I, k, N) \subset Q(I_1, k, N)$, where $I_1 = [-\alpha-1, \alpha + 1]$. Hence, if 4.4 is false, then some $Q(I, k, N)$ contains a ball

W of radius $\delta > 0$. By 4.3, we may suppose that, if a is the center of W, then $\int_0^t a(\omega \cdot s)ds$ is in some compact interval I_2 for all t, ω. Let $I \cup I_2 \subset [-\alpha_1, \alpha_1]$, then let $I_3 = [-2\alpha_1/\delta, 2\alpha_1/\delta]$. Apply 4.2 with I_3 replacing I and $1/k$ replacing ε. We obtain a function c such that $a + \delta c \in W$ and $A(n, I, \omega, a + \delta c) < 1/k$ for all ω if n is sufficiently large. We have arrived at a contradiction, and proved 4.4.

REMARKS 4.5. (a) By 4.4 and 3.9, residually many $b \in C_0(\Omega)$ admit no μ_0-measurable r with $r(\omega \cdot t) - r(\omega) = \int_0^t b(\omega \cdot s)ds$.

(b) A theorem analogous to 4.4 holds for *integer* a.p. minimal flows (Ω, T). The statement of this theorem is obtained (as in 3.10(d)) by simply replacing $\int_0^t b(\omega \cdot s)ds$ with $g_\omega(m) = \sum_{k=0}^m b(\omega \cdot T^k)$ $(\sum_{k=0}^{-n} b(\omega \cdot T^{-k})$ if $n<0)$, and replacing $A(n, I, \omega, b)$ by $1/2n$ card $\{m \in [-n, n] | g_\omega(m) \in I\}$. We must assume (Ω, T) is not periodic; i.e., that $T^j = \text{idy}$ on Ω for no j.

REFERENCES

1. N. Bourbaki, *Integration*, Chpts. I-IV, 2nd ed., Hermann, Paris, 1965.
2. ———, *Integration*, Chpt. V, 2nd ed., Hermann, Paris, 1967.
3. ———, *Integration*, Chpt. VI, 1st ed., Hermann, Paris, 1959.
4. N. Dunford and J. B. Pettis, *Linear operations on summable functions*, Trans. Amer. Math. Soc., **47** (1940), 323-392.
5. R. Ellis, *Lectures on Topological Dynamics*, Benjamin, New York, 1969.
6. A. M. Fink, *Almost Periodic Differential Equations*, Lecture Notes in Mathematics # 377, Springer-Verlag, New York, 1974.
7. H. Furstenberg, *Strict ergodicity and transformations of the torus*, Amer. J. Math., **83** (1961), 573-601.
8. A. and C. Ionescu-Tulcea, *On the existence... locally compact group*, Proc. Fifth Berk. Symp. Math. Stat. and Prob., 1965, V. 2, pt. 1, 63-97.
9. ———, *Topics in the Theory of Lifting*, Springer-Verlag, New York, 1969.
10. R. Johnson, *A measurable, non-continuous function on the 2-torus*, preprint.
11. ———, *Disintegration of measures on compact transformation groups*, to appear in Trans. Amer. Math. Soc.
12. J. L. Kelley, *General Topology*, D. Van Nostrand Co., Princeton, 1955.
13. V. V. Nemyckii and V. V. Stepanov, *Qualitative Theory of Differential Equations*, English trans., Princeton Math. series, #22, Princeton Univ. Press, Princeton, 1960.
14. R. Sacker and G. Sell, *Existence of dichotomies and invariant splittings for linear differential systems I*, J. Differential Equations, **15** (1974), 429-458.
15. G. Sell, *Linear Differential Systems*, Lecture Notes, University of Minnesota, 1975.

Received February 15, 1978 and in revised form March 28, 1979.

UNIVERSITY OF SOUTHERN CALIFORNIA
LOS ANGELES, CA 90007