PACIFIC JOURNAL OF MATHEMATICS
Vol. 88, No. 1, 1980

QUASI-METRIZABLE SPACES

JACOB KOFNER

A construction is given which yields to any quasi-metrizable not non-archimedeanly quasi-metrizable space another quasi-metrizable space which is not σ-orthocompact. It is shown that (σ-)orthocompactness does not imply non-archimedean quasi-metrizability and is neither summable nor multiplicative nor (CH) hereditary in completely regular quasi-metric spaces.

It is proved that quasi-metric spaces are preserved under perfect mappings.

0. Let T be the completely regular quasi-metric space without any σ-interior preserving base presented in [8]. T has been invented to show that a sufficient condition for quasi-metrizability due to S. Nedev ([12]) and to P. Fletcher and W. F. Lindgren ([3]), namely the existence of a σ-interior preserving base, is not necessary. In §1 of the present paper some further analogs of well known metric theorems are proved to be false. A general construction on quasi-metric spaces is given, which when applied to the space T yields a (completely regular perfect subparacompact submetrizable) quasi-metric non-σ-orthocompact extention $T^{\vee}$, while T is shown to be hereditarily orthocompact. $T^{\vee}$ supples i.a. an answer to a question of P. Fletcher concerning the σ-orthocompactness of quasi-metric spaces [private communication].

It is shown further that (σ-)orthocompactness is neither multiplicative nor summable in completely regular quasi-metric spaces. In fact, the product of the space T with the Sorgenfrey line is shown to be non-σ-orthocompact, and $T^{\vee}$ is shown to be the union of an open set homeomorphic to T and a discrete set of cardinality of the continuum. Together with the continuum hypothesis the above construction provides an example of a regular Lindelöf quasi-metric space that is not hereditarily σ-orthocompact.

In §2[1] it is shown that a perfect image of quasi-metrizable space is quasi-metrizable, in analogy to the metric case. This result answers a question posed first by S. Nedev and M. M. Čoban ([13]). It is further proved that non-archimedean quasi-metric spaces are preserved under perfect mappings. In [13] S. Nedev and M. M. Čoban have proved the same result for γ-spaces. Hence each of the three increasing classes of spaces, namely non-archimedean quasi-metric spaces, quasimetric spaces and γ-spaces, is preserved under perfect mappings.

[1] The results of §2 had been included in [9].

All spaces below are T_1. D denotes the set $\{0\} \cup \{1/j: j = 1, 2, \cdots\}$.

1. A generalized metric d on a space X is a *quasi-metric* (*non-archimedean quasi-metric*) provided that always $d(x, z) \leqq d(x, y) + d(y, z)(d(x, z) \leqq \max\{d(x, y), d(y, z)\})$.

A collection α of subsets of a space X is *interior preserving* provided that $\mathrm{int} \cap \{A: A \in \alpha'\} = \bigcap \{\mathrm{int}\, A: A \in \alpha'\}$ for every $\alpha' \subset \alpha$, and it is *σ-interior preserving* provided that α is countable union of interior preserving collections. A space is non-archimedeanly quasi-metrizable iff it has a σ-interior preserving base ([8], [3]). A space is *(σ-)orthocompact* provided that every open cover has a (σ-) interior preserving open refinement ([14], [2]). A space X is *perfect*, provided that any open set of X is F_σ and *subparacompact* provided that every cover of X has a σ-discrete closed refinement. A space X is *submetrizable* provided that there exists a metrizable topology which is coarser than that of X.

The space T has the complex plane as its underlying set. A base of neighborhoods in a point $t \in T$ consists of the sets $C(t, r) = \{t\} \cup \{t': |t' - (t + ri)| < r\}$, $r > 0$, i.e., $C(t, r)$ is an open circle with radius r together with its "southern pole" t. It is shown in [8] that the space T is submetrizable, quasi-metrizable via a quasi-metric which is continuous in the second variable, but not non-archimedeanly quasi-metrizable. Moreover, the same arguments as in [8] prove the following lemma.

LEMMA. *Let T_0 be a subset of the second category in the plane topology and let $\mathscr{C} = \{U(t): t \in T_0\}$ be a collection of subsets of T such that for each $t \in T_0$, $U(t)$ is a T-neighborhood of t contained in $C(t, r_t)$. Then $\mathscr{C}$ is not σ-interior preserving in T and $\{U(t) \cap T_0: t \in T_0\}$ is not σ-interior preserving in the subspace T_0 of T.*

PROPOSITION 1. *T is a perfect, subparacompact and hereditarily orthocompact space*[2].

Proof. Let us show that any open cover ζ of an open set V has a closed σ-discrete refinement in T, so that T is perfect and subparacompact.

For each $G \in \zeta$ let G^0 denote the interior of G in the plane topology. Set $\zeta^0 = \{G^0: G \in \zeta\}$ and set $V' = \bigcup \zeta^0$. Note that ζ^0 has a closed σ-discrete refinement even in the plane topology. Set $F = V - V'$ and set $F_n = \{t \in F: C(t, 1/n) \subset G$ for some $G \in \zeta\}$. We have $F = \bigcup F_n$. Let us show that any F_n is σ-discrete.

For $t \in T$, $r > 0$ set $C^{-1}(t, r) = \{t': t \in C(t', r)\}$. Note that $C^{-1}(t, r)$

[2] After this result was obtained, I learned from H. Junnila that he also has proved that the space T is orthocompact in a different way.

is an open circle with radius r together with its "northern pole" t. Set $S(t, r) = C(t, r) \cup C^{-1}(t, r)$. The space X with the plane as its underlying set with the basic neighborhoods $S(x, r)$ is semimetrizable ([6], [7]) and its topology is coarser than that of T yet finer than the plane topology. Now for any $t \in F_n$ we have $S(t, 1/n) \cap F_n = \{t\}$. Otherwise pick some $t' \in S(t, 1/n) \cap F_n$, $t' \neq t$. Then $t' \in C(t, 1/n)$ or $t' \in C^{-1}(t, 1/n)$ and $t \in C(t', 1/n)$. In the first case, for instance, $t' \in C(t, 1/n) - \{t\} \subset G^0 \in \zeta^0$, and $t' \in V'$ and this contradicts $t' \in F$. Hence for every $t \in F_n$ the trace of $S(t, 1/n)$ on F_n is $\{t\}$.

Since the open collection $\{S(t, 1/n): t \in F_n\}$ in the semi-metrizable space X has a closed σ-discrete refinement, F_n is a union of countably many sets that are closed and discrete in X, and hence in T.

We have proved that ζ has a σ-discrete closed refinement. Thus T is perfect and subparacompact. A perfect space is hereditarily orthocompact if it is σ-orthocompact ([2]). Let η be an open cover of T. For each $H \in \eta$ let H^0 denote the interior of H in the plane topology. Set $\eta^0 = \{H^0: H \in \eta\}$ and set $T_0 = \bigcup \eta^0$. Note η^0 has a σ-locally finite open refinement even in the plane topology. Let $E = T - T_0$, $E_n = \{t \in E: C(t, 1/n) \in H$ for some $H \in \eta\}$. We have $E = \bigcup_{n=1}^{\infty} E_n$. We shall construct for every E_n an open interior preserving collection which refines $\{C(t, 1/n): t \in E_n\}$ and covers E_n.

Let β be a base of the plane topology, $\beta = \bigcup_{k=1}^{\infty} \beta_k$, where β_k are point finite and for every $U \in \beta_k$ $\operatorname{diam} U = \sup \{|t - t'|: t, t' \in U\} < 1/k$.

For $t \in E_n$, $t' \in C(t, 1/2n)$ let $k(t, t')$ be the smallest k such that there exists $U \in \beta_k$ with $t' \in U \subset C(t, 1/n)$, and let $U(t, t')$ be such a U. Let us note that if one has sequences $t_j \in E_n$, $t'_j \in C(t, 1/2n)$, $t_j \neq t'_j$, then (1) $k(t_j, t'_j) \to \infty \Rightarrow |t_j - t'_j| \to 0$.

We put $U(t) = \bigcup \{U(t, t'): t' \in C(t, 1/2n)\} \cup \{t\}$. Obviously, $U(t) \subset C(t, 1/n)$.

The collection $\{U(t): t \in E_n\}$ is interior preserving. Otherwise pick some $t_0 \in T$ such that $\bigcap \{U(t): t_0 \in U(t)\}$ is not a neighborhood of t_0. Since any $U(t)$ is an union of some $U(t, t') \in \beta_{k(t,t')}$ and any β_k is interior preserving, there exist sequences $t_j \in E_n$ and $t'_j \in C(t_j, 1/2n)$ such that (2) $t_0 \in U(t_j, t'_j)$ for any $j = 1, 2, \cdots$ and (3) $k(t_j, t'_j) \to \infty$. From (3) and (1) it follows that $|t_j - t'_j| \to 0$, from (2) and from the definitions of $U(t, t')$ and β_k it follows that $|t'_j - t_0| < \operatorname{diam} U(t_j, t'_j) < 1/k(t_j, t'_j) \to 0$, so $|t_0 - t_j| \to 0$ and since all $t_j \in E$ and E is closed in the plane topology we have $t_0 \in E$. However, for some t_j we have $t_0 \neq t_j$, and $t_0 \in U(t_j) - \{t_j\} \subset C(t_j, 1/n) - \{t_j\} \subset H^0 \in \eta^0$, hence $t_0 \in T_0$. Thus $t_0 \in T_0 \cap E$ — *a* contradiction.

We have proved that η has a σ-interior preserving refinement. Hence T is σ-orthocompact, and therefore, as mentioned above, it is hereditary orthocompact.

REMARK. By the same arguments it can be proved that the space X of H. W. Martin's Example 3 of [11] is orthocompact; this answers Question 1 of [11].

Let (X, d) be a quasi-metric space, $B(x, r)$ be a d-sphere, and set $X^{\vee} = X \times D$. We define a generalized metric $d^{\vee}$ on $X^{\vee}$ such that for $r \leqq 1$ the $d^{\vee}$-spheres $B^{\vee}(\langle x, 1/j\rangle, r) = B(x, r/j) \times \{1/j\}$ and $B^{\vee}(\langle x, 0\rangle, r) = \bigcup_{1/j<r} B^{\vee}(\langle x, 1/j\rangle, r) \cup \{\langle x, 0\rangle\}$. For $r > 1$ we put all $d^{\vee}$-spheres $B^{\vee}(\langle x, y\rangle, r) = X^{\vee}$. It follows that $d^{\vee}$ is a quasi-metric and that if X is Hausdorff that so is $X^{\vee}$.

THEOREM 1. (i) *$X^{\vee}$ is a union of countably many disjoint clopen subspaces homeomorphic to X and a discrete subspace of the same cardinality as that of X.*

(ii) *If d is continuous in the second variable then so is $d^{\vee}$.*

(iii) *If X is perfect (subparacompact, submetrizable) then so is $X^{\vee}$.*

(iv) *If X is not non-archimedeanly quasi-metrizable then $X^{\vee}$ is not σ-orthocompact.*[3]

Proof. (i) is obvious. (ii) follows from the following criterion due to R. Stoltenberg [15]: a quasi-metric d is continuous in the second variable iff for every $x \in X$, $0 < r < r'$ one has $\operatorname{cl} B(x, r) \subset B(x, r')$. (iii) The topology of the product of X with the metric space D is coarser than that of $X^{\vee}$, hence $X^{\vee}$ is submetrizable if X is. The rest follows from (i). (iv) Let ζ be a σ-interior preserving refinement of the open cover $\{B^{\vee}(\langle x, 0\rangle, 1): x \in X\}$, and let $\pi_j: X \to X^{\vee}$ be defined by $\pi_j(x) = \langle x, 1/j\rangle$. Then $\{\pi_j^{-1}G: G \in \zeta, j = 1, 2, \cdots\}$ is a σ-interior preserving open collection in X. It is also a base in X because if U is a neighborhood of $x \in X$, $B(x, 1/j) \subset U$ and $\langle x, 0\rangle \in G \in \zeta$, where $G \subset B^{\vee}(\langle x, 0\rangle, 1)$, then one has $x \in \pi_j^{-1}(G) \subset \pi_j^{-1}(B^{\vee}(\langle x, 0\rangle, 1)) = B(x, 1/j) \subset U$, $\pi_j^{-1}(G) \in \pi_j^{-1}(\zeta)$. This completes the proof.

The following proposition is a consequence of Proposition 1 and Theorem 1.

PROPOSITION 2. *The space $T^{\vee}$ is perfect, subparacompact, submetrizable, quasi-metrizable via a quasi-metric which is continuous in the second variable, but is not σ-orthocompact.*

The notion of neighbornet due to H. Junnila (]5]) helps to unify some definitions.

A reflexive binary relation U on a space X is a *neighbornet* provided that for any $x \in X$ the set $U(x)$ is a neighborhood of x. A sequence $\langle U_n\rangle$ of neighbornets is *basic* provided that for any

[3] Some constructions non-orthocompact *non-quasi-metrizable* spaces based on quite a different idea have been given in [4].

$x \in X$ the sequence $\langle U_n(x) \rangle$ is a base of neighborhoods of x ([5]). A neighbornet U (a sequence of neighbornets $\langle U_n \rangle$) *refines* a cover ζ of X provided that for any $x \in X$ there exists some $G \in \zeta$ such that $U(x) \subset G(U_n(x) \subset G$ for some n). A neighbornet U_0 is *double* (*normal*) provided that there exists another neighbornet U_1 (a sequence of neighbornets $U_1, U_2, \cdots$) such that $U_0 \supset U_1^2$ ($U_n \supset U_{n+1}^2$ any $n \geqq 0$). A space X is orthocompact (has the *Lebesque property*, is *pre-orthocompact*) iff for any open cover ζ of X there exists a transitive (normal, double) neighbornet which refines ζ, *countably orthocompact* (*countably preorthocompact*) iff for any countably open cover ζ of X there exists a transitive (double) neighbornet which refines X; σ-orthocompact (has the *σ-Lebesque property, is σ-refinable*) iff for any open cover ζ of X there exists a sequence of transitive (normal, double) neighbornets which refines ζ, and non-archimedeanly quasi-metrizable (quasi-metrizable, a *γ-space*) iff there exists a basic sequence transitive (normal, double) neighbornets ([5], [2], [3], [8]).

REMARKS. H. Junnila mentioned in a letter to the author that the $^{\vee}$-construction preserves the Lebesgue property, any since T is orthocompact and therefore has the Lebesgue property, $T^{\vee}$ is an example of a quasi-metric space with the Lebesgue property which is not *σ-orthocompact*.

We note that σ-orthocompactness does not imply the orthocompactness in quasi-metric spaces. A σ-orthocompact non-orthocompact quasi-metric space was found by P. Fletcher. E. K. van Douwen and H. H. Wicke [1] have constructed an example of a regular non-archimedianly quasi-metrizable space Γ' which is not countably orthocompact. Moreover, it can be shown that Γ' is not even countably preorthocompact.

$$
\begin{tikzcd}
\text{orthocompact} \arrow[dr, Rightarrow, "(\Gamma')"] \arrow[dd, Rightarrow, "(T^{\vee})"] & & \text{n.-a.quasi-metrizable} \arrow[dl, Rightarrow, "(C)"'] \arrow[dd, Rightarrow, "(T)"] \\
& \sigma\text{-orthocompact} \arrow[dd, Rightarrow, "(T^{\vee})"] & \\
\text{Lebesgue property} \arrow[dr, Rightarrow, "(\Gamma')"] \arrow[dd, Rightarrow, "(?)"] & & \text{quasi-metrizable} \arrow[dl, Rightarrow, "(C)"'] \arrow[dd, Rightarrow, "(?)"] \\
& \sigma\text{-Lebesgue property} \arrow[dd, Rightarrow, "(?)"] & \\
\text{preorthocompact} \arrow[dr, Rightarrow, "(\Gamma')"] & & \gamma\text{-space} \arrow[dl, Rightarrow, "(C)"'] \\
& \sigma\text{-refinable} &
\end{tikzcd}
$$

Let us consider the following diagram:

All implications of the diagram are obvious. Those marked with T, $T^{\vee}$, Γ', and C are irreversible by the counterexamples indicated, where C is an arbitrary compact nonmetrizable space. The problem of the reversibility of the other implications is open.

PROPOSITION 3. *(σ-)orthocompactness is not summable in quasi-metric spaces, namely, $T^{\check{}}$ is a sum of an open set homeomorphic to T and a discrete subspace.*

Proof. Since T is the countable union of disjoint open and closed mutually homeomorphic subsets, the desired result follows from Theorem 1.

PROPOSITION 4. *(σ-)orthocompactness is not multiplicative, namely the product of the space T with the Sorgenfrey line Z is not σ-orthocompact; neither is $T \times T$.*

Proof. Z is homeomorphic to closed subspace of T, hence it is enough to show that $T \times Z$ is not σ-orthocompact. Let $t \in T$, $z \in Z$, $a = \langle t, z\rangle \in T \times Z$. The sets $S(a, r) = C(t, r) \times [z, z + r)$, $r > 0$ form a base of neighborhoods of a. The set $\{t\} \times [z, z + r)$ will be called $I(a, r)$.

The underlying set of T is a plane. Any line parallel to x-axis is a discrete set in T. Hence the plane $P_1 = \{\langle x, y, z\rangle : y + z = 0, \langle x, y\rangle \in T, z \in Z\}$ is a discrete set in $T \times Z$ while any subspace of $T \times Z$ whose underlying set is a plane P parallel to $P_2 = \{\langle x, y, z\rangle : y - z = 0, \langle x, y\rangle \in T, z \in Z\}$ is homeomorphic to T, and the orthogonal projection $\pi : P \to T$ is a homeomorphism.

Let us show that the open cover $\zeta = \{S(a, 1): a \in P_1\}$ of a clopen set $F = U\zeta$ has no σ-interior preserving refinement η. Otherwise let $S(a)$ be some member of η containing $a \in P_1$. One has $S(a) \subset S(a, 1)$. For some k the subset $P_1(k) = \{a \in P_1 : S(a, 1/k) \subset S(a)\}$ is of the second category in the plane topology of P_1. Since the sets $I(a, 1/k)$ with $a \in P_1(k)$ have the same "height", hence the intersection of $\cup \{I(a, 1/k): a \in P_1(k)\}$ with some plane P parallel to P_2 denoted by P_0 is of the second category in the plane topology of P. Therefore the set $T_0 = \pi(P_0)$ with $T_0 \subset T$ is of the second category in the plane topology. Let $t \in T_0$, $t = \pi(b)$ and $\{b\} = I(a, 1/k) \cap P$. The set $\pi(S(a) \cap P)$ will be called $U(t)$. Since $t \in \pi(I(a, 1/k) \cap P) \subset \pi(S(a, 1/k) \cap P) \subset U(t)$ hence $U(t)$ is a neighborhood of t in T. Since the collection of all $S(a)$ is σ-interior preserving in $T \times Z$, hence the collection of all $S(a) \cap P$ is σ-interior preserving in the subspace P of $T \times Z$, and since π is a homeomorphism, we get that the collection of all $U(t)$, where $t \in T_0$, is σ-interior preserving. We also have $U(t) = \pi(S(a) \cap P) \subset \pi(S(a, 1) \cap P) \subset 0(t, 1)$, and since T_0 is of the second category in the plane topology, we get a contradiction to the lemma. Hence ζ has no σ-interior preserving refinement and $T \times Z$ is not σ-orthocompact.

PROPOSITION 5. *(CH) The (σ-)orthocompactness is not hereditary in quasi-metric spaces, namely there exists a regular Lindelöf non-archimedean quasi-metric space T^L which is not hereditary σ-orthocompact.*

Proof. If the continuum hypothesis is valid, then there exists an uncountable subspace T_0 of T such that the trace of T_0 on each nowhere dense subset in the plane topology is countable ([10]). Note that $T - T_0$ is dense. T_0 is of the second category in the plane topology, and the subspace T_0 of T is Lindelöf. Indeed, if ζ is an open cover of T_0 and ζ' is a subcollection of ζ which covers some dense set in T, then the complement of $U\zeta'$ in T_0 is countable by the definition of T_0. The same arguments imply that the spaces obtained from the plane by scattering the points of T_0 is also Lindelöf.

Let $B(t, r)(B_0(t, r))$ be spheres of some quasi-metric of the plane (of the subspace T_0 of T) and $B(t, r) \subset B_0(t, r)$ for $t \in T_0$. We define a space T^L with the underlying set $(T_0 \times D) \cup (S - T_0) \times \{0\})$, and with the generalized metric d^L on T^L such that for $r \leqq 1$ d^L-spheres $B^L(\langle t, 0\rangle, r) = B((t, r) \times [0, r)) \cap T^L$ for $t \in S - T_0$ and, $B^L(\langle t, x\rangle, r) = B_0^{\vee}(\langle t, x\rangle, r)$ for $t \in T_0$. For $r > 1$ all $B^L(\langle t, x\rangle, r) = T^L$. It follows that the subspace $T_0 \times D$ of T^L is isometric to $T_0^{\vee}$, d^L is a quasi-metric and T^L is regular. Since T_0 is of the second categoy in the plane topology, it follows from the lemma that T_0 is not non-archimedeanly quasi-metrizable and $T_0^{\vee}$ is not σ-orthocompact, neither is the subspace $T_0 \times D$ of T^L. Since $S \times \{0\}$ is Lindelöf and for each $x \in D - \{0\}$, $T_0 \times \{x\}$ is Lindelöf, the desired space T^L is Lindelöf.

2. A quasi-uniformity on X is *transitive* provided that it has a base consisting of transitive binary relations ([3]).

THEOREM 2. *Let f be a perfect map from X onto Y. If the space X is quasi-uniformizable via a (transitive) quasi-uniformity with a base of cardinality $\leqq m$, then so is Y.*

Proof. Let $\mathscr{U}$ be a quasi-uniformity on X with a base $\mathscr{B}$ of cardinality $\leqq m$. For any $U \in \mathscr{U}$ the binary relation $U^Y = \{(y, y') \in Y \times Y: U(f^{-1}(y)) \supset f^{-1}(y')\}$ is reflexive in Y. If $U_1 \subset U_2$ then $U_1^Y \subset U_2^Y$. If $U_1 \circ U_1 \subset U_2$, then $U_1^Y \circ U_1^Y \subset U_2^Y$. Thus $\{U^Y | U \in \mathscr{B}\}$ is a base of cardinality $\leqq m$ for a quasi-uniformity $\mathscr{U}^Y$ on Y which is transitive if $\mathscr{U}$ is transitive.

We shall show that $\mathscr{U}^Y$ is compatible with the topology of Y, i.e., for any $y \in Y$, $E \subset Y$ one has that $y \in \operatorname{cl} E$ if, and only if, for each $U \in \mathscr{U}$, $U^Y(y) \cap E \neq \emptyset$.

For $U \in \mathscr{U}$, $y \in Y$ define $U_Y(y) = f(U(f^{-1}(y)))$. As $U^Y(y) = Y -$

$f(X - U(f^{-1}(y)))$ and f is a surjection, we obtain $U^Y(y) \subset U_Y(y)$.

If now $U^Y(y) \cap E \neq \varnothing$ for every $U \in \mathscr{U}$, then also $U_Y(y) \cap E \neq \varnothing$ and $A(U) = \{z \in f^{-1}(y)\colon U(z) \cap f^{-1}(E) \neq \varnothing\} \neq \varnothing$.

Since $A(U_1) \subset A(U_2)$, whenever $U_1 \subset U_2$, it follows that $\{A(U) \mid U \in \mathscr{U}\}$ is a filter base on the compact set $f^{-1}(y)$ with a limit point $x \in f^{-1}(y)$.

For any $U \in \mathscr{U}$ let $U' \in \mathscr{U}$, $U' \circ U' \subset U$ and let $z \in U'(x) \cap A(U')$. One obtains $U'(z) \cap f^{-1}(E) \neq \varnothing$ and $U(x) \cap f^{-1}(E) \neq \varnothing$. Since $\mathscr{U}$ is compatible with the topology of X, $x \in \operatorname{cl} f^{-1}(E)$, and since f is continuous, $y = f(x) \in \operatorname{cl} ff^{-1}(E) = \operatorname{cl} E$.

Let now $y \in \operatorname{cl} E$ and let $U \in \mathscr{U}$. Then $U^Y(y) \cap E \neq \varnothing$. Indeed, $(f^{-1}(y))$ is a neighborhood of $f^{-1}(y)$, i.e., $f^{-1}(y) \subset \operatorname{int} U(f^{-1}(y))$. Thus $E \not\subset f(X - U(f^{-1}(y)))$. Otherwise $E \subset f(X - \operatorname{int} U(f^{-1}(y)))$ and since f is closed, $\operatorname{cl} E \subset f(X - \operatorname{int}(f^{-1}(y))) \subset f(X - f^{-1}(y)) = Y - \{y\}$.

Therefore, $\varnothing = E \cap (Y - f(X - U(f^{-1}(y)))$ and as it was mentioned above, $Y - f(X - U(f^{-1}(y))) = U^Y(y)$, so that $\varnothing \neq E \cap U^Y(y)$. The theorem is proved.

Since a space is (non-archimedeanly) quasi-metrizable iff it is uniformizable via a (transitive) quasi-uniformity with a countable base, the two following corollaries are valid.

COROLLARY 1. *A perfect image of a quasi-metrizable space is quasi-metrizable.*

COROLLARY 2. *A perfect image of a non-archimedeanly quasi-metrizable space is non-archimedeanly quasi-metrizable.*

REMARK. A direct proof of the last result is given by the author in [9][4].

ACKNOWLEDGMENTS. The author thanks Peter Fletcher and Heikki Junnila in particular for calling his attention to the subjects of §1 and in general for their helpful cooperation. The author gratefully acknowledges the privilege of having seen [4] and [5] prior to publication.

REFERENCES

1. E. K. van Douwen and H. H. Wicke, *A real, weird topology on the reals,* Houston J. Math., **3**, No. 1 (1977), 141-152.

2. P. Fletcher and W. F. Lindgren, *Orthocompactness and strong Cech completeness in Moore spaces,* Duke Math., **39** No. 4 (1972), 753-766.

3. ———, *Transitive quasi-uniformities,* J. Math. Anal. Appl., **39** No. 2 (1972), 397-405.

[4] P. Fletcher informed me that the same result was proved by C. Aull.

4. R. W. Heath and W. F. Lindgren, *On generating non-orthocompact spaces. Set-Theoretic Topology* (Inst. Med. Math., Ohio University, Ohio), Academic Press (1977), 225-237.

5. H. Junnila, *Neighbornets*, Pacific J. Math., (1978), to appear.

6. J. Kofner, *On a new class of spaces and some problems of symmetrizability theory*, Dokl. Akad. Nauk SSSR, **187** No. 2 (1969), 270-272 = Soviet Math. Dokl., No. 4 (1969), 845-848.

7. ———, *Pseudo-stratifiable spaces* (Russian), Fund. Math., **70** No. 1 (1971), 25-45.

8. ———, *On Δ-metrizable spaces*, Mat. Zametki, **13** No. 2 (1973), 277-287 = Math. Note, **13** No. 2 (1973), 168-174.

9. ———, *Semi-stratifiable spaces and spaces with generalized metrics*, Ph. D. thesis, the Technion-IIT, Haifa, Israel, (1975).

10. N. Lusin, *Sur un problème de M. Baire*, C. R. Acad. Sci. Paris, **158** (1914), 1258-1261.

11. H. W. Martin, *Local connectedness in developable spaces*, Pacific J. Math., **61** (1975), 219-224.

12. S. I. Nedev, *Generalized-metrizable spaces* (Russian), C. R. Acad. Bulgare Sci., **20** No. 6 (1967), 513-516.

13. S. I. Nedev and M. M. Čoban, *On the theory of* 0-*metrizable spaces, III*, (Russian), Vestnik Moskov, Univ., Ser. 1, Mat. Meh., **27** No. 3 (1972), 10-15.

14. M. Sion and R. C. Willmott, *Hausdorff measures on abstract spaces*, Trans. Amer. Math. Soc., **123** (1966), 275-309.

15. R. A. Stoltenberg, *On quasi-metric spaces*, Duke Math. J., **36** No. 1 (1969), 65-71.

Received March 3, 1978 and in revised form May 31, 1978.

UNIVERSITY OF HAIFA
HAIFA, ISRAEL