PACIFIC JOURNAL OF MATHEMATICS
Vol. 89, No. 1, 1980

SQUARE-FREE AND CUBE-FREE COLORINGS OF THE ORDINALS

JEAN A. LARSON, RICHARD LAVER AND GEORGE F. MCNULTY

We prove: Theorem 1. The class of all ordinals has a square-free 3-coloring and a cube-free 2-coloring. Theorem 2. Every kth power-free n-coloring of α can be extended to a maximal kth power-free n-coloring of β, for some $\beta \times \alpha \cdot \omega$, where $k, n \in \omega$.

Every ordinal is conceived as the set of all smaller ordinals; ω is the least infinite ordinal. By an *interval of ordinals* we mean any set $\{\delta: \beta \leqq \delta < \gamma\}$ where β and γ are ordinals; $[\beta, \gamma)$ abbreviates $\{\delta: \beta \leqq \delta < \gamma\}$. If S and T are intervals then there can be at most one order isomorphism from S onto T.

Let S be an interval of ordinals and κ be a cardinal. A *κ-coloring* of S is just a function with domain S and range included in κ. Suppose S and T are intervals of ordinals and that f is a coloring of S while g is a coloring of T. Then the coloring f of S is *similar* to the coloring g of T provided S and T are order isomorphic and $f(\alpha) = g(h(\alpha))$ for all $\alpha \in S$ where h is the unique order isomorphism from S onto T; if f and g are clear from the context we say that S is similar to T. A coloring f of the ordinal α is *square-free* if no two adjacent nonempty intervals of α are similar; it is *cube-free* if no three consecutive nonempty intervals are all similar to each other. All these notions extend naturally to the class of all ordinals.

In Bean, Ehrenfeucht, and McNulty [1] it was shown that α has a square-free 3-coloring and a cube-free 2-coloring whenever $\alpha < (2^{\aleph_0})^+$ and the question of extending this result to all ordinals was left open. This question is resolved here.

THEOREM 1. *The class of all ordinals has a square-free* 3*-coloring and a cube-free* 2*-coloring.*

If I is a class of ordinals and α_β is an ordinal for each $\beta \in I$, then $\sum_{\beta \in I} \alpha_\beta$ denotes the *ordinal sum* of the α_β's with respect to I. (See Sierpinski [2] for details.) Finite ordinal sums are written like $\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}$. For each $\beta \in I$, let $\mathrm{Int}(\beta) = [\mu, \mu + \alpha_\beta)$ where $\mu = \sum_{\gamma \in J} \alpha_\gamma$ and $J = I \cap \beta$. For each $\beta \in I$, $\mathrm{Int}(\beta)$ is order isomorphic with α_β. In fact, $\sum_{\beta \in I} \alpha_\beta$ can be construed as the disjoint union of the $\mathrm{Int}(\beta)$'s as $\beta \in I$ where the intervals are given the order type of I. This means that if f_β is a κ-coloring of α_β, for each $\beta \in I$, then there is a κ-coloring f of $\sum_{\beta \in I} \alpha_\beta$ such that $f \upharpoonright \mathrm{Int}(\beta)$ is similar to f_β.

An ordinal α is (*additively*) *indecomposable* provided $\alpha \neq \beta + \gamma$ whenever $\beta < \alpha$ and $\gamma < \alpha$. It is known (cf. Sierpinski [2]) that every ordinal is the ordinal sum of finitely many indecomposable ordinals and that the infinite indecomposable ordinals are exactly the ordinal powers of ω.

LEMMA 0. *If α is the class of all ordinals or α is an indecomposable ordinal with $\alpha > \omega$, then α is the sum of a strictly increasing sequence of smaller limit ordinals.*

Proof. There are three cases. First, suppose $\alpha = \omega^\beta$ where β is a limit ordinal. So $\alpha = \omega^\beta = \sum_{\gamma<\beta} \omega^\gamma$. Second, suppose $\alpha = \omega^{\beta+1}$. Then $\alpha = \omega^{\beta+1} = \omega^\beta \cdot \omega = \sum_{n \in \omega}(\omega^\beta \cdot n)$. Third, the class of all ordinals is $\sum_{\kappa \in I} \kappa$, where I is the class of cardinals. In each case the lemma holds.

Let f be a coloring of the interval S of ordinals and let g be a coloring of the interval T. S and T are *mismatched* provided that U and V fail to be similar whenever U is an infinite subinterval of S and V is an infinite subinterval of T. Theorems 1.8 and 1.16 from Bean, Ehrenfeucht, and McNulty [1] are collected in the next lemma.

LEMMA 1. (a) *There is a collection $\mathscr{F}$ of square-free 3-colorings of ω such that $|\mathscr{F}| = 2^{\aleph_0}$ and C and D are mismatched whenever $C, D \in \mathscr{F}$ with $C \neq D$.*

(b) *There is a collection $\mathscr{S}$ of cube-free 2-colorings of ω such that $|\mathscr{S}| = 2^{\aleph_0}$ and C and D are mismatched whenever $C, D \in \mathscr{S}$ with $C \neq D$.*

Proof of Theorem 1. We will provide a proof that the class of all ordinals has a square-free 3-coloring. This proof can be easily modified to establish that the class of all ordinals has a cube-free 2-coloring. The property of having a square-free 3-coloring is hereditary in the sense that if α has a square-free 3-coloring and $\beta < \alpha$, then β has a square-free 3-coloring. Below we are concerned with providing each limit ordinal with a square-free 3-coloring and we proceed by induction.

Induction hypothesis. If α is an infinite limit ordinal or the class of all ordinals, and $f_0, f_1, \cdots$ are countably many square-free 3-colorings of ω such that f_i and f_j are mismatched whenever $i, j \in \omega$ with $i \neq j$, then there is a 3-coloring g of α such that

(i) g is square-free.

(ii) g and f_i are mismatched for each $i \in \omega$.

(iii) Any two similar infinite intervals of α are separated by an infinite interval.

Suppose the induction hypothesis holds for all infinite limit ordinals less than α and that $f_0, f_1, f_2, \cdots$ are countably many pairwise mismatched square-free 3-colorings of ω. There are two cases.

Case 1. $\alpha = \rho_0 + \rho_1 + \cdots + \rho_n$ where $\rho_0, \cdots, \rho_n$ are indecomposable and $0 < n \in \omega$.

According to Lemma 1 there must be $h_0, \cdots, h_n$, all square-free 3-colorings of ω, such that $h_0, h_1, \cdots, h_n, f_0, f_1, \cdots$ are all pairwise mismatched. By the induction hypothesis there are 3-colorings $d_0, \cdots, d_n$ of $\rho_0, \cdots, \rho_n$ respectively such that for each $i \leqq n$

(i)′ d_i is square-free.

(ii)′ $d_i, h_0, h_1, \cdots, h_n, f_0, f_1 \cdots$ are all pairwise mismatched.

(iii)′ Any two similar infinite intervals of ρ_i are separated by an infinite interval.

For each $i \leqq n$ and each $\gamma \in \rho_i$, let

$$d_i^*(\gamma) = \begin{cases} h_i(\gamma) & \text{if } \gamma \in \omega \\ d_i(\gamma) & \text{otherwise}, \end{cases}$$

and let g be the coloring of α induced by $d_0^*, \cdots, d_n^*$.

Condition (ii) of the induction hypothesis holds by (ii)′. To check condition (iii) suppose S and T are distinct similar infinite intervals of α. Since h_i and d_j are mismatched whenever $i, j \leqq n$ and since $h_0, h_1, \cdots, h_n$ are pairwise mismatched, for each $i \leqq n$ there is exactly one interval U of α (of order type ω) such that $f|U$ is similar to h_i. Since S and T are distinct but similar neither can have a subinterval similar to any of $h_0, h_1, \cdots, h_n$ or *any of their final segments*. Consequently there are $i, j \leqq n$, with finite initial segments δ of ρ_{i+1} and ε of ρ_{j+1} such that S is a subinterval of $\rho_i + \delta$ missing the initial segment of ρ_i of order-type ω, while T is a subinterval of $\rho_j + \varepsilon$ missing the initial segment of ρ_j of order-type ω. If $i \neq j$ then (iii) follows immediately, so suppose $i = j$. There must be cofinite initial segments S' of S and T' of T such that S' and T' are distinct yet similar and both S' and T' are subintervals of ρ_i missing the initial segment of ρ_i of order type ω. So S' and T' are colored by d_i and by (iii)′ they are separated by an infinite interval and therefore S and T are separated by an infinite interval as well.

To see that g is a square-free coloring of α, observe that (iii) forces any two similar adjacent intervals to be finite. But g was devised so that *all* intervals of α of order type ω are colored in a square-free manner. Hence g is square-free and Case I of the induction is complete.

Case II. α is indecomposable with $\alpha > \omega$.

By Lemma 0 $\alpha = \sum_{\gamma \in \beta} \rho_\gamma$ for some β where $\rho_\gamma < \rho_\delta < \alpha$ and ρ_γ is an infinite limit ordinal, if $\gamma < \delta < \beta$. According to Lemma 1 there must be h_0 and h_1, both square-free 3-colorings of ω such that $h_0, h_1, f_0, f_1, f_2, \cdots$ are pairwise mismatched. By the induction hypothesis for each $\gamma \in \beta$ there is a 3-coloring d_γ of ρ_γ such that

(i)″ d_γ is square-free.

(ii)″ $d_\gamma, h_0, h_1, f_0, f_1, f_2, \cdots$ are pairwise mismatched.

(iii)″ Any two similar infinite intervals of ρ_γ are separated by an infinite interval

$$d_\gamma^*(\delta) = \begin{cases} h_0(\delta) & \text{if} \quad \delta \in \omega \quad \text{and} \quad \gamma \text{ is even} \\ h_1(\delta) & \text{if} \quad \delta \in \omega \quad \text{and} \quad \gamma \text{ is odd} \\ d_\gamma(\delta) & \text{otherwise}\,, \end{cases}$$

and let g be the coloring of α induced by $\langle d_\gamma^* \colon \gamma \in \beta \rangle$.

Conditions (i) and (ii) of the induction hypothesis can be established as in Case I. We argue that (iii) holds. Suppose S and T are similar infinite intervals in α. If S contains an interval of type ω colored the way h_0(or h_1) colors some final segment of ω then the same is true for T. According to the construction of g these kinds of colorings occur only on the initial segments of each ρ_γ of type ω. Since the ρ_γ's form an increasing sequence, no interval between an interval colored with h_0 and the next colored with h_1 occurs twice. So if S contains an interval of type ω colored the way h_0 (or h_1) colors some final segment of ω, then S does *not* contain an interval of type ω colored the way h_1 (alternatively h_0) colors some final segment of ω. The same is true for T. Consequently, if S and T were separated by a finite interval, then both S and T would lie entirely in $\rho_\gamma + \delta$ for some γ where δ is a finite initial segment of $\rho_{\gamma+1}$. From this point the argument proceeds as in Case I.

Since Lemma 1 guarantees the theorem when $\alpha = \omega$, the induction is complete and the theorem established.

For any $k \in \omega$, kth power-free colorings have definitions analogous to those of square-free and cube-free colorings. Every square-free coloring is kth power-free for all $k \geqq 2$. A kth power-free κ-coloring f of α is *maximal* provided f cannot be extended to a kth power-free κ-coloring of $\alpha + 1$. In Bean, Ehrenfeucht, and McNulty [1] it is shown that every kth power-free n-coloring f of m can be extended to a maximal kth power-free n-coloring of some natural number, whenever $k, n, m \in \omega$. We remark that the following theorem holds. The proof differs in no important way from the proof of Theorem 2.0 in [1].

THEOREM 2. *For any natural numbers n and k and any ordinal α, every kth power-free n-coloring of α can be extended to a maximal kth power-free n-coloring of β for some $\beta \in \alpha \cdot \omega$.*

References

1. D. Bean, A. Ehrenfeucht, and G. McNulty, *Avoidable patterns in strings of symbols*, Pacific J. Math., **85** (1979), 261-294.
2. V. Sierpinski, *Cardinal and Ordinal Numbers*, Warsawa 1958.

Received February 7, 1978.

UNIVERSITY OF FLORIDA
GAINESVILLE, FL 32611
UNIVERSITY OF COLORADO
BOULDER, CO 80302
AND
UNIVERSITY OF SOUTH CAROLINA
COLUMBIA SC 29208