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ON THE ZEROS OF CONVEX COMBINATIONS OF POLYNOMIALS

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Given monic nth degree polynomials $P_0(z)$ and $P_1(z)$, let $P_A(z) = (1 - A)P_0(z) + AP_1(z)$. If the zeros of P_0 and P_1 all lie in a circle $\mathscr{C}$ or on a line L, necessary and sufficient conditions are given for the zeros of P_A $(0 \leq A \leq 1)$ to all lie on $\mathscr{C}$ or L. This describes certain convex sets of monic nth degree polynomials having zeros in $\mathscr{C}$ or L. If the zeros of P_0 and P_1 lie in the unit disk and P_0 and P_1 have real coefficients, then the zeros of P_A $(0 \leq A \leq 1)$ lie in the disk $|z| < \cos(\pi/2n)/\sin(\pi/2n)$. A set is described which includes the locus of zeros of $P_A (0 \leq A \leq 1)$ as P_0 and P_1 vary through all monic nth degree polynomials having all their zeros in a compact set K. When K is path-connected, this locus is exactly the set described.

Given polynomials $P_0(z)$ and $P_1(z)$, let $P_A(z)$ denote the polynomial:

$$P_A(z) = (1 - A)P_0(z) + AP_1(z) .$$

P_A is defined for any complex value of A and the zeros of $P_A(z)$ are continuous functions of A. In particular, if A is varied through the reals between 0 and 1, the locus of zeros of $P_A(z)$ is a network of paths in the plane starting at the zeros of $P_0(z)$ and terminating in the zeros of $P_1(z)$. If the degree of P_0 is higher than that of P_1 then some of the paths of zeros must tend to infinity as A tends to one. It is the aim of this note to describe these loci of zeros when P_0 and P_1 are monic, have the same degree and are constrained to have their zeros on a circle, on a line or in a disk.

First, let P_0 and P_1 be real and have their zeros in $S^1 = \{z \in C: |z| = 1\}$ where C denotes the complex numbers. The following lemma gives a necessary and sufficient condition for the locus of zeros of $P_A(z)$. $(0 \leqq A \leqq 1)$ to be contained in S^1.

LEMMA 1. *Let $P_0(z)$ and $P_1(z)$ be real monic polynomials of degree n with their zeros contained in $S^1 - \{-1, 1\}$. Denote the zeros of $P_0(z)$ by $w_1, w_2, \cdots, w_n$ and of $P_1(z)$ by $z_1, z_2, \cdots, z_n$ and assume:*

$$w_i \neq z_j \quad (1 \leqq i, j \leqq n)$$

and

$$0 < \arg(w_i) \leqq \arg(w_j) < 2\pi$$
$$0 < \arg(z_i) \leqq \arg(z_j) < 2\pi \quad (1 \leqq i < j \leqq n) .$$

Let α_i be the smaller open arc of S^1 bounded by w_i and $z_i(i = 1, \cdots, n)$. Then the locus of zeros of $P_A(z)$ $(0 \leqq A \leqq 1)$ is contained in S^1 if and only if the arcs α_i are disjoint.

Proof. If P_0 and P_1 are fixed, then for each $z \in C$ such that $P_0(z) \neq P_1(z)$ there is a unique value of $A = A(z)$ such that $P_{A(z)}(z) = 0$. The function $A(z)$ is given by:

$$(*) \qquad A(z) = \frac{P_0(z)}{P_0(z) - P_1(z)} = \frac{1}{1 - \dfrac{P_1(z)}{P_0(z)}} = \frac{1}{1 - \dfrac{(z - z_1) \cdots (z - z_n)}{(z - w_1) \cdots (z - w_n)}}$$

if $P_0(z) \neq 0$.

First assume that $P_A(z)$ has all its zeros in S^1 for $0 \leqq A \leqq 1$. When $A = 0$, the zeros of $P_A(z)$ are the w_i. Perturbing A from 0 to 1 will give a trajectory of zeros eminating from each w_i. Each trajectory will pass through a z_i at $A = 1$. Equation (*) implies that no z can be a zero of $P_A(z)$ for two different values of A (unless $P_0(z) = P_1(z) = 0$ which is not the case here). Two trajectories can intersect only at z's which are multiple zeros of P_A for some A. The set of all z which are multiple zeros of $P_A(z)$ for some $A \in C$ is a finite set, as this is the set of $z \in C$ for which $P_A(z)$ and $P'_A(z)$ are both zero. $P'_A(z) = 0$ implies $A(z) = P'_0(z)/(P'_0(z) - P'_1(z))$ if $P'_0(z) \neq P'_1(z)$ and equating this formula for $A(z)$ with that in (*) gives a polynomial that z must satisfy if it is a multiple root of $P_A(z)$ for some A. Hence, two trajectories can cross but not coincide over a curve. If the trajectories are constrained to a circle, they can only intersect at their endpoints. The n disjoint open arcs covered by these trajectories minus their endpoints are clearly the arcs α_i.

Now assume that the arcs α_i are disjoint. Let θ_i denote the angle of the arc α_i. Consider the quotients

$$q_i = \frac{z - z_i}{z - w_i}$$

and

$$q_{n+1-i} = \frac{z - z_{n+1-i}}{z - w_{n+1-1}} = \frac{z - \bar{z}_i}{z - \bar{w}_i} .$$

If $z \in S^1$ and $z \notin \alpha_i \cup \alpha_{n+1-i}$ then:

$$\arg(q_i) = \pm \frac{\theta_i}{2}$$

while

$$\arg(q_{n+1-i}) = \mp\frac{\theta_i}{2}\ .$$

Hence $\arg(q_i q_{n+1-i}) = 0$ and $q_i q_{n+1-i}$ is a positive real. If $z \in S^1 - \bigcup_{i=1}^n \bar{\alpha}_i$ then $P_1(z)\, P_0(z)$ is positive and except at the finite number of $z \in S^1$ where $P_0(z) = P_1(z)$, $A(z)$ is real with either $A(z) < 0$ or $A(z) > 1$. If $z \in \alpha_i$ for some i, then for $j \neq i$, $q_j q_{n+1-j}$ is a positive real. On the other hand,

$$\arg(q_i) = \pm\left(\pi - \frac{\theta_i}{2}\right)$$

while

$$\arg(q_{n+1-i}) = \pm\frac{\theta_i}{2}\ .$$

In this case $\arg(q_i q_{n+1-i}) = \pi$ so $P_1(z)/P_0(z)$ is negative and $A(z)$ is real with $0 < A(z) < 1$.

$A(z)$ is a continuous real-valued function of z on each arc α_i. $A(z)$ takes on the values 0 and 1 at the endpoints w_i and z_i of α_i. $A(z)$ must then take on all values between 0 and 1 on each arc α_i. That is, for each $A(0 \leqq A \leqq 1)$ there is a zero of $P_A(z)$ in each arc α_i. This accounts for all n zeros of $P_A(z)$ so there can be no zeros of $P_A(z)$ outside S^1.

Note that a similar lemma holds for polynomials P_0 and P_1 having their zeros in any circle whose center is on the real line.

THEOREM 1. *Let $\mathscr{C}$ be any circle whose center is on the real line and let γ_i be open arcs in $\mathscr{C} \cap \{z \mid \operatorname{Im} z > 0\}$ for $i = 1, \cdots, k$. The set of (real) monic polynomials of degree $2k$ with zeros $z_1, \bar{z}_1, \cdots, z_k, \bar{z}_k$ where $z_i \in \gamma_i$ $(i = 1, \cdots, k)$ is a convex set of polynomials if and only if the arcs γ_i are disjoint.*

Proof. All that remains is to consider what happens when P_0 and P_1 have zeros in common. In this case,

$$P_0(z) = Q(z)\tilde{P}_0(z)\ ,$$
$$P_1(z) = Q(z)\tilde{P}_1(z)$$

and

$$P_A(z) = Q(z)((1 - A)\tilde{P}_0(z) + A\tilde{P}_1(z))$$

where $\tilde{P}_0(z)$ and $\tilde{P}_1(z)$ satisfy the conditions of Lemma 1. This lemma applied to $(1 - A)\tilde{P}_0(z) + A\tilde{P}_1(z)$ implies the theorem.

COROLLARY 1. *Let P_0, P_1 and α_i be as in Lemma 1. For each*

$z \in S^1$. *Let* $n(z) = \text{card}\{\alpha_i | z \in \alpha_i\}$. *For all* $z \in S^1$ *such that* $P_0(z) \neq P_1(z)$, z *is a zero of* $P_A(z)$ *for some real value of* $A = A(z)$ *and* $0 \leqq A(z) \leqq 1$ *if and only if* $n(z)$ *is odd or* z *is a zero of* P_0 *or* P_1.

Proof. This follows easily from the proof of Lemma 1.

The techniques used in the proof of Lemma 1 applied to polynomials whose zeros lie on a straight line give the following result.

THEOREM 2. *Let* $I_j (j = 1, \cdots, n)$ *be open intervals in a line* $L \subseteqq C$. *The set of monic polynomials of degree* n *having zeros* $\zeta_j (j = 1, \cdots, n)$ *where* $\zeta_j \in I_j$ *is a convex set of polynomials if and only if the intervals* I_j *are disjoint.*

Proof. Let $P_0(z)$ and $P_1(z)$ have zeros $w_1, w_2, \cdots, w_n$ and $z_1, \cdots, z_n$, respectively, where w_j and z_j are in $L(j = 1, \cdots, n)$. Assume that L is directed and that the zeros w_i and z_j are ordered in this direction. Define intervals α_j and quotients q_j as in Lemma 1 and its proof. If P_0 and P_1 are monic and $w_i \neq z_j (i, j = 1, \cdots, n)$ then

$$\arg(q_i) = 0 \quad \text{or} \quad 2\pi \quad z \in L - \alpha_i$$
$$\arg(q_i) = \pi \quad \text{or} \quad -\pi \quad z \in \alpha_i \,.$$

The arguments of Lemma 1 imply that the zeros of $P_A (0 \leqq A \leqq 1)$ are contained in L if and only if the intervals α_i are disjoint and the theorem follows.

Theorem 2 is similar to a result of N. Obreschkoff [2] which states: Let $P(x)$ and $Q(x)$ be two polynomials without common zeros whose degrees differ by at most one. A necessary and sufficient condition that P and Q have only real zeros which separate each other is that the equations $aP + bQ = 0$ have real zeros for all real a and b. In the proof of Theorem 2, the zeros of P_0 and P_1 need not separate each other for $P_A (0 \leqq A \leqq 1)$ to have all its zeros on the line L. The zeros do, however, need to be "paired" which is the condition that the invervals α_i are disjoint. Theorem 2 can be restated in the flavor of Obreschkoff as follows.

THEOREM 2'. *Let* P_0 *and* P_1 *be monic polynomials of the same degree. A necessary and sufficient condition that* P_0 *and* P_1 *have only paired zeros lying on one line* L *is that the polynomials* $P_A = (1 - A)P_0 + AP_1$ *have all their zeros on the line* L *for* A *real,* $0 \leqq A \leqq 1$.

Lemma 1 is essentially Theorem 1 stated in this form. Theorem

3 (below) can also be so formulated.

In Theorem 2, the polynomials P_0 and P_1 are not required to be real and there was no need of the symmetry obtained by having complex conjugate zeros. As linear transformations take circles into lines, there should be a version of Theorem 1 that does not require P_0 and P_1 to be real.

THEOREM 3. *Let* $\mathscr{C}$ *be a circle in the complex plane and let* γ_i $(i = 1, \cdots, n)$ *be disjoint open arcs in* $\mathscr{C}$. *Let* $z_0 \in \mathscr{C} - \bigcup_{i=1}^{n} \gamma_i$. *Then for any* $w_0 \in C$, $w_0 \neq 0$, *the set of polynomials* P *having zeros* $z_i (i = 1, \cdots, n)$ *where* $z_i \in \gamma_i$ *and satisfying* $P(z_0) = w_0$ *is a convex set of polynomials.*

Proof. A transformation of the form

$$w(z) = \alpha\left(\frac{z-\beta}{z-\delta}\right)$$

will take a given circle through β and δ onto a line, sending β to the origin and δ to the point at infinity. The inverse of this transformation is given by

$$z(w) = \delta\left(\frac{w - \alpha\beta/\delta}{w-\alpha}\right).$$

If $P(z)$ is the polynomial, $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ then

$$\begin{aligned} P(z(w)) &= a_n\left(\delta\left(\frac{w-\alpha\beta/\delta}{w-\alpha}\right)\right)^n + \cdots + a_1\left(\delta\left(\frac{w-\alpha\beta/\delta}{w-\alpha}\right)\right) + a_0 \\ &= \frac{1}{(w-\alpha)^n}[a_n\delta^n(w-\alpha\beta/\delta)^n + \cdots + a_1\delta(w-\alpha\beta/\delta)(w-\alpha)^{n-1} \\ &\quad + a_0(w-\alpha)^n] = \frac{1}{(w-\alpha)^n}Q(w)\,. \end{aligned}$$

$Q(w)$ is a polynomial with leading coefficient $P(\delta)$ and $P(z) = 0$ if and only if $Q(w(z)) = 0$. Take $\delta = z_0$ and $\mathscr{C}$ to be the given circle.

If P_0 and P_1 are polynomials in the set described in the statement of this theorem, let Q_0, Q_1 and Q_A be the polynomials associated with P_0, P_1 and P_A by the above. $Q_A = (1 - A)Q_0 + AQ_1$ and the proof of Theorem 2 applies to Q_A as Q_0 and Q_1 have the same leading coefficient. $P_A(0 \leqq A \leqq 1)$ has zeros in the arcs γ_i and $P_A(z_0) = w_0$ so P_A is in the set described.

REMARK. Theorem 3 is not stronger than Theorem 1. Take, for example, $P_0(z) = z^2 + z + 1$ and $P_1(z) = z^2 - z + 1$ then $P_0(z) \neq P_1(z)$ for any $z \in S^1$.

The results presented so far show, in particular, that if P_0 and P_1 are monic real polynomials of the same degree having zeros in S^1 or $\boldsymbol{R}^1$ then any convex combination P_A of P_0 and P_1 will have all its zeros in S^1 or $\boldsymbol{R}^1$ if and only if the zeros of P_0 and P_1 are paired. There remains the question of where the zeros of P_A must lie in general. A special case of a theorem of J.L. Walsh ([1] p. 77) says that if P_0 and P_1 are monic polynomials of degree n with all their zeros contained in the disk $\{z \in C: |z| \leqq 1\}$, then all the zeros of $P_A (0 \leqq A \leqq 1)$ are contained in $\{z \in C: |z| \leqq 1/\sin(\pi/2n)\}$.

This bound on the moduli of the zeros of $P_A (0 \leqq A \leqq 1)$ is optimal. If $|z| = 1/\sin(\pi/2n)$, construct the lines through z which are tangent to the circle S^1 and let w_1 and z_1 be the points of tangency. Then z will be a zero of $P_{1/2}$ if $P_0 = (z - z_1)^n$ and $P_1 = (z - w_1)^n$. If, however, P_0 and P_1 are real polynomials there is a slightly smaller bound on the moduli of zeros of $P_A (0 \leqq A \leqq 1)$.

THEOREM 4. *Let P_0 and P_1 be real monic polynomials with their zeros contained in the unit disk $\{z \in C: |z| \leqq 1\}$. Then the zeros of $P_A (0 \leqq A \leqq 1)$ are contained in the disk*

$$\left\{z \in C: |z| \leqq \frac{\cos(\pi/2n)}{\sin(\pi/2n)}\right\} .$$

Proof. Let the zeros of P_0 and P_1 be denoted by $z_1, z_2, \cdots, z_n$ and $w_1, w_2, \cdots, w_n$ and assume that if $z_i(w_i)$ is not real then $z_{n+1-i} = \bar{z}_i (w_{n+1-i} = \bar{w}_i)$. Let $q_i = (z - z_i)/(w - w_i)$. As in the proof of Lemma 1, z is a solution of P_A for $0 \leqq A \leqq 1$, if and only if

$$\arg(q_1 q_2 \cdots q_n) = \pi + 2k\pi \quad \text{for some} \quad k \in Z .$$

The following two lemmas show that $|\arg(q_i q_{n+1-i})|$ is maximal for $|z|$ fixed and greater than 1 when z is pure imaginary and $\{z_i, w_i\} = \{-1, +1\} = \{\bar{z}_i, \bar{w}_i\}$. In this case

$$|\arg(q_i)| = 2 \arctan \frac{1}{|z|} ,$$

if $|z| > \cot(\pi/2n)$ then $1/|z| < \tan(\pi/2n)$ and

$$0 < \arg(q_1 \cdots q_n) \leqq 2n \arctan \frac{1}{|z|} < 2n \arctan\left(\tan \frac{\pi}{2n}\right) = \pi$$

which is a contradiction.

LEMMA 2. *Let a and b be two points on a circle of radius 1 with center c. Let p be at distance $r > 1$ from c, then angle apb is maximal (minimal if negative) when angle acp equals angle pcb.*

LEMMA 3. *Let a and b be two points on a circle or radius 1 with center c and let p and p' be the two points on the perpendicular bisector of the segment ab at equal distances from c. Then angle apb + angle $bp'a$ is maximal (minimal if negative) when ab is a diameter of the circle.*

To prove these two lemmas, I had to resort to the Law of Cosines and taking derivatives. The calculations are straightforward but tedious so I omit them here.

The following result shows what happens when the circle is replaced by a given compact set. The result is essentially the same as a theorem due to Nagy and generalized by Marden ([1] p. 32), though they state their results only for polynomials having their zeros in a given convex set.

Let $K \subseteqq C$ be compact. Given $z \in C$, there is a minimal closed sector with vertex z that includes K. Let $\theta(K, z)$ denote the angle of this sector. $0 \leqq \theta(K, z) \leqq 2\pi$ $(z \in C)$.

THEOREM 5. *Let $K \subseteqq C$ be compact. The locus of zeros of P_A $(0 \leqq A \leqq 1)$ as P_0 and P_1 run through all nth degree monic polynomials having all their zeros in K is included in the set*

$$S(K, \pi/n) = \{z \in C \mid \theta(K, z) \geqq \pi/n\} .$$

If K is path-connected, this locus is exactly $S(K, \pi/n)$.

Proof. Let $q_i (i = 1, \cdots, n)$ be defined as in the proof of Lemma 1. If $0 < \theta(K, z) < \pi/n$ then $0 \leqq \arg(q_i \cdots q_i) < \pi$ and z cannot be a zero of $P_A (0 \leqq A \leqq 1)$.

If K is path-connected and $\theta(K, z) \geqq \pi/n$ there exist z_1 and w_1 in S such that $\arg((z - z_1)/(z - w_1)) = \pi/n$. z is a zero of a $P_A(0 \leqq A \leqq 1)$ when $P_0(z) = (z - z_1)^n$ and $P_1(z) = (z - w_1)^n$.

Finally, we return to polynomials having their zeros in a line. The following result is stated for the interval $[-1, +1]$ in R but it generalizes easily to polynomials having their zeros in any line segment in C.

COROLLARY 2. *If P_0 and P_1 are monic nth degree polynomials having all their zeros in $[-1, +1]$ then the locus of zeros of $P_A(0 \leqq A \leqq 1)$ is included in the union of the two disks with diameter $\cot(\pi/2n) + \tan(\pi/2n)$ whose boundaries pass through the points -1 and $+1$.*

Proof. Observing that an inscribed angle on a circle is measured by half the arc it subtends shows that $S([-1, +1], \pi/n)$ is the union of the two disks described.

REFERENCES

1. Morris Marden, *Geometry of Polynomials*, Mathematical Surveys, Number 3, A.M.S. 1966.

2. N. Obreschkoff, *Verteilung und Berechnung der Nullstellen Reeller Polynome*, Deutscher Verlag der Wissenschaften, Berlin, 1963.

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