PACIFIC JOURNAL OF MATHEMATICS
Vol. 90, No. 1, 1980

$[0, \infty]$-VALUED, TRANSLATION INVARIANT MEASURES ON N AND THE DEDEKIND COMPLETION OF $^*R$

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This paper investigates $\{0, \infty\}$-valued, translation invariant measures on the set N of positive integers. The main tool in this investigation is Nonstandard Analysis and especially the completion, $^\# R$, in the sense of Dedekind of the Nonstandard Reals, $^*R$. The algebraic and topological properties of $^\# R$ are developed and exploited to obtain a classification theorem for a particularly nice class of $\{0, \infty\}$-valued, translation invariant measures on N.

1. Introduction. One of the basic problems in mathematics is to define a measure for a suitable collection of subsets of a given set X. When X is the real line there is a unique (up to scale) natural, countably additive, translation invariant measure, namely Lebesgue measure. However, if X is the set of positive integers, $N = \{1, 2, 3, \cdots\}$ then the situation is not so neat. First, since N is countable the only (up to scale) countably additive, translation invariant measure is the measure which assigns $+\infty$ to every infinite set and to every finite set assigns its cardinality. Although, this measure is very important it fails to distinguish between infinite sets in any way. One way to obtain a measure which does make some distinction between different infinite sets is to apply Zorn's lemma to find a nonprincipal ultrafilter $\mathscr{D}$ on N. Such an ultrafilter is a collection of subsets of N satisfying the following properties

(1) $\varnothing \notin \mathscr{D}$, $N \in \mathscr{D}$

(2) $A \in \mathscr{D}$, $A \subseteq B \subseteq N$ implies $B \in \mathscr{D}$

(3) $A, B \in \mathscr{D}$ implies $A \cap B \in \mathscr{D}$

(4) $\cap \{A \mid A \in \mathscr{D}\} = \varnothing$

(5) $\mathscr{D}$ is maximal with respect to properties (1)-(3).

Properties (1)-(3) are the defining properties for a filter on N. Property (4) says that the filter is not principal, and Property (5) says that $\mathscr{D}$ is an ultrafilter. Property (5) is equivalent to (5′).

$$(5') \; A \cup B \in \mathscr{D} \quad \text{implies} \quad A \in \mathscr{D} \quad \text{or} \quad B \in \mathscr{D}.$$

Intuitively, one thinks of the sets in $\mathscr{D}$ as "big" and assigns them measure one while the sets outside of $\mathscr{D}$ are given measure zero. This yields a finitely additive $\{0, 1\}$-valued measure which is extremely useful for many purposes. However, this measure lacks some properties which are intuitively natural. In particular this measure is not translation invariant. Indeed, translation invariance implies that the two sets $A = \{1, 3, 5, \cdots\}$ and $A + 1 = \{2, 4, 6, \cdots\}$ have the same measure contradicting (1), (3) and (5′). The purpose of this paper is to investigate a class of $\{0, \infty\}$-valued measures on N which are translation invariant and satisfy properties (1), (2), (4) and (5′).

DEFINITION I.1. A premeasure on N is a set $\mathscr{E}$ of subsets of N which satisfies the following properties.

(A) $N \in \mathscr{E}$, $\varnothing \notin \mathscr{E}$.

(B) $A \in \mathscr{E}$, $A \subseteqq B \subseteqq N$ implies $B \in \mathscr{E}$.

(C) $A \cup B \in \mathscr{E}$ implies $A \in \mathscr{E}$ or $B \in \mathscr{E}$.

(D) $A \in \mathscr{E}$, $k \in Z$ implies $A + k \in \mathscr{E}$

where Z denotes the set $\{0, +1, -1, +2, -2, \cdots\}$ of all integers.

The sets in $\mathscr{E}$ are thought of as "big" and are assigned infinite measure while those outside of $\mathscr{E}$ are given zero measure. Thus, $\mathscr{E}$ corresponds to a $\{0, \infty\}$-valued, finitely additive, translation invariant measure on N.

EXAMPLES I.2.

(i) Let $\mathscr{E}$ be the collection of all infinite subsets of N. Then $\mathscr{E}$ is easily seen to be a premeasure on N. In fact, $\mathscr{E}$ is the unique maximal premeasure on N since if F is a finite set, $F = \{x_1, x_2, \cdots, x_k\}$ with $x_1 < x_2 < \cdots < x_k$, then F is in no premeasure since if it were, $\varnothing = F - x_k$ would be also, violating (A).

(ii) Let ν be a fixed infinite integer in a Nonstandard Model $^*R$ of the reals. For $A \subseteqq N$ let $A_\nu = \{a \in {}^*A \mid a \leqq \nu\}$ and let $||A_\nu||$ denote the *cardinality of A_ν. Then let $\mathscr{E}_1^\nu$ denote the set of all subsets $A \subseteqq N$ such that $||A_\nu||/\nu$ is not infinitesimal. The collection $\mathscr{E}_1^\nu$ is a premeasure.

(iii) Using the notation above, let $\mathscr{E}_2^\nu$ denote the set of all subsets $A \subseteqq N$ such that $||A_\nu||/\sqrt{\nu}$ is not infinitesimal. Then $\mathscr{E}_2^\nu$ is a premeasure containing $\mathscr{E}_1^\nu$. Notice the set $\{1, 4, 9, 16, \cdots\}$ is contained in $\mathscr{E}_2^\nu$ but not $\mathscr{E}_1^\nu$.

(iv) Let $\mathscr{F}$ be the set of all subsets $A \subseteqq N$ such that $\operatorname{Lim\,inf}_{n\to\infty} (||A_n||/n)$ is positive. That is, $A \in \mathscr{F}$ if and only if $||A_\nu||/\nu$ is not infinitesimal for every infinite ν. $\mathscr{F}$ is not a premeasure. To see this let a_n be any sequence such that $a_1 < a_2 < a_3 \cdots$ and $\operatorname{Lim}_{n\to\infty} a_{n+1}/a_n = \infty$. Then let

$$A = \bigcup_{n=1}^{\infty} [a_{2n}, a_{2n+1}) \cap N$$

$$B = \bigcup_{n=1}^{\infty} [a_{2n-1}, a_{2n}) \cap N .$$

Then $A, B \notin \mathscr{F}$ but $N = A \cup B \in \mathscr{F}$ contradicting (C).

Examples (ii), (iii) and (iv) above indicate the importance of Nonstandard Analysis in the study of premeasures. This is not surprising in view of the fact that one way of representing ultrafilters is via the correspondence between an infinite nonstandard integer ν and the ultrafilter $\mathscr{D}_\nu = \{A \subseteq N \mid \nu \in {}^*A\}$.

In § 5 of this paper we will obtain a similar representation for a particularly nice class of premeasures. However, in this case the correspondence will be between such premeasures and integers in the completion (in the Dedekind sense), ${}^\#R$, of ${}^*R$. §§ 2–4 of this paper are concerned with the construction and investigation of ${}^\#R$.

One of the uncomfortable facts about ultrafilters is their high degree of arbitrariness. In fact, except for finite sets (which are in no nonprincipal ultrafilters) and cofinite sets (which are in every nonprincipal ultrafilter), given any set A there are infinitely many ultrafilters containing A as well as infinitely many which exclude A. Premeasures are also very far from being unique. However, we do have the following lemma.

DEFINITION AND LEMMA I.3. Suppose $A = \{a_1, a_2, a_3, \cdots\}$ is a subset of N with $a_1 < a_2 < a_3 < \cdots$. Then A is said to be *universally big* provided $\mathrm{Sup}(a_{n+1} - a_n) < \infty$. A is universally big if and only if A is contained in every premeasure on N.

Proof.

($\Rightarrow$) Suppose A is universally big, so $\mathrm{Sup}(a_{n+1} - a_n) = k < \infty$. Then $N = \bigcup_{i=0}^{k-1}(A + i)$. Let $\mathscr{E}$ be any premeasure. By (A) $N = \bigcup_{i=0}^{k-1}(A + i) \in \mathscr{E}$, so by (C) for some i, $(A + i) \in \mathscr{E}$. Hence, by (D), $A \in \mathscr{E}$.

($\Leftarrow$) Suppose $\mathrm{Sup}(a_{n+1} - a_n) = \infty$. Then there is an infinite ν such that $a_{\nu+1} - a_\nu$ is infinite. Let $\mathscr{E}$ be the set of all subsets $E \subseteq N$ such that $\exists e \in {}^*E$ such that $|e - (a_{\nu+1} + a_\nu)/2|$ is finite. $\mathscr{E}$ is a premeasure which excludes A.

COROLLARY I.4. *There is no premeasure which is contained in every premeasure.*

Proof. If there were such a premeasure $\mathscr{E}$ it would have to consist entirely of universally big sets. But this is impossible by Example I.2(iv).

However, a straightforward Zorn's lemma argument gives us the following proposition.

PROPOSITION I.5. *Every premeasure contains a minimal premeasure.*

For ultrafilters conditions (1) and (3) imply that if $A \in \mathscr{D}$ then $N \backslash A \notin \mathscr{D}$. As we have seen this violates translation invariance. However, for minimal premeasures the following proposition gives us a weaker but analogous property.

PROPOSITION I.6. *Suppose $\mathscr{E}$ is a premeasure. Then the following conditions are equivalent.*

(P_1) *$\mathscr{E}$ is minimal.*

(P_2) *For every $A \in \mathscr{E}$ there is a $k \in N$ such that*

$$N \Big/ \left[\bigcup_{i=0}^{k} (A + i)\right] \notin \mathscr{E} .$$

Proof. (P_1 implies P_2) suppose P_2 is false. Then there is a set $A \in \mathscr{E}$ such that for every k, $N \backslash [\bigcup_{i=0}^{k} (A + i)] \in \mathscr{E}$. This implies that for every $S \notin \mathscr{E}$ and every $i_1, \cdots, i_k$

$$N \neq \left[\bigcup_{j=1}^{k} (A + i_j)\right] \cup S .$$

Let $\mathscr{S}$ be the set of all subsets $E \subseteqq N$ contained in sets of the form

$$\bigcup_{j=1}^{k} (A + i_j) \cup S , \quad S \notin \mathscr{E} .$$

Let $\mathscr{E}' = \{E \mid E \notin \mathscr{S}\}$. It is straightforward to verify that $\mathscr{E}'$ is a premeasure and $\mathscr{E}' \subset \mathscr{E}$. Thus $\mathscr{E}$ was not minimal.

(P_2 implies P_1) Suppose $\mathscr{E}$ is not minimal. Then there is a premeasure $\mathscr{E}' \subsetneqq \mathscr{E}$. Let $A \in \mathscr{E} \backslash \mathscr{E}'$. By ($P_2$) there is a k such that $N \backslash [\bigcup_{i=0}^{k} (A + i)] \notin \mathscr{E}$ and hence not in $\mathscr{E}'$ as well. But since $A \notin \mathscr{E}'$, $\bigcup_{i=0}^{k} (A + i) \notin \mathscr{E}'$ by Properties (C) and (D). But now $N = \{N \backslash [\bigcup_{i=0}^{k} (A + i)]\} \cup [\bigcup_{i=0}^{k} (A + i)] \notin \mathscr{E}'$ contradicting (A). This completes the proof.

In § 5 we will show that minimal premeasures have additional desirable properties. However, before continuing the study of premeasures we will construct and study the completion of $^*R$.

II. **The Dedekind completion of $^*R$.** For the remainder of this paper $^*R$ will denote the set of nonstandard real numbers in a κ-saturated, higher order, nonstandard model $^*\mathscr{M}$ of the complete structure $\mathscr{M}$ on the reals, R, where κ is any cardinal greater than that of the universe of $\mathscr{M}$. If P denotes a given entity in $\mathscr{M}$,

$^*P$ will denote the corresponding entity in $^*\mathscr{M}$. Thus, for example, $^*N$ denotes the set of nonstandard positive integers and $^*[\ ]$: $^*R \to {}^*Z$ denotes the extension to $^*R$ of the greatest integer function. We use the usual notation $a \approx b$, for a infinitely close to b, and $\mathrm{St}(a)$ for the unique standard number infinitely close to a finite nonstandard number a. The monad of a, the set $\{x \in {}^*R \mid x \approx a\}$ is denoted, $\mu(a)$. See, for example, [1], [2], [3], or [4] for the necessary material on Nonstandard Analysis.

It is well-known that for arbitrary subsets $^*R$ is not complete. For example, $\mu(0)$ and R are bounded subsets of $^*R$ which have no suprema or infima. In this section we use the method of Dedekind cuts to construct and study a completion $^\#R$ of $^*R$. The set $^\#R$ inherits some but by no means all of the structure on $^*R$. For example, $^\#R$ is not a group with respect to addition since if μ denotes the supremum of $\mu(0)$ then $\mu + \mu = \mu + 0 = \mu$. Thus, one must proceed somewhat cautiously. In this section more details than is customary will be included in proofs because propositions which at first glance appear clear often at second glance reveal themselves to be false.

DEFINITION. II.1. A $\#$-real is a subset $\alpha \subsetneqq {}^*R$ such that

(i) For every $a \in \alpha$ and $b < a$, $b \in \alpha$.

(ii) $\alpha \neq \emptyset$, $^*R$.

(iii) α has no greatest element.

$^\#R$ is the set of all $\#$-reals.

We embed $^*R$ in $^\#R$ in the obvious way. If $a \in {}^*R$ the corresponding element, $^\#a$, of $^\#R$ is

$$^\#a = \{x \in {}^*R \mid x < a\} .$$

Condition (iii) above is included only to avoid nonuniqueness. Without it $^\#a$ would be represented by both $^\#a$ and $^\#a \cup \{a\}$. If E is a subset of $^*R$ satisfying (i) and (ii) then $\langle E \rangle$ will denote E if E has no greatest element and $E \backslash \{e\}$ if e is the greatest element of E.

Two elements of $^\#R$ will be particularly useful for examples,

$$\mu = {}^*(-\infty, 0] \cup \mu(0)$$
$$\phi = \bigcup_{n \in N} {}^*(-\infty, n) .$$

For $a \in R$ we will often by abuse of notation write $^\#a$ or even just a for $^{\#*}a$.

If α, $\beta \in {}^\#R$ we define the sum $\alpha + \beta$ by

$$\alpha + \beta = \{a + b \mid a \in \alpha,\ b \in \beta\} .$$

Note, for example, that $\mu + \mu = \mu$ and $\phi + \phi = \phi$ so $+$ is not a group operation.

LEMMA II.2.
(i) *$+$ is commutative and associative in $^{\#}R$.*
(ii) $\forall \alpha \in {}^{\#}R \ \alpha + {}^{\#}0 = \alpha$.
(iii) $\forall a, b \in {}^{*}R \ {}^{\#}(a + b) = {}^{\#}a + {}^{\#}b$.

Proof.
(i) is clear,
(ii) $\alpha + {}^{\#}0 \subseteq \alpha$ is clear.
Now, suppose $a \in \alpha$. Since α has no greatest element $\exists b > a$ $b \in \alpha \therefore a - b \in {}^{\#}0$ and $a = (a - b) + b \in {}^{\#}0 + \alpha$.
(iii) ${}^{\#}a + {}^{\#}b \subseteq {}^{\#}(a + b)$ is clear since

$$x < a,\ y < b \quad \text{implies} \quad x + y < a + b\ .$$

Now, suppose $x < (a + b)$

$$\therefore \qquad a - \frac{(a + b) - x}{2} < a$$

and

$$b - \frac{(a + b) - x}{2} < b\ .$$

So

$$x = \left[a - \frac{(a + b) - x}{2}\right] + \left[b - \frac{(a + b) - x}{2}\right] \in \# a + \# b\ .$$

This completes the proof.

DEFINITION II.3. Suppose $\alpha, \beta \in R$. Then $\alpha \leqq \beta$ if and only if $\alpha \subseteq \beta$. Notice, here again something is lost going from $^{*}R$ to $^{\#}R$ since $\alpha < \beta$ does *not* imply $\alpha + \alpha < \beta + \alpha$ since $0 < \mu$ but $0 + \mu = \mu = \mu + \mu$. However, we do have the following.

LEMMA II.4.
(i) *$\leqq$ is a linear ordering on $^{\#}R$, which extends the usual ordering on $^{*}R$.*
(ii) $\alpha \leqq \alpha',\ \beta \leqq \beta' \rightarrow \alpha + \beta \leqq \alpha' + \beta'$.
(iii) $\alpha < \alpha',\ \beta < \beta' \rightarrow \alpha + \beta < \alpha' + \beta'$.
(iv) *If $A \subseteq {}^{\#}R$ is bounded above then*

$$\operatorname{Sup} A = \operatorname*{Sup}_{\alpha \in A} \alpha = \bigcup_{\alpha \in A} \alpha \quad \textit{exists in } {}^{\#}R\ .$$

If $A \subseteq {}^{\#}R$ *is bounded below then*

$$\operatorname{Inf} A = \operatorname*{Inf}_{\alpha \in A} \alpha = \langle \bigcap_{\alpha \in A} \alpha \rangle \quad \textit{exists in} \quad {}^{\#}R\ .$$

(v) *${}^{*}R$ is dense in ${}^{\#}R$. That is if $\alpha < \beta$ in ${}^{\#}R$, there is an* $a \in {}^{*}R$ s.t. $\alpha < {}^{\#}a < \beta$.

Proof.
(i), (ii), (iv), (v) are clear.
(iii) $\alpha < \alpha'$, $\beta < \beta'$ imply $\exists\, a, a', b, b' \in {}^{*}R$

$$\text{s.t.} \quad \alpha < {}^{\#}a < {}^{\#}a' < \alpha' \quad \text{and} \quad \beta < {}^{\#}b < {}^{\#}b' < \beta'$$

$$\therefore \alpha + \beta \leqq {}^{\#}a + {}^{\#}b < {}^{\#}a' + {}^{\#}b' \leqq \alpha' + \beta'$$

$$\therefore \alpha + \beta < \alpha' + \beta' \quad \text{completing the proof}\ .$$

Our next task is to define $-\alpha$ for $\alpha \in {}^{\#}R$. By earlier remarks $-\alpha$ can only have some of the properties of an additive inverse. In particular, $\alpha + (-\alpha)$ need not be ${}^{\#}0$.

DEFINITION II.5. Suppose $\alpha \in {}^{\#}R$. Then $-\alpha$ is defined by

$$-\alpha = \langle \{a \in {}^{*}R \mid -a \notin \alpha\} \rangle\ .$$

LEMMA II.6.
(i) $-({}^{\#}a) = {}^{\#}(-a)$
(ii) $-(-\alpha) = \alpha$
(iii) $\alpha \leqq \beta \leftrightarrow -\beta \leqq -\alpha$
(iv) $(-\alpha) + (-\beta) \leqq -(\alpha + \beta)$
(v) *if* $a \in {}^{*}R$, ${}^{\#}(-a) + (-\beta) = -({}^{\#}a + \beta)$
(vi) $\alpha + (-\alpha) \leqq {}^{\#}0$.

Proof.
(i)-(iii) are clear.
(iv) Suppose $a \in (-\alpha)$ and $b \in (-\beta)$.
Then $\exists\, a_1, b_1$ $a < a_1 \in (-\alpha)$, $b < b_1 \in (-\beta)$

$$\therefore -a_1 \notin \alpha\ , \quad -b_1 \notin \beta$$

$$\therefore \alpha < {}^{\#}(-a_1)\ , \quad \beta < {}^{\#}(-b_1)$$

$$\therefore \alpha + \beta < {}^{\#}(-a_1) + {}^{\#}(-b_1) = {}^{\#}(-a_1 - b_1)$$

$$\therefore -a_1 - b_1 \notin \alpha + \beta$$

$$\therefore \text{since} \quad a + b < a_1 + b_1\ , \quad a + b \in -(\alpha + \beta)\ .$$

(v) By (iv) ${}^{\#}(-a) + (-\beta) \leqq -({}^{\#}a + \beta)$.
Suppose $c \in -({}^{\#}a + \beta)$

$$\therefore \exists c_1 \quad \text{s.t.} \quad c < c_1 \in -({}^{\#}a + \beta)$$

$$\therefore -c_1 \notin {}^{\#}a + \beta$$

$$\therefore -c - a \notin \beta \quad (\text{since} \quad -c - a \in \beta$$

and

$$a - (c_1 - c) \in {}^{\#}a \quad \text{imply}$$
$$-c_1 = a - (c_1 - c) + (-c - a) \in {}^{\#}a + \beta)$$

$$\therefore c + a \in -\beta \, .$$

By similar reasoning $c_1 + a \in -\beta$.
But $-a - (c_1 - c) \in {}^{\#}(-a)$.
So

$$c = -a - (c_1 - a) + c_1 + a \in {}^{\#}(-a) + (-\beta) \, .$$

(vi) Suppose $a \in \alpha$ and $b \in -\alpha$

$$\therefore -b \notin \alpha$$

$$\therefore a < -b \quad \text{so} \quad a + b < -b + b = 0$$

$$\therefore \alpha + (-\alpha) \leqq {}^{\#}0 \, .$$

This completes the proof.

Next we consider a few examples which show that Lemma II.6 is best possible.

EXAMPLES II.7.
(i) $\mu + (-\mu) = -\mu$ and $\phi + (-\phi) = -\phi$. So Lemma II.6 (vi) is best possible.
(ii) $[-(-\phi) + -\phi] = \phi + (-\phi) = -\phi < \phi = -(-\phi) = -((-\phi) + \phi)$. So Lemma II.6 (iv) is best possible.
(iii) One might conjecture that equality would hold in Lemma II.6(iv) whenever both α, β were positive. However, the following a counterexample.
Let $\alpha = {}^{\#}1 + \mu$ $\beta = {}^{\#}1 - \mu$ then

$$\begin{aligned}(-\alpha) + (-\beta) &= {}^{\#}(-1) + (-\mu) + {}^{\#}(-1) + \mu \\ &= {}^{\#}(-2) + (-\mu) + \mu \\ &= {}^{\#}(-2) + (-\mu) \, .\end{aligned}$$

But $\alpha = {}^*(-\infty, 1] \cup \mu(1)$ $\beta = {}^*(-\infty, 1] \backslash \mu(1)$.

So

$$\alpha + \beta = {}^*(-\infty, 2]\backslash\mu(2)$$

and

$$-(\alpha + \beta) = {}^*(-\infty, -2] \cup \mu(-2) = {}^\#(-2) + \mu$$

$$\therefore (-\alpha) + (-\beta) < -(\alpha + \beta) .$$

From now on we will write $\alpha - \beta$ for $\alpha + (-\beta)$. Also, by abuse of notation we will sometime write a instead of ${}^\#a$. Next we define absolute value in ${}^\#R$. Although the definition is completely straightforward, its properties are not. In fact, the triangle inequality fails.

DEFINITION II.8. Suppose $\alpha \in {}^\#R$. The *absolute value* of α, written $|\alpha|$ is defined as follows.

$$|\alpha| = \begin{cases} \alpha & \text{if } \alpha \geqq 0 \\ -\alpha & \text{if } \alpha < 0 . \end{cases}$$

EXAMPLE II.9. The triangle inequality fails in ${}^\#R$. Let $\alpha = -1 - \mu$ and $\beta = -1 + \mu$ then

$$\alpha + \beta = -2 - \mu \quad \text{so} \quad |\alpha + \beta| = -(-2 - \mu) = 2 + \mu$$
$$|\alpha| = 1 + \mu \qquad |\beta| = 1 - \mu$$

and

$$|\alpha| + |\beta| = 1 + \mu + 1 - \mu = 2 - \mu < 2 + \mu = |\alpha + \beta| .$$

DEFINITION II.10. Suppose $\alpha, \beta \in {}^\#R$. The *product* $\alpha \cdot \beta$, is defined as follows.

Case (i). $\alpha, \beta > 0$

$$\alpha \cdot \beta = \{a \cdot b \mid 0 < {}^\#a < \alpha , \quad 0 < {}^\#b < \beta\} \cup {}^*(-\infty, 0] .$$

Case (ii). $\alpha = 0$ or $\beta = 0$

$$\alpha \cdot \beta = 0 .$$

Case (iii). $\alpha < 0$ or $\beta < 0$

$$\begin{aligned} \alpha \cdot \beta &= |\alpha| \cdot |\beta| && \text{if both } \alpha, \beta < 0 \\ \alpha \cdot \beta &= -|\alpha| \cdot |\beta| && \text{if } \alpha < 0 , \ \beta > 0 \\ & && \text{or } \alpha > 0 , \ \beta < 0 . \end{aligned}$$

The bad news for multiplication is that it is not a group operation (since $\phi \cdot \phi = \phi$) and the distributive law does not hold in full generality (if it did ${}^{\#}(-1) \cdot \alpha = -\alpha$ would imply $\alpha + (-\alpha) = 0$). The good new is.

LEMMA II.11.
(i) $\forall a, b \in {}^*R \; {}^{\#}(ab) = {}^{\#}a \cdot {}^{\#}b$.
(ii) *Multiplication is associative and commutative.*
(iii) ${}^{\#}1 \cdot \alpha = \alpha \; {}^{\#}(-1) \cdot \alpha = -\alpha$.
(iv) $|\alpha||\beta| = |\alpha\beta|$.
(v) *If* $\alpha, \beta, \gamma \geqq 0 \; \alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
(vi) $0 < \alpha < \alpha', \; 0 < \beta < \beta' \rightarrow \alpha\beta < \alpha'\beta'$.

Proof.
(i) First, suppose $a, b > 0$ then clearly ${}^{\#}a{}^{\#}b \leqq {}^{\#}(ab)$ since $0 < x < a, \; 0 < y < b$ implies $xy < ab$. Now suppose $0 < c < ab$ let

$$a' = a\sqrt{\frac{c}{ab}} < b$$

$$b' = b\sqrt{\frac{c}{ab}} < b$$

$$c = a'b' \in {}^{\#}a{}^{\#}b$$

so

$${}^{\#}(ab) \leqq {}^{\#}a{}^{\#}b \; .$$

The other cases follow from (iv).
(iv) Immediate from Definition II.10 and Lemma II.6(ii).
(ii) is immediate from the definition for $\alpha, \beta, \gamma \geqq 0$ and (iv) otherwise.
(iii) We may assume $\alpha > 0$. Clearly ${}^{\#}1 \cdot \alpha \leqq \alpha$. Now suppose $a \in \alpha$ then $\exists a' \in \alpha \; a < a'$

$$\therefore a/a' < 1 \quad \text{so} \quad a/a' \in {}^{\#}1$$

and

$$a' \cdot (a/a') = a \in {}^{\#}1 \cdot \alpha \; .$$

By the definition ${}^{\#}(-1)\alpha = -({}^{\#}1\alpha) = -\alpha$.
(v) Clearly $\alpha(\beta + \gamma) \leqq \alpha\beta + \alpha\gamma$.
Suppose $d \in \alpha\beta + \alpha\gamma$

$$\therefore d = ab + a'c \quad a, a' \in \alpha \quad b \in \beta \quad c \in \gamma$$

Without loss of generality we may assume $a \leqq a'$.
Hence $d = ab + a'c \leqq a'b + a'c = a'(b + c) \in \alpha(\beta + \gamma)$.
(vi) $0 < \alpha < \alpha' \; 0 < \beta < \beta'$ implies

$$\exists\, a, a', b, b' \quad \text{s.t.} \quad 0 < \alpha < {}^{\#}a < {}^{\#}a' < \alpha'$$

$$0 < \beta < {}^{\#}b < {}^{\#}b' < \beta'$$

$$\therefore\ \alpha\beta \leqq {}^{\#}(ab) < {}^{\#}(a'b') \leqq \alpha'\beta'$$

$$\therefore\ \alpha\beta < \alpha'\beta' .$$

This completes the proof.

The next step is to define α^{-1}. As we have seen above α^{-1} cannot have all the properties of a multiplicative inverse.

DEFINITION II.12. Suppose $\alpha \in {}^*R$ and $\alpha \neq 0$ then α^{-1} is defined as follows.

Case (i). $\alpha > 0$

$$\alpha^{-1} = \mathrm{Inf}\{a^{-1} \mid 0 < a \in \alpha\} .$$

Case (ii). $\alpha < 0$

$$\alpha^{-1} = -(-\alpha)^{-1} .$$

LEMMA II.13.
(i) $({}^{\#}a)^{-1} = {}^{\#}(a^{-1})$.
(ii) $(\alpha^{-1})^{-1} = \alpha$.
(iii) $0 < \alpha \leqq \beta \Rightarrow \beta^{-1} \leqq \alpha^{-1}$.
(iv) $\alpha, \beta > 0 \Rightarrow (\alpha^{-1})(\beta^{-1}) \leqq (\alpha\beta)^{-1}$.
(v) $\forall a \in {}^*R$, $a \neq 0$ *implies* $({}^{\#}a)^{-1}\beta^{-1} = ({}^{\#}a\beta)^{-1}$.
(vi) $\alpha(\alpha^{-1}) \leqq {}^{\#}1$.

Proof.
(i)-(iii) are clear.
(iv) Suppose $x \in (\alpha^{-1})(\beta^{-1})$

$$\therefore\ x = hg \quad h \in \alpha^{-1} , \quad g \in \beta^{-1} , \quad h, g > 0$$

$$\therefore\ \forall\, a \in \alpha , \quad a > 0\, h \leqq a^{-1} , \quad \forall\, b \in \beta , \quad b > 0\, g \leqq b^{-1}$$

$$\therefore\ h^{-1} \geqq a , \quad g^{-1} \geqq b$$

$$\therefore\ (hg)^{-1} \geqq ab$$

$$\therefore\ \forall\, t \in \alpha\beta \quad (hg)^{-1} \geqq t$$

$$\therefore\ \forall\, t \in \alpha\beta \quad t > 0 \quad \text{implies} \quad hg \leqq t^{-1}$$

$$\therefore\ hg \leqq (\alpha\beta)^{-1} .$$

(v) We may assume $a > 0$. The case for $a < 0$ follows from this case.

By (iv) $({}^{\#}a)^{-1}\beta^{-1} \leqq ({}^{\#}a\beta)^{-1}$.

Now suppose $x \in ({}^{\#}a\beta)^{-1}$ $\therefore \exists y \in ({}^{\#}a\beta)^{-1}$ $x < y$

$$\therefore \forall\, b \in \beta\,, \quad b > 0 \quad y \leqq (ab)^{-1}$$

$$\therefore ay \leqq b^{-1}$$

$$\therefore ay \in \beta^{-1}$$

$$\therefore x = a^{-1}ax = \left[a^{-1}\left(\frac{x}{y}\right)\right]ay \in {}^{\#}(a^{-1})\beta^{-1}\,.$$

(vi) We may assume $\alpha > 0$. Suppose $a \in \alpha$, $b \in \alpha^{-1}$ and $a, b > 0$

$$b < a^{-1}$$

$$ab < aa^{-1} = 1$$

$$\therefore \alpha\alpha^{-1} \leqq {}^{\#}1\,.$$

This completes the proof.

Example II.14.

(i) $\phi^{-1} = \mu$, $\mu^{-1} = \phi$.

(ii) $\phi\phi^{-1} = \phi\mu = \mu < 1$ so Lemma II.13(vi) is best possible.

(iii) $\mu\phi = \mu$

So $(\mu\phi)^{-1} = \mu^{-1} = \phi > \mu^{-1}\phi^{-1} = \phi\mu = \mu$.

So Lemma II.13(iv) is best possible.

III. The topology of ${}^{\#}R$.

Topologically, ${}^{\#}R$ has many properties strongly reminiscent of R itself. We proceed as follows.

Definition III.1. Suppose $U \subseteqq {}^{\#}R$. Then U is *open* if and only if for every $u \in U$, $\exists\, \alpha, \beta \in {}^{\#}R$ $\alpha < u < \beta$ such that

$$u \in (\alpha, \beta) \subseteqq U\,.$$

(Notice because of the peculiarities of addition this is not equivalent to $\forall\, u \in U\, \exists\, \varepsilon > 0$ such that $(u - \varepsilon, u + \varepsilon) \subseteqq U$.)

Lemma III.2.

(i) *${}^{*}R$ is dense in R.*

(ii) *${}^{\#}R \backslash {}^{*}R$ is dense in ${}^{\#}R$.*

Proof.

(i) is just Lemma II.4(v).

(ii) Suppose $U \neq \varnothing$ is open.

Then $\exists\, a < b$ s.t. $(a, b) \subseteq U$.
We may assume $a, b \in {}^*R$.
Let $\alpha = \operatorname{Inf}_{n \in N} (a + (b - a)/n)$.
Then by a straightforward saturation argument $\alpha \in (a, b)$ and $\alpha \notin {}^*R$.
This completes the proof.

LEMMA III.3. *Suppose $A \subseteq {}^\sharp R$. Then A is closed if and only if*
(i) $\forall E \subseteq A$ *E bounded above implies* $\operatorname{Sup} E \in A$, *and*
(ii) $\forall E \subseteq A$ *E bounded below implies* $\operatorname{Inf} E \in A$.

Proof.
($\Longrightarrow$) Suppose A is closed and $E \subseteq A$ is bounded above.
Let $s = \operatorname{Sup} E$.
If $s \notin A\, \exists\, \alpha, \beta$ s.t. $s \in (\alpha, \beta)$ and $(\alpha, \beta) \cap A = \varnothing$

$$\therefore (\alpha, \beta) \cap E = \varnothing\ .$$

So s cannot be the $\operatorname{Sup} E$

$$\therefore s \in A\ .$$

(ii) is proved similarly.
($\Longleftarrow$) Suppose $x \notin A$.
Set

$$\alpha = \operatorname{Sup}\{t \mid t < x,\ t \in A\}\ ,$$

and

$$\beta = \operatorname{Inf}\{t \mid x < t,\ t \in A\}\ .$$

Clearly $\alpha \leqq x \leqq \beta$ and (i) and (ii) imply $\alpha < x < \beta$.
Clearly $(\alpha, \beta) \cap A = \varnothing$.
Thus the complement of A is open.
This completes the proof.

PROPOSITION III.4. *${}^\sharp R$ is connected.*

Proof. Suppose ${}^\sharp R = A \cup B$ where $A, B \neq \varnothing$ are both closed and $A \cap B = \varnothing$.
Choose $a \in A$, $b \in B$. We may assume $a < b$.
Let $x = \operatorname{Sup}\{t \in A \mid [a, t] \subseteq A\}$.
Note, x exists and is $\leqq b$.
Since A is closed $x \in A$. Hence $x \neq b$.
Since B is closed $\exists\, \alpha, \beta\ \alpha < x < \beta$
s.t. $(\alpha, \beta) \subseteq A \therefore \exists\, e\ x < e < \beta$ s.t.
$[t, e] \subseteq A \therefore [a, e] \subseteq A$ contradicting the definition of x.

COROLLARY III.5. (Intermediate Value Theorem). *Suppose* $f: [\alpha, \beta] \to {}^{\#}R$ *is continuous and* $f(\alpha) < \gamma < f(\beta)$. *Then* $\exists \gamma' \in [\alpha, \beta]$ *such that* $f(\gamma') = \gamma$.

Proof. Straightforward.

PROPOSITION III.6. *For* $\alpha < \beta$ *in* ${}^{\#}R$. $[\alpha, \beta]$ *is compact.*

Proof.
Let $\mathscr{U} = \{U_\sigma\}_{\sigma \in S}$ be an open covering of $[\alpha, \beta]$.
Let $A = \{x \in [\alpha, \beta] | \exists \sigma_1, \cdots, \sigma_k\ [\alpha, x] \subseteqq U_{\sigma_1} \cup \cdots \cup U_{\sigma_k}\}$.
A straightforward argument shows A is both open and closed, so $\beta \in A$ completing the proof.

COROLLARY III.7. *Suppose* $A \subseteqq {}^{\#}R$. *Then* A *is compact if and only if* A *is closed and bounded.*

Proof. Straightforward.

COROLLARY III.8. ${}^{\#}R$ *is normal.*

Proof. Straightforward.

DEFINITION AND LEMMA III.9.
(i) Suppose $\alpha \in (-\phi, \phi)$. Then there is a unique standard x, called $ST(\alpha)$, such that $x \in [\alpha - \mu, \alpha + \mu]$.
(ii) $\alpha \leqq \beta$ implies $ST(\alpha) \leqq ST(\beta)$.
(iii) ST is continuous.
(iv) $ST(\alpha + \beta) = ST(\alpha) + ST(\beta)$
$ST(\alpha\beta) = ST(\alpha)ST(\beta)$
$ST(-\alpha) = -ST(\alpha)$
$ST(\alpha^{-1}) = [ST(\alpha)]^{-1}$ if $\alpha \notin [-\mu, \mu]$.

Proof.
(i) $ST(\alpha)$ is clearly unique if it exists.
Let $X = \{x \in R \,|\, {}^{\#}x \leqq \alpha\}$.
Since X is bounded above X has a supremum x.

$$\alpha \leqq x + \mu \quad \text{since} \quad \alpha > x + \mu$$

implies $\exists a \in {}^{*}R\ x + \mu < a < \alpha$

$$\therefore x < \frac{x + St(a)}{2} < \alpha$$

$$\therefore x \neq \text{Sup}\, X\,.$$

Also, $\alpha \geqq x - \mu$ since $\alpha < x - \mu$
implies $\exists a \in {}^*R$, $\alpha < {}^\# a < x - \mu$
implies X is bounded by $St(a) < x$

$$\therefore \alpha \in [x - \mu, x + \mu] .$$

(ii)-(iv) are completely straightforward.

Despite the fact that the topology on ${}^\# R$ is quite nice, one must still be cautious. For example, mappings which one might expect to be continuous may not be. In particular, the map $\alpha \mapsto \alpha + (-\alpha)$ is not continuous since if it were it would be zero since it is zero on the dense subset ${}^*R$. The next two propositions show that the problem is not in the map $\alpha \mapsto -\alpha$ but in the map $(\alpha, \beta) \mapsto \alpha + \beta$.

PROPOSITION III.10.
(i) *The map $\alpha \mapsto -\alpha$ is continuous.*
(ii) *The map $\alpha \rightarrow \alpha^{-1}$ is continuous.*

Proof.
(i) Suppose $-\alpha \in (\beta, \gamma)$ then

$$\alpha \in (-\gamma, -\beta)$$

and

$$\theta \in (-\gamma, -\beta) \Longrightarrow -\theta \in (\beta, \gamma) .$$

(ii) The proof is identical.

PROPOSITION III.11. *The maps $(\alpha, \beta) \mapsto \alpha + \beta$ and $(\alpha, \beta) \mapsto \alpha \cdot \beta$ are not continuous.*

Proof. We will show addition is not continuous. The proof for multiplication is similar. Notice $(\phi, -\phi) \mapsto -\phi$.

Now, $(-\infty, -1)$ is an open neighborhood of $-\phi$.

Suppose $(\alpha, \beta) \times (\alpha', \beta')$ is a basic open neighborhood of $(\phi, -\phi)$. We may assume α and β' are finite. Hence, there is a finite $a \in {}^*R$ such that $(a, -a) \in (\alpha, \beta) \times (\alpha', \beta')$.

But $(a, -a) \mapsto 0 \notin (-\infty, -1)$.

The next question we wish to consider is when a "continuous" function on ${}^*R$ can be extended to ${}^\# R$.

THEOREM III.12. *Suppose $f \colon [a, b] \rightarrow A \subseteqq {}^*R$ is internal, *continuous, and monotonic. Then f has a unique continuous extension ${}^\# f \colon [{}^\# a, {}^\# b] \rightarrow \bar{A} \subseteqq R$, where $\bar{A}$ denotes the closure of A in ${}^\# R$.*

Proof.

(i) Uniqueness is immediate since $[a, b]$ is dense in $[^{\#}a, {}^{\#}b]$.

(ii) We may assume f is monotonically increasing.

Define ${}^{\#}f(\alpha) = \mathrm{Sup}\{f(x) | x \leqq \alpha \text{ and } x \in [a, b]\}$.

($\sharp$) *Claim.* ${}^{\#}f(\alpha) = \beta = \mathrm{Inf}\{f(x) | \alpha \leqq x \text{ and } x \in [a, b]\}$.

Clearly ${}^{\#}f(\alpha) \leqq \beta$.

Now suppose ${}^{\#}f(\alpha) < \beta \therefore \exists y \in {}^*R \; {}^{\#}f(\alpha) < y < \beta$.

Then by the *Intermediate Value Theorem $\exists x \in [a, b]$ such that $f(x) = y$.

But either $x < \alpha$ or $x \geqq \alpha$ and either case leads to an immediate contradiction.

Now suppose $\theta \in [^{\#}a, {}^{\#}b]$ and ${}^{\#}f(\theta) \in (\alpha, \beta)$. In view of ($\sharp$) $\exists u, v \in {}^*R$, $u < \theta < v$ such that $\alpha < f(u) < f(\theta) < f(v) < \beta$.

Hence ${}^{\#}f$ maps $({}^{\#}u, {}^{\#}v)$ into (α, β).

Thus, ${}^{\#}f$ is continuous.

COROLLARY III.13. *The conclusion above holds if* $[a, b]$ *is replaced by* (a, b) *even if* $a = -\infty$ *or* $b = +\infty$.

COROLLARY III.14. *The conclusion above holds if f is piecewise monotonic* (i.e., *the domain can be decomposed into a finite* (*not *finite*) *number of intervals on each of which f is monotonic*).

Next, we consider some examples showing that the assumptions above are necessary.

EXAMPLES III.15.

(i) The internality of f is needed (via the *Intermediate Value Theorem). Consider the function

$$f(x) = \begin{cases} x & x \in \mu(0) \\ 1 + x & {}^{\#}x > \mu \\ -1 + x & {}^{\#}x < -\mu \end{cases}$$

which clearly has no continuous extension to ${}^{\#}R$.

(ii) Let ν be a fixed infinite *integer. The function

$$f(x) = \begin{cases} \sin x & |x| < 2\pi\nu \\ 0 & |x| > 2\pi\nu \end{cases}$$

can not be defined at ϕ, but is *piecewise monotonic.

PROPOSITION III.16. *Suppose f, g are *continuous, piecewise monotonic functions then*

(i) $f \circ g$ *is also and*

(ii) ${}^{\#}(f \circ g) = {}^{\#}f \circ {}^{\#}g$.

Proof.

(i) is straightforward.

(ii) follows from continuity and the fact that $^*R$ is dense in $^\#R$.

IV. **#-Integers.** The set $^\#R$ has within it a set $^\#Z$ of #-integers which behave very much like $^*Z$ inside $^*R$. In particular the greatest integer function $^*[\ ]: {}^*R \to {}^*Z$ extends in a natural way to $^\#[\ ]: {}^\#R \to {}^\#Z$. To simplify the notation we will denote $^*[\ ]$ by $[\ ]$.

DEFINITION AND LEMMA IV.1. Suppose $\alpha \in {}^\#R$. Then the following two conditions on α are equivalent. If α satisfies these conditions α is said to be a *#-integer*. The set of #-integers is denoted $^\#Z$ and $^\#Z \cap [\#1, \infty)$ is denoted $^\#N$.

(i) $\alpha = \mathrm{Sup}\{^\#\nu \mid \nu \in {}^*Z \text{ and } \nu \leqq \alpha\}$

(ii) $\alpha = \mathrm{Inf}\{^\#\nu \mid \nu \in {}^*Z \text{ and } \alpha \leqq \nu\}$.

Proof.

Let

$$\beta_1 = \mathrm{Sup}\{^\#\nu \mid \nu \in {}^*Z \quad \text{and} \quad \nu \leqq \alpha\}$$

and

$$\beta_2 = \mathrm{Inf}\{^\#\nu \mid \nu \in {}^*Z \quad \text{and} \quad \alpha \leqq \nu\}\,.$$

Clearly $\beta_1 \leqq \beta_2$.

Suppose $\beta_1 < \beta_2$.

Then $\exists\, b \in {}^*R$ s.t. $\beta_1 < {}^\#b < \beta_2$

$$\therefore [b] < \alpha \quad \text{and} \quad [b] + 1 > \alpha$$

$$\therefore \beta_1 = {}^\#[b] \quad \text{and} \quad \beta_2 = {}^\#[b] + 1$$

so $\alpha \neq \beta_1$ and $\alpha \neq \beta_2$.

This completes the proof.

LEMMA IV.2.

(i) *$^\#Z$ is the closure in $^\#R$ of $^*Z$.*

(ii) *$^\#N$ is the closure in $^\#R$ of $^*N$.*

Hence both $^\#Z$ and $^\#N$ are closed with respect to taking Suprema and Infima.

Proof. We prove (i), (ii) follows immediately.

Clearly $^\#Z \subseteq \overline{^*Z}$.

Suppose $\alpha \notin {}^\#Z$.

Let $\beta_1 = \mathrm{Sup}\{\nu \in {}^*Z \mid \nu \leqq \alpha\}$

$\beta_2 = \mathrm{Inf}\{\nu \in {}^*Z \mid \alpha \leqq \nu\}$.
Since $\alpha \notin {}^\#Z$, $\beta_1 < \alpha < \beta_2$.
But, clearly $(\beta_1, \beta_2) \cap {}^*Z = \varnothing$.
So, $\alpha \notin \overline{{}^*Z}$ which completes the proof.

LEMMA IV.3. *Suppose* α, $\beta \in {}^\#Z$. *Then,*
(i) $\alpha + \beta \in {}^\#Z$.
(ii) $-\alpha \in {}^\#Z$.
(iii) $\alpha \cdot \beta \in {}^\#Z$.

Proof. Completely straightforward.

CONSTRUCTION IV.4. Suppose $\alpha \in {}^\#R$. Then, we define $[\alpha] \in {}^\#Z$ by

$$[\alpha] = \mathrm{Sup}\{\nu \mid \nu \in {}^*Z, \nu \leqq \alpha\} .$$

There are two possibilities.

(i) $\{\nu \mid \nu \in {}^*N, \nu \leqq \alpha\}$ has no greatest element. In this case $[\alpha] = \alpha$ since $[\alpha] < \alpha$ implies $\exists a \in {}^*R$ such that $[\alpha] < a < \alpha$. But then $[a] < \alpha$ which implies $[a] + 1 < \alpha$ contradicting $[\alpha] < a \leqq [a]+1$.

(ii) $\{\nu \mid \nu \in {}^*N, \nu \leqq \alpha\}$ has a greatest element, ν.
In this case $[\alpha] = \nu \in {}^*N$.
Notice in case (i) $[\alpha] = \alpha$ and in case (ii) $[\alpha] \leqq \alpha < [\alpha] + 1$.

LEMMA AND DEFINITION IV.5. Suppose A is a *standard* subset of N. Suppose $\nu \in {}^\#N/{}^*N$. Then the following are equivalent.
(i) $\nu = \mathrm{Sup}\{a \in {}^*A \mid a < \nu\}$.
(ii) $\nu = \mathrm{Inf}\{a \in {}^*A \mid \nu < a\}$.
When these two equivalent conditions hold, ν is said to be in the *tail of* A, written $\nu \in \tau(A)$.

Proof.
Let $\beta_1 = \mathrm{Sup}\{a \in {}^*A \mid a < \nu\}$
and $\beta_2 = \mathrm{Inf}\{a \in {}^*A \mid \nu < a\}$.
If $\beta_1 < \beta_2$ then $\exists b \in {}^*R$ such that $\beta_1 < b < \beta_2$.
Let $A = \{a_1, a_2, \cdots\}$ $a_1 < a_2 < \cdots$.
Let ρ be the greatest integer such that $a_\rho < b$. (ρ exists since $b \in {}^*R$.)
Thus $a_\rho \leqq \beta_1$ and $a_{\rho+1} \leqq \beta_2$.
So $a_\rho = \beta_1 < \beta_2 = a_{\rho+1}$.
But now since $\nu \in {}^\#N/{}^*N$

$$\nu \neq a_\rho , \quad a_{\rho+1} .$$

On the other hand if $\beta_1 = \beta_2$ then $\nu = \beta_1 = \beta_2$.

EXAMPLES IV.6.

(i) A is infinite $\Leftrightarrow \phi \in \tau(A) \Leftrightarrow \tau(A) \neq \emptyset$.

(ii) $\tau(A) = \tau(N) \Leftrightarrow A$ is universally big.

Proof.

(i) is clear.

(ii) First, suppose A is not universally big. Then $\mathrm{Sup}(a_{n+1} - a_n) = \infty$ and there is an infinite ν such that $a_{\nu+1} - a_\nu$ is infinite.

Let $\alpha = \mathrm{Sup}_{n \in N}(a_\nu + n)$.

Then $\alpha \notin \tau(A)$ but $\alpha \in \tau(N)$.

Now, suppose $\mathrm{Sup}_n(a_{n+1} - a_n) = k < \infty$.

If $\alpha \in \tau(N)$ then $\alpha \in {}^{\#}N/N$ and the set $\{\nu \in {}^*N | \nu < \alpha\} = E$ has no maximum. Thus $\nu \in E$ implies $\nu + 1, \nu + 2, \cdots, \nu + k \in E$. Clearly $\mathrm{Sup}\{a \in {}^*A | a < \alpha\} \leqq \alpha$.

If $\mathrm{Sup}\{a \in {}^*A | a < \alpha\} < \alpha$ then there is an $x \in {}^*R$ such that $\mathrm{Sup}\{a \in {}^*A | a < \alpha\} < x < \alpha$.

But then $[x] < \alpha$ and

$$^{\#}[x] + 1\,, \quad ^{\#}[x] + 2\,, \cdots, \, ^{\#}[x] + k < \alpha$$

and at least one of these *integers must be in *A contradicting $\mathrm{Sup}\{a \in {}^*A | a < \alpha\} < [x] + 1$.

This completes the proof.

PROPOSITION IV.7 *Suppose $A \subseteqq N$. Then $\tau(A)$ is closed.*

Proof. Suppose $\alpha \notin \tau(A)$. There are two cases.

(i) $\alpha = {}^{\#}a, a \in {}^*N$

then $({}^{\#}a - 1/2, {}^{\#}a + 1/2) \cap (A \backslash \{a\}) = \emptyset$ and since $\tau(A) \subseteqq \overline{A \backslash \{a\}}$, $({}^{\#}a - 1/2, {}^{\#}a + 1/2) \cap \tau(A) = \emptyset$.

(ii) $\alpha \neq \mathrm{Sup}\{a \in {}^*A | a < \alpha\}$.

Hence $\exists x \in {}^*R$ such that

$$\mathrm{Sup}\{a \in {}^*A | a < \alpha\} < x < \alpha\,.$$

Hence, $\{\nu | a_\nu < x\}$ being an internal set in *N has a maximum ν.

$$\therefore (a_\nu, a_{\nu+1}) \cap {}^*A = \emptyset$$

$$\therefore (a_\nu, a_{\nu+1}) \cap \tau(A) = \emptyset$$

But $\alpha \in (a_\nu, a_{\nu+1})$.

This completes the proof.

LEMMA IV.8.

(i) $\alpha \in \tau(A)$ *implies* $\forall n \in Z \; \alpha + n = \alpha$.

(ii) $\forall n \in Z\ \tau(A) = \tau(A + n)$.
(iii) $A \subseteq B$ *implies* $\tau(A) \subseteq \tau(B)$.
(iv) $\tau(A \cap B) \subseteq \tau(A) \cap \tau(B)$.
(v) $\tau(A \cup B) = \tau(A) \cup \tau(B)$.

Proof.
(i) $\alpha \in \tau(A)$ implies $\alpha \notin {}^*N$ and

$$\alpha = \operatorname{Sup}\{\nu \in {}^*A \mid \nu < \alpha\} .$$

So $\{\nu \in {}^*A \mid \nu < \alpha\}$ has no maximum.
This is sufficient to imply (i).
(ii) follows from (i)
(iii) and (iv) are immediate.
(v) If $\alpha = \operatorname{Sup}\{\nu \in {}^*(A \cup B) \mid \nu < \alpha\}$ then either

$$\alpha = \operatorname{Sup}\{\nu \in {}^*A \mid \nu < \alpha\}$$

or

$$\alpha = \operatorname{Sup}\{\nu \in {}^*B \mid \nu < \alpha\}$$

so $\alpha \in \tau(A)$ or $\alpha \in \tau(B)$.
This completes the proof.

Notice that $\tau(A) \cap \tau(B)$ need not equal $\tau(A) \cap \tau(B)$. In fact, if $A = \{1, 3, 5, \cdots\}$ and $B = A+1$ then $\tau(A) \cap \tau(B) = {}^*N \setminus {}^*N$ but $A \cap B = \varnothing$ so $\tau(A \cap B) = \varnothing$. However, given A, $B \subseteq N$ we can construct a set $A \wedge B$ such that $\tau(A \wedge B) = \tau(A) \cap \tau(B)$ as follows.

CONSTRUCTION AND LEMMA IV.9. Suppose $A_1, A_2, \cdots, A_k$ are infinite subsets of N. Write, $A_n = \{a_{n1}, a_{n2}, \cdots\}$ with $a_{n1} < a_{n2} < \cdots$. Then we define $A_1 \wedge A_2 \wedge \cdots \wedge A_k = \{c_1, c_2, \cdots\}$ as follows.

$$c_1 = a_{11}$$
$$c_{n+1} = \text{least } a_{pj} \text{ such that } c_n < a_{pj} \text{ where } p \equiv n + 1 \pmod{k} .$$

Then,

$$\tau(A_1 \wedge A_2 \wedge \cdots \wedge A_k) = \tau(A_1) \cap \tau(A_2) \cap \cdots \cap \tau(A_k) .$$

Proof.
(i) $\tau(A_1 \wedge A_2 \wedge \cdots \wedge A_k) \subseteq \tau(A_1) \cap \tau(A_2) \cap \cdots \cap \tau(A_k)$.
Suppose $\alpha \in \tau(A_1 \wedge A_2 \wedge \cdots \wedge A_k)$. Then

$$\alpha = \operatorname{Sup}\{c \in A_1 \wedge A_2 \wedge \cdots \wedge A_k \mid c < \alpha\} .$$

Hence the set $\{c \in A_1 \wedge A_2 \wedge \cdots \wedge A_k \mid c < \alpha\}$ has no greatest element. By the construction of $A_1 \wedge A_2 \wedge \cdots \wedge A_k$ this implies

$$\alpha = \text{Sup}\{a \in A_n \mid a < \alpha\} .$$

So,

$$\alpha \in \tau(A_1) \cap \tau(A_2) \cap \cdots \cap \tau(A_k) .$$

(ii) $\tau(A_1) \cap \tau(A_2) \cap \cdots \cap \tau(A_k) \subseteqq \tau(A_1 \wedge A_2 \wedge \cdots \wedge A_k)$.
Suppose $\alpha \in \tau(A_1) \cap \tau(A_2) \cap \cdots \cap \tau(A_k)$.
Hence $\alpha = \text{Sup}\{a \in {}^*A_n \mid a < \alpha\}$.
Now, suppose $\alpha > \text{Sup}\{c \in {}^*(A_1 \wedge A_2 \wedge \cdots \wedge A_k) \mid c < \alpha\}$.
Then, $\exists\, x \in {}^*R$ such that

$$\alpha > x > \text{Sup}\{c \in {}^*(A_1 \wedge A_2 \wedge \cdots \wedge A_k) \mid c < \alpha\} .$$

Let ρ be the largest *integer such that $c_\rho < x$. Hence $c_\rho < x < c_{\rho+1}$. Without loss of generality we may assume $\rho \equiv 1$ (modulo k). But since $\alpha = \text{Sup}\{a \in {}^*A_2 \mid a < \alpha\}$, $c_{\rho+1} < \alpha$. This contradiction completes the proof.

V. **Premeasures.** The work of the preceding section immediately gives rise to a large number of examples of premeasures.

DEFINITION AND EXAMPLES V.1. Suppose $\alpha \in {}^*N/{}^*N$.
Let $\mathscr{E}_\alpha$ be the set

$$\mathscr{E}_\alpha = \{A \subseteqq N \mid \alpha \in \tau(A)\} .$$

It is an immediate consequence of Lemma IV.8 that $\mathscr{E}_\alpha$ is a premeasure.

Notice, for example, that $\mathscr{E}_\phi$ is the maximal premeasure of all infinite sets.

Premeasures of the form $\mathscr{E}_\alpha$ have two additional interesting properties.

DEFINITION V.2. Suppose $A, B \subseteqq N$; $A = \{a_1, a_2, \cdots\}$, $B = \{b_1, b_2, \cdots\}$, $a_1 < a_2 < \cdots$, $b_1 < b_2 < \cdots$. Then B is said to be *dense in* A iff $\forall n\ [a_n, a_{n+1}) \cap B \neq \varnothing$.

Suppose $\mathscr{E}$ is a premeasure. $\mathscr{E}$ is said to satisfy property (E) iff

(E) $\forall A \in \mathscr{E}$, $B \subseteqq N$ if B is dense in A then $B \in \mathscr{E}$.

Suppose $\mathscr{E}$ is a premeasure. $\mathscr{E}$ is said to satisfy property (F) iff

$$\text{(F)}\quad \forall A_1, A_2, \cdots, A_k \in \mathscr{E}\ , \quad A_1 \wedge A_2 \wedge \cdots \wedge A_k \in \mathscr{E}\ .$$

The following examples show the properties (E) and (F) are independent of the defining properties (A)-(D) for premeasures.

EXAMPLES V.3.

(i) Let $s_1, s_2, \cdots$ be an increasing sequence such that

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \infty .$$

Let

$$A = \bigcup_{n=1}^{\infty} [s_{2n-1}, s_{2n}) \cap N$$

$$B = \bigcup_{n=1}^{\infty} [s_{2n}, s_{2n+1}) \cap N .$$

Notice as $n \to \infty ||A_n||/n$ fluctuates back and forth between close to zero and close to 1 and that $||B_n||/n = 1 - ||A_n||/n$. Choose an infinite $\nu \in {}^*N$ such that $||A_\nu||/\nu \approx 1/2$. Hence $||B_\nu||/\nu \approx 1/2$.

Let $\mathscr{E}_1^\nu = \{E \subseteq N \mid ||E_\nu||/\nu \neq 0\}$.

Hence $A, B \in \mathscr{E}_1^\nu$.

But $A \wedge B = \{s_1, s_2, \cdots\} \notin \mathscr{E}_1^\nu$.

Thus $\mathscr{E}_1^\nu$ does not satisfy Property (F).

However $\mathscr{E}_1^\nu$ does satisfy Property (E) since D dense in C implies $\forall n ||D_n|| \geqq ||C_n||$.

(ii) Choose an infinite $\nu \in {}^*N$. Let

$$\mathscr{E} = \{A \subseteq N \mid \exists e \in {}^*A, |e - \nu^2| \text{ is finite}\} .$$

In particular the set $A = \{1, 4, 9, 16, \cdots, n^2, \cdots\} \in \mathscr{E}$.

Let $B = \{2, 6, 12, \cdots [a_{n+1} + a_n]/2, \cdots\}$.

Then B is dense in A but $B \notin \mathscr{E}$.

Hence $\mathscr{E}$ does not satisfy Property (E).

On the other hand $\mathscr{E}$ does satisfy Property (F).

Suppose $A_1, A_2, \cdots, A_k \in \mathscr{E}$, $C = A_1 \wedge A_2 \wedge \cdots \wedge A_k$.

$C = \{c_1, c_2, \cdots\}$.

Let c_i be the largest element of the internal set

$$\{c_i \in {}^*C \mid c_i < \nu^2\} .$$

If $|c_i - \nu^2|$ is finite $C \in \mathscr{E}$ and we're done.

If $|c_i - \nu^2|$ is infinite we may assume without loss of generality that $c_i \in {}^*A_1$.

Hence c_{i+1} is the least element of ${}^*A_2$ wihch is greater than c_i. But ${}^*A_2$ has an element b such that $|b - \nu^2|$ is finite. So $|c_{i+1} - \nu^2|$ must also be finite.

This brings us to the following representation theorem for some premeasures.

THEOREM V.4. *Suppose $\mathscr{E}$ is a premeasure. Then there exists* *an* $\alpha \in {}^{\#}N \backslash {}^{*}N$ *such that* $\mathscr{E} = \mathscr{E}_{\alpha}$ *if and only if* $\mathscr{E}$ *satisfies Properties* (E) *and* (F).

Proof.

($\Rightarrow$) By Lemma IV.9 $\mathscr{E}_{\alpha}$ satisfies Property (F).

Now, suppose $A \in \mathscr{E}_{\alpha}$ and B is dense in A. Since $A \in \mathscr{E}_{\alpha}$

$$\alpha = \operatorname{Sup}\{a \in {}^{*}A \,|\, a < \alpha\}$$

and since $\alpha \notin {}^{*}N$, $\{a \in {}^{*}A \,|\, a < \alpha\}$ has no last element. Thus since B is dense in A

$$\alpha = \operatorname{Sup}\{b \in {}^{*}B \,|\, b < \alpha\}$$

so $B \in \mathscr{E}_{\alpha}$ and $\mathscr{E}_{\alpha}$ satisfies Property (E).

($\Leftarrow$) (i) Claim. Property (E) implies that $\forall B \in \mathscr{E}$, $\forall S \notin \mathscr{E}$, $\forall n \in N$

$$\exists k [b_k, b_{k+n}) \cap S = \varnothing \ .$$

Proof. Suppose no such k exists. Then S is dense in each of the sets $B_i = \{b_i, b_{i+n}, b_{i+2n}, \cdots\}$ $i = 1, 2, \cdots, n$

But $B = B_1 \cup B_2 \cup \cdots \cup B_n$.

Thus $\exists i$ such that $B_i \in \mathscr{E}$.

But now Property (E) implies $S \in \mathscr{E}$ contradicting $S \notin \mathscr{E}$ and completing the proof.

(ii) Claim. Suppose $B_1, B_2, \cdots, B_n \in \mathscr{E}$ and $S_1, S_2, \cdots, S_k \notin \mathscr{E}$.

Let $B = B_1 \wedge B_2 \wedge \cdots \wedge B_n = \{b_1, b_2 \cdots\}$.

Let $S = S_1 \cup S_2 \cup \cdots \cup S_k$.

Then $\forall p \in N\ \exists g$ such that $[b_g, b_{p+g}) \cap S = \varnothing$.

Proof. By Property (F) $B \in \mathscr{E}$ and the claim follows from (i).

(iii) By saturation choose a *finite set $\{B_1, B_2, \cdots, B_{\nu}\} \subseteqq {}^{*}\mathscr{E}$ containing ${}^{*}B$ for every $B \in \mathscr{E}$.

Also, by saturation choose a *finite set

$\{S_1, S_2, \cdots, S_{\lambda}\} \subseteqq {}^{*}(P(N) \backslash \mathscr{E})$ containing ${}^{*}S$ for every $S \notin \mathscr{E}$.

Let

$$B = B_1 \wedge B_2 \wedge \cdots \wedge B_{\nu}$$

and

$$S = S_1 \cup S_2 \cup \cdots \cup S_{\lambda} \ .$$

Let $B = \{b_1, b_2, \cdots\}$ with $b_1 < b_2 < \cdots$.

By *(ii) $\exists g$ such that

$$[b_g, b_{g+\nu}\lambda) \cap S = \varnothing$$

where $\lambda = \nu^2$.

Let $\alpha = \mathrm{Sup}\{b_{g+n\nu} \mid n \in N\}$.

Claim. $\mathscr{E} = \mathscr{E}_\alpha$.
Notice, first if $T \notin \mathscr{E}$ then ${}^*T \subseteq S$ and hence

$$[b_g, b_{g+\nu}\lambda) \cap {}^*T = \varnothing .$$

So $\alpha \neq \mathrm{Sup}\{t \in {}^*T \mid t < \alpha\} \leqq b_g$.
Thus, $\mathscr{E}_\alpha \subseteq \mathscr{E}$.
Conversely, suppose $T \in \mathscr{E}$. Thus for some ρ, ${}^*T = B_\rho$.
But for each n

$$B_\rho \cap [b_{g+n\nu}, b_{g+(n+1)\nu}) \neq \varnothing .$$

Hence $\alpha = \mathrm{Sup}\{x \in {}^*T \mid x < \alpha\}$.
Thus, $T \in \mathscr{E}_\alpha$ and $\mathscr{E} \subseteq \mathscr{E}_\alpha$.
This completes the proof.

LEMMA V.5. *Suppose $\mathscr{E}$ is any premeasure. Then there exists an $\alpha \in {}^\sharp N/N$ such that $\mathscr{E}_\alpha \subseteq \mathscr{E}$.*

Proof. If $S \notin \mathscr{E}$ then S is not universally big so $\mathrm{Sup}(s_{n+1} - s_n) = \infty$. Hence there are infinite $\nu, \lambda \in {}^*N$ such that

$$[\nu, \nu + \lambda] \cap {}^*S = \varnothing .$$

Hence, if $S_1, S_2, \cdots, S_k \notin \mathscr{E}$ then there are infinite $\nu, \lambda \in {}^*N$ such that $[\nu, \nu + \lambda] \cap (S_1 \cup S_2 \cup \cdots \cup S_k) = \varnothing$ and by a straightforward saturation argument there exists an infinite $\nu, \lambda \in {}^*N$ such that

$$\forall S \notin \mathscr{E} \quad [\nu, \nu + \lambda] \cap {}^*S = \varnothing .$$

Let $\alpha = \mathrm{Sup}_{n \in N}\, \nu + \eta$.
Then $S \notin \mathscr{E}$ implies $S \notin \mathscr{E}_\alpha$ so that $\mathscr{E}_\alpha \subseteq \mathscr{E}$, completing the proof.

COROLLARY V.6. *Suppose $\mathscr{E}$ is a minimal premeasure. Then*
(i) *there is a $\alpha \in {}^\sharp N/{}^*N$* s.t. $\mathscr{E} = \mathscr{E}_\alpha$.
(ii) *$\mathscr{E}$ satisfies Properties* (E) *and* (F).

Proof. Immediate.

EXAMPLE V.7. Notice Properties (E) and (F) do not imply $\mathscr{E}$ is minimal since $\mathscr{E}_\phi$ has properties (E) and (F) and is not minimal.

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Received April 25, 1977 and in revised form July 10, 1979. This research was partially supported by NSF Grant No. MCS77-02262. It was carried out while the author was visiting the University of Warwick. The author would like to thank the NSF for its support and the University of Warwick for its hospitality and lively mathematical environment.

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