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INDUCED p-ELEMENTS IN THE SCHUR GROUP

RICHARD ANTHONY MOLLIN

The main result of this paper gives necessary and sufficient conditions for the p-primary part $S(K)_p$ of the Schur group $S(K)$ to be induced from $S(F)_p$ for any subfield F of K where K is contained in $Q(\varepsilon_n)$, under the restriction that ε_{p^2} is not in K if $p>2$ and n is odd if $p=2$, where ε_n is a primitive nth root of unity.

Moreover we completely answer the question: "When is $S(Q(\varepsilon_n+\varepsilon_n^{-1}))$ induced from $S(Q)$?" for any n, and also the question: "When are the quaternion division algebras in $S(Q(\varepsilon_n))$ induced from $S(Q(\varepsilon_n+\varepsilon_n^{-1}))$?" for any n. Finally, in the last section we investigate the "generalized group of algebras with uniformly distributed invariants" which we introduced in an earlier paper. We obtain, for the first time, a sufficient condition for the group to be induced from a certain subgroup.

Preliminaries. Let $L=Q(\varepsilon_n)$ and let K be a subfield of L. Although it is not necessary for all results in the paper it is convenient to choose n as small as possible for a given K. The Schur subgroup $S(K)$ of the Brauer group $B(K)$ consists of those equivalence classes $[A]$ which contain an algebra which is isomorphic to a simple summand of the group algebra KG for some finite group G. An elegant proof of the following result was given by Janusz [18, Prop. 6.2, p. 89]:

(1.1) Let $[A]\in S(K)$ where $[A]$ has exponent n then ε_n, a primitive nth root of unity is in K. (In fact (1.1) holds for any field K.)

For K over Q finite abelian, Benard and Schacher [2, Th. 6.1, p. 89] proved the following:

If $[A]\in S(K)$ then:

(1.2) If the index of A is n then ε_n is in K.

(1.3) If $\mathfrak{q}$ is a K-prime above the rational prime q and $\sigma\in G(K/Q)$, the Galois group of K over Q, with $\sigma(\varepsilon_n)=\varepsilon_n^{b^2}$ then the Hasse $\mathfrak{q}$-invariant of A satisfies:

$$\mathrm{inv}_{\mathfrak{q}}\, A\equiv b_\sigma\, \mathrm{inv}_{\mathfrak{q}^\sigma} A (\mathrm{mod}\, 1)\ .$$

If $[A]\in B(K)$ and A satisfies (1.2)-(1.3) then A is said to have *uniformly distributed invariants*. These algebras form a subgroup $U(K)$ of $B(K)$. For a treatment of this group see Mollin [7, 14, 15, 16]. We note from (1.2)-(1.3) that $S(K)$ is a subgroup of $U(K)$. For a generalization of $U(K)$ to the algebraic number field case and consequences thereof (including, therefore, results for $S(K)$ see Mollin [8]-[13]).

Now, if $[A] \in U(K)$ and $\mathfrak{q}'$ and $\mathfrak{q}$ are K-primes above q then $A \otimes_K K_{\mathfrak{q}'}$ and $A \otimes_K K_{\mathfrak{q}}$ have the same index where $K_{\mathfrak{q}}$ denotes the completion of K at $\mathfrak{q}$. We call the common value of the indices of $A \otimes_K K_{\mathfrak{q}}$ for all $K =$ primes above q the *q-local index* of A and denote it by $\mathrm{ind}_q(A)$.

We shall have need of the following formula which can be found in Deuring [3]:

(1.4) Let $[A] \in B(K)$. Let K/F be finite and let $\hat{\mathfrak{q}}$ be a K-prime above the F-prime $\mathfrak{q}$. Then:

$$\mathrm{inv}_{\hat{\mathfrak{q}}}(A \otimes_F K) \equiv |K_{\hat{\mathfrak{q}}} : F_{\mathfrak{q}}| \, \mathrm{inv}_{\mathfrak{q}} A (\mathrm{mod}\ 1) .$$

Henceforth, when we write a tensor product it shall be assumed to be taken over the center of the algebra in the left factor. Moreover, by the symbol $S(F) \otimes K$ we mean the image of $S(F)$ under the map which extends the center to K. The symbol $\sim$ denotes equivalence in the Brauer group.

If $\mathfrak{q}$ is an F-prime above q, then any reference to the decomposition of $\mathfrak{q}$ in K over F (abelian), shall be referred to as the decomposition of q in K/F since the decomposition essentially depends on q and not on $\mathfrak{q}$. For example if $\mathfrak{q}$ is unramified in K over F we say q is unramified in K over F.

Finally, for groups G and H contained in G, $a \in G - H$ means $a \in G$ but $a \notin H$.

For most basic results concerning $S(K)$ the reader is referred to [18].

2. Induced p-elements. Let $Q(\varepsilon_n)$ be the smallest cyclotomic field containing K. We may assume $n \not\equiv 2 (\mathrm{mod}\ 4)$ since $Q(\varepsilon_n) = Q(\varepsilon_{2n})$ whenever n is odd. Let p be a prime such that if p is odd then ε_{p^2} is not in K and if $p = 2$ then n is odd. Let F be a subfield of K and set $G_0 = G(Q(\varepsilon_n)/F)$. Now we present for the first time necessary and sufficient conditions for $S(K_p)$ to be induced from $S(F)_p$. In the following theorem we maintain the above notation and assumptions. To avoid the trivial case $S(K)_p = 1$ we assume ε_p is in K.

THEOREM 2.1. *$S(K)_p = S(F)_p \otimes K$ if and only if*
(1) ε_p is in F and
(2) $G_0^p \cap G = G^p$.

Proof. Since $S(K)_p \neq 1$ then equality holds only if ε_p is in F.

We show that the equality of the theorem is equivalent to (2) when (1) holds.

By [6, Th. 2, Th. 3, Th. 5] and [17, Th. 2.2, Th. 2.3], an algebra class in $S(K)_p$ is determined by a skew-pairing ψ^K on G to $\langle \varepsilon_p \rangle$ and by certain elements in $S(Q(\varepsilon_p))_p \otimes K$ (which also lie in $S(F) \otimes K$). A similar statement holds for $S(F)_p$. Therefore $S(K)_p = S(K)_p \otimes K$ if and only if every skew-pairing on G is the restriction of a skew-pairing on G_0. Since the values lie in $\langle \varepsilon_p \rangle$, this is equivalent to the assertion that the inclusion of G into G_0 induces an inclusion of G/G^p into G_0/G_0^p. This is equivalent to (2).

The following result obtained in Mollin [14, Corollary 2.3, p. 165] is immediate.

COROLLARY 2.2. *If K/Q is real of even degree and K is in $Q(\varepsilon_n)$ where n is odd and no prime congruent to* 1 *modulo* 4 *divides n then $S(K) = S(Q) \otimes K$.*

Before presenting a sequence of results anchored to Theorem 2.1 we demonstrate that the theorem does not hold if n is even and ε_4 is not in K. We shall need a result which we isolate as a lemma since it verifies remarks made in Mollin [14, p. 165], (remarks follow Theorem 2.2 therein).

LEMMA 2.3. *Let $n = 2^a h$, $a \geqq 2$, $(2, h) = 1$, and let $K = Q(\varepsilon_n + \varepsilon_n^{-1})$.*
(i) *If $a = 2$ then $S(K) = S(Q) \otimes K$,*
(ii) *If $a > 2$ then $S(K) \neq S(Q) \otimes K$.*

Proof. (i) If $a = 2$ and $h = 1$ the result is clear. We assume that $h > 1$. We see easily that in order to obtain $S(K) = S(Q) \otimes K$ it suffices to prove $\operatorname{ind}_p A = 1$ for $[A] \in S(K)$ whenever $|K_\mathfrak{p}: Q_p|$ is even where $\mathfrak{p}$ is a K-prime above p. If $p \mid n$ then by Yamada [18, Th. 1, p. 591], $\operatorname{ind}_p A = 1$ for any $[A] \in S(K)$. If p does not divide n and $|K_\mathfrak{p}: Q_p|$ is even then Yamada's aforementioned result says that if $[A] \in S(K)$ with $\operatorname{ind}_p A = 2$ then $p^{f/2} \equiv -1 \pmod n$ where f is the residue class degree of p in $Q(\varepsilon_n)/Q$. This means that p is inert in $Q(\varepsilon_n)/K$ so that $f/2$ must be even in order that $|K_\mathfrak{p}: Q_p|$ is even. Thus, $p^{f/2} \equiv -1 (\operatorname{mod} n)$ implies that -1 is a square modulo 4, which is absurd. This establishes (i).

(ii) If $h = 1$ then the result follows from Yamada [18, Th. 2.2, p. 586] and Mollin [7, Th. 2.6, p. 277]. We assume $h > 1$ and let $n = p_1^{a_1} \cdots p_s^{a_s}$ where the p_i's are distinct primes. Choose a prime p such that $p \equiv -1 (\operatorname{mod} h)$ and $p \equiv 5 (\operatorname{mod} 2^a)$. Such a choice is allowed by the Chinese Remainder theorem. Now we show that there exists $[A] \in S(K)$ with $\mathrm{ind}_p A = 2$ and such that $[A] \notin S(Q) \otimes K$.

The smallest positive integer f such that $p^f \equiv 1(\mathrm{mod}\, n)$ is 2^{a-2}; i.e., the residue class degree of p in K over Q is 2^{a-2}. Thus, by the choice of p we have: p does not divide n, f is even, $p^{f/2} \not\equiv -1(\mathrm{mod}\, n)$ and $p^{f/2} \not\equiv \pm 1(\mathrm{mod}\, 2^a)$. By Yamada [18, Th. 1, p. 591] this guarantees the existence of $[A]$ is $S(K)$ with $\mathrm{ind}_p A = 2$. Now, since $p^{f/2} \not\equiv -1(\mathrm{mod}\, n)$ then p splits completely in $Q(\varepsilon_n)$ over K. Hence, $f = 2^{a-2}$ equals the residue class degree of p in K over Q. Since $a > 2$ it follows that $[A] \notin S(Q) \otimes K$ from (1.4).

Now we present the aforementioned example to show that the theorem does not hold if n is even and ε_4 is not in K. We maintain the above notation; i.e., $K = Q(\varepsilon_n + \varepsilon_n^{-1})$ where $n = 2^a h$, $h > 1$, $a > 2$, $(2, h) = 1$, and $G_0 = GQ(\varepsilon_n)/(Q)$. Let:

$$G_0 = \langle\theta\rangle \times \langle\alpha\rangle \times \langle\phi_1\rangle \times \cdots \times \langle\phi_s\rangle$$

where:

$$\varepsilon_{2^a}^{\theta} = \varepsilon_{2^a}^{5}\,, \qquad \varepsilon_h^{\theta} = \varepsilon_h\,;$$
$$\varepsilon_{2^a}^{\alpha} = \varepsilon_{2^a}^{-1}\,, \qquad \varepsilon_h^{\alpha} = \varepsilon_h\,;$$

and

$$\theta^{2^{a-2}} = \alpha^2 = \phi_i^{h_i} = 1$$

where:

$$\phi_i(\varepsilon_n) = \varepsilon_n^{s_i}\,; \quad s_i \equiv r_i(\mathrm{mod}\, p_i^{a_i})\,; \quad s_i \equiv 1(\mathrm{mod}\, n/p_i^{a_i})$$

where r_i is a primitive root modulo p_i, and $h_i = p_i^{a_i-1}(p_i - 1)$ for $i = 1, 2, \cdots, s$. By Lemma 2.3, $S(K) \neq S(Q) \otimes K$. We have $G_0^2 = \langle\theta^2\rangle \times \langle\phi_1^{h_1/2}\rangle \times \cdots \times \langle\phi_s^{h_s/2}\rangle$ and $G = \langle\alpha\phi_1^{h_1/2} \cdots \phi_s^{h_s/2}\rangle$. Therefore $G_0^2 \cap G = \langle 1\rangle = G^2$. This establishes the counterexample.

Now we establish a series of results tied to Theorem 2.1. In the introduction to [19] Yamada remarks that if K is a real subfield of $Q(\varepsilon_n)$ such that $G(Q(\varepsilon_n)/K)$ is cyclic; then the structure of $S(K)$ does not depend on whether or not n is divisible by a prime congruent to 3 modulo 4. Lemma 2.3 indicates that Yamada is correct in general. However, if we restrict our attention to maximal real subfields of $Q(\varepsilon_n)$ for n odd the result goes through. The following theorem therefore is the exact analogue of Yamada's result on real quadratic fields [18].

Moreover, this theorem generalizes and simplifies the proof of the result obtained in Mollin [14, Th. 2.2, p. 164]. Finally it completes the answer to the 'Tensoring question' for the maximal real subfield of $Q(\varepsilon_n)$ for any n.

THEOREM 2.4. *Let* $K = Q(\varepsilon_n + \varepsilon_n^{-1})$ *where* $n > 1$ *is odd. Then* $S(K) = S(Q) \otimes K$ *if and only if there exists a prime* q *dividing* n *such that* $q \equiv 3 (\mathrm{mod}\ 4)$.

Proof. To establish the necessity assume $S(K) = S(Q) \otimes K$. We have: $G = G(Q(\varepsilon_n)/K) = \langle \phi_1^{h_1/2} \cdots \phi_s^{h_s/2} \rangle$ where the ϕ_i and h_i are defined as in the above example. If we assume $p_i \equiv 1 (\mathrm{mod}\ 4)$ for all $i = 1, 2, \cdots, s$ then it is clear that $h_i \equiv 0 (\mathrm{mod}\ 4)$ for all $i = 1, 2, \cdots, s$. Thus: $G_0^2 \cap G = \langle (\phi_1^{h_1/4})^2 \rangle \times \cdots \times \langle (\phi_s^{h_s/4})^2 \rangle = G$. However $G^2 = \langle 1 \rangle$, so $G_0^2 \cap G \neq G^2$ which implies by Theorem 2.1 that $S(K) \neq S(Q) \times K$ contradicting the hypothesis, thereby establishing the necessity.

Conversely if $S(K) \neq S(Q) \otimes K$ then by Theorem 2.1 we have: $G_0^2 \cap G \neq G^2 = \langle 1 \rangle$. Therefore $G_0^2 \cap G = G$. This forces $h_i \equiv 0 (\mathrm{mod}\ 4)$ for each $i = 1, 2, \cdots, s$, which establishes the theorem.

It is reasonable to ask whether or not a similar result holds for an arbitrary real subfield $Q(\varepsilon_n)$ for n odd. If n is a prime-power it does, (see [18]).

However, if n is divisible by at least 2 distinct primes it does not. The following counterexample illustrates this fact.

Let $n = 65$ and let:

$$\phi_{13}\colon \varepsilon_{13} \longrightarrow \varepsilon_{13}^2 ; \qquad \phi_{13}\colon \varepsilon_5 \longrightarrow \varepsilon_5 ;$$
$$\phi_5\colon \varepsilon_5 \longrightarrow \varepsilon_5^2 ; \qquad \phi_5\colon \varepsilon_{13} \longrightarrow \varepsilon_{13} .$$

Let $\phi = \phi_{13}^3 \cdot \phi_5$ and let K equal the fixed field of $\langle \phi \rangle$. We note $G_0^2 \cap G = \langle \phi_{13}^6 \cdot \phi_5^2 \rangle = G^2$ which yields $S(K) = S(Q) \otimes K$ from Theorem 2.1. This completes the counterexample.

Now, let K over Q be finite imaginary and abelian with M as maximal real subfield. From [1, Th. 2.1, p. 161] it follows that $[A] \in S(K)$ with index 2 satisfies $A \sim B \otimes K$ where $[B] \in B(M)$ and B is also quaternion. A natural question to ask is whether or not $[B] \in S(M)$. The following theorem answers this question for certain fields.

THEOREM 2.5. *Let* K *be contained in* $Q(\varepsilon_n)$ *with* n *odd such that* K *over* Q *is finite, imaginary and abelian, then*

$$S(K)_2 = S(M) \otimes K .$$

Proof. If $G_0^2 \cap G \neq G^2$ then there is a cyclic subgroup of G_0 of 2 power order such that

(i) $H \cap G^2 \neq H \cap G$

(ii) $H \cap G \neq \langle 1 \rangle$ and

(iii) $H \cap G \neq H$.

Therefore by Pendergrass [17, Th. 2.3, p. 433] there exists $[A] \in S(K)_2$ with $\mathrm{ind}_p A = 2$ where p has Frobenius automorphism corresponding to a generator of H. By [1], op. cit., $A \sim B \otimes K$ where $[B] \in B(M)$ is quaternion. Thus, for a K-prime $\mathfrak{P}$ above an M-prime $\mathfrak{p}$ which in turn sits above the rational prime p; we have from (1.4) that:

$$\mathrm{inv}_{\mathfrak{P}} A = \mathrm{inv}_{\mathfrak{P}} B \otimes K \equiv |K_{\mathfrak{P}}: M_{\mathfrak{p}}| \, \mathrm{inv}_{\mathfrak{p}} B (\mathrm{mod}\, 1) \ .$$

However, by (iii) above we have $|K_{\mathfrak{P}}: M_{\mathfrak{p}}| = 2$ so it follows that $\mathrm{inv}_{\mathfrak{p}} A = 0$, a contradiction which secures the theorem.

We note that the above theorem includes the case where $M = Q(\varepsilon_n + \varepsilon_n^{-1})$ for n odd. The following theorem establishes that for n even the result does not hold. Moreover it yields necessary and sufficient conditions for elements of order 2 in $S(Q(\varepsilon_n))$ to be induced from $S(M)$.

THEOREM 2.6. *Let*

$$K = Q(\varepsilon_n) \ , \quad M = Q(\varepsilon_n + \varepsilon_n^{-1}) \ .$$

All elements of order 2 *in* $S(K)$ *are induced from* $S(M)$ *if and only if* n *is odd or a power of* 2.

Proof. First we prove the necessity of the condition. Assume that $n = 2^a m$, where $(2, m) = 1$, $a > 1$, $m > 1$. We now prove that there exists an element of order 2 in $S(K)$ which is not induced from $S(M)$. Choose a prime $p \equiv 1 (\mathrm{mod}\, 2^a)$ and $p \equiv -1 (\mathrm{mod}\, m)$. Thus the residue class degree of p in K over Q is 2; i.e., the smallest integer f such that $p^f \equiv 1 (\mathrm{mod}\, n)$ is $f = 2$. However, $p = p^{f/2} \not\equiv -1 (\mathrm{mod}\, n)$ so p has inertial degree 1 in K over M. This fact together with $p = p^{f/2} \equiv 1 (\mathrm{mod}\, 2^a)$ is enough to ensure that there does not exist an element in $S(M)$ with p-local index 2, by Yamada [18, Th. 1, p. 591]. Now,

$$S(K)_2 = S(Q(\varepsilon_{2^a})) \otimes K \ ,$$

by Janusz [5, Th. 1, p. 346]. Since $p \equiv 1 (\mathrm{mod}\, 2^a)$ then there exists $[A] \in S(Q(\varepsilon_{2^a}))$ with $\mathrm{ind}_p A = 2^a$ by Yamada [18, pp. 135-139]. Let $[A]^{2^{a-2}} = [B]$. Then $\mathrm{ind}_p B = 4$ and if $\mathfrak{P}$ is a K-prime above p then:

$$\mathrm{inv}_{\mathfrak{P}} B \otimes K \equiv |K_{\mathfrak{P}}: Q_p(\varepsilon_{2^a})| \, \mathrm{inv}_p B (\mathrm{mod}\, 1) \ .$$

But $|K_{\mathfrak{P}}: Q_p(\varepsilon_{2^a})| = 2$ so that $\mathrm{ind}_p(B \otimes K) = 2$. We have $[B \otimes K] \in S(K)$ having p-local index 2 but $B \otimes K$ is not induced from $S(M)$. This establishes the necessity.

Conversely, if n is odd then we are done by Theorem 2.1, so we assume n is a power of 2. Given $[A] \in S(K)$ with $\operatorname{ind}_p A = 2$ we have $|K_{\mathfrak{P}}: M_p| = 1$ which follows from the fact that $A \sim B \otimes K$ with $[B] \in B(M)$ being quaternion. Now it suffices to show that there exists $[C] \in S(M)$ with $\operatorname{ind}_p C = 2$, but this is immediate from Yamada [18, Th. 2.2, p. 586].

3. The tensoring question for $U_F(K)$. Let K/F be finite Galois where F is an algebraic number field. We define $U_F(K)$ to be the subset of $B(K)$ consisting of $[A] \in B(K)$ such that:

(3.1) If the index of A is m then, ε_m is in K, and

(3.2) If $\mathfrak{P}$ is a K-prime lying over the F-prime $\mathfrak{p}$ and

$$\tau \in G(K/F) \quad \text{with} \quad \varepsilon_m^\tau = \varepsilon_m^{b_\tau} \quad \text{then:}$$
$$\operatorname{inv}_{\mathfrak{P}}(A) \equiv b_\tau \operatorname{inv}_{\mathfrak{P}^\tau}(A) (\operatorname{mod} 1) .$$

For a treatment of this subgroup, which we call the 'group of algebras with uniformly distributed invariant for K relative to F', see Mollin [9]. We note here that $S(K)$ is a subgroup of $U_F(K)$.

We need a definition before stating the next result. If K and E are number fields and D is a K-division ring; i.e., D is a division ring with $[D] \in B(K)$ then we say that D is 'E-adequate' if there exists an E-division ring containing D.

THEOREM 3.1. *Let E/F be a Galois extension of number fields and K/F any extension of number fields. If D is a K-adequate division ring with $[D] \in U_F(E)$ where D has exponent n, then ε_n is in K and for all $p \mid n$ we have:*

$$U_K(KE)_p = U_F(E)_p \otimes KE .$$

Proof. From Mollin [9, Th. 3.2, p. 263] we have ε_n is in K and from Mollin [9, Lemma 31, p. 262] we have that $U_F(E)_p \otimes KE$ is contained in $U_K(KE)_p$. From the proof of [9, Th. 2.10, p. 260] and from [9, Lemma 3.1, p. 262] it is easily seen that it suffices to prove that there are no higher p-power roots of unity in KE than in E and that p does not divide $|KE: E|$.

Now, let D_1 be a K-division ring containing D. Then $D \otimes KE$ is isomorphic to the division ring, of index n in D_1, generated by D and K. Therefore p does not divide $|KE: E|$. Now, if ε_{p^a} is in KE but not in E then $|E(\varepsilon_{p^a}): E| = p$ and $E(\varepsilon_{p^a}) \subseteq KE$. Thus $p \mid |KE: E|$, a contradiction which establishes the theorem.

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University of Lethbridge
4401 University Drive
Lethbridge, Alberta T1K 3M4