

MOIRÉ PHENOMENA IN ALGEBRAIC GEOMETRY: RATIONAL ALTERNATIONS IN R^2

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This paper investigates rational alternations, principally in R^2 . Rational alternations in R^n generalize the polynomial alternations studied in the author's *Moiré Phenomena in Algebraic Geometry: Polynomial Alternations in R^n* . Rational alternations, like polynomial alternations, have the spirit of diffraction gratings, but may possess singularities, where grating bands flow together. Both alternations carry with them more information than ordinary varieties. As in the polynomial case, the systems of varieties making up two rational alternations generate new systems of varieties under union (or dually, intersection), corresponding to systems of moiré fringes of various orders. This paper investigates density functions naturally associated with these fringes, and studies the behavior of the fringes at points of indeterminacy of the defining functions.

1. Introduction. For the sake of motivation, we begin not with a general rational function, but with a polynomial. Thus, let p be an element of a polynomial ring $k[X_1, \dots, X_n]$ over a field k . Then p defines an algebraic variety $V(p) = \{(x_1, \dots, x_n) \mid p(x_1, \dots, x_n) = 0\}$ in $k^n = k_{x_1, \dots, x_n}$. It is natural to ask how operations on polynomials translate into operations on the associated varieties. For instance, if $p, q \in k[X_1, \dots, X_n]$, then

$$(1.1) \quad V(pq) = V(p) \cup V(q) .$$

This is especially satisfying because the variety $V(pq)$ is so easily obtained from the original pieces $V(p)$ and $V(q)$, just by taking their union. The simple form in (1.1) of course holds for ideals—that is, for ideals a, b in $k[X_1, \dots, X_n]$, we have

$$(1.2) \quad V(ab) = V(a) \cup V(b) .$$

What about sum? That is, how is $V(p + q)$ related to $V(p)$ and $V(q)$? $V(p + q)$ does not have that same kind of simple geometric relationship to $V(p)$ and $V(q)$. Of course, one cannot expect a purely geometric answer to this since, for example, $V(p) = V(ap)$ ($a \in k \setminus \{0\}$), but in general $V(p + q) \neq V(ap + q)$. Although “ $V(a + b) = V(a) \cap V(b)$ ” answers an analogous ideal-theoretic question, there is still a natural polynomial question, and polynomials should not be neglected. The trouble is that since $V(p) = V(ap)$ ($a \in k \setminus \{0\}$), it seems that in taking the variety, we lose too much geometric information to hope

to relate $V(p + q)$ with $V(p)$ and $V(q)$, and that polynomials are “really” trying to define something more informative. In this paper, we let a polynomial p define a more informative geometric object “ $V^*(p)$ ”, which does not ignore multiplication by elements of the groundfield. $V^*(p)$ will be generalized to $V^*(r)$ (r , a rational function) in §3. For these more informative objects, one can then reasonably hope to relate $V^*(p + q)$ with $V^*(p)$ and $V^*(q)$.

To keep things visual, we take $k = \mathbf{R}$, though it appears that much of what we say can be extended to any complete valued field. We begin by noting that $V(p)$ may be regarded as the inverse image $p^{-1}(\{0\})$ of $\{0\} \in \mathbf{R}$. In place of $\{0\}$, we consider the set $A = [0, 1] + 2\mathbf{Z} \subseteq \mathbf{R}$. Thus A consists of “half of \mathbf{R} ”—unit intervals separated by unit intervals. For $p \in \mathbf{R}[X_1, \dots, X_n]$, we define $V^*(p)$ by

$$(1.3) \quad V^*(p) = p^{-1}(A) (\subseteq \mathbf{R}_{x_1, \dots, x_n}).$$

EXAMPLE 1.4.

1.4.1. $V^*(X_1) \subseteq \mathbf{R}_{x_1, x_2}$ is just the cylindrization along \mathbf{R}_{x_2} of A and thus consists of equally-spaced bands parallel to \mathbf{R}_{x_2} . To a physicist, this could be considered as a “straight-line diffraction grating”.

1.4.2. $V^*(X_1^2 + X_2^2) \subseteq \mathbf{R}_{x_1, x_2}$ consists of circular bands centered at $(0, 0)$, the bounding circles having radii \sqrt{n} ($n = 1, 2, 3, \dots$). $V^*(X_1^2 + X_2^2)$ can thus be regarded as a “Fresnel diffraction plate”. The bands all have equal area.

Now it is a familiar fact that superimposing diffraction rulings creates alternately light and dark fringes called “moiré fringes”. Let us begin by looking at a specific example in \mathbf{R}_{x_1, x_2} : $p = aX_1$ ($a \neq 0$), and $q = X_1^2 + X_2^2$. From a purely geometric viewpoint, it is not very obvious “why” the line $V(aX_1)$ and the point $V(X_1^2 + X_2^2)$ should yield a circle $V(aX_1 + X_1^2 + X_2^2)$ centered at $(-a/2, 0)$ and passing through $(0, 0)$. Now look at $V^*(aX_1)$ and $V^*(X_1^2 + X_2^2)$. In the superposition of these two diffraction rulings, there appear moiré fringes, and we see that something remarkable happens: there is a system of circular fringes centered at $(-a/2, 0)$ and the bands of $V^*(aX_1 + X_1^2 + X_2^2)$ appear to run along these fringes. In fact, $V(aX_1 + X_1^2 + X_2^2)$ is the unique “fringe curve” (cf. §2) of this system passing through $V(aX_1) \cap V(X_1^2 + X_2^2)$. Fig. 3 in [2] shows these fringes; a is approximately 9.5. This figure also shows a similar system of fringes centered at $(a/2, 0)$; this, it turns out, corresponds to subtracting $X_1^2 + X_2^2$ from aX_1 . If this figure were extended out further, one could even detect fainter systems further out, which correspond to adding (or subtracting) various integral multiples of $X_1^2 + X_2^2$ from aX_1 .

REMARK 1.5. Both $V(p)$ and $V^*(p)$ have strengths and weaknesses, and these tend to complement each other. For instance, at the ground field level, we have:

- V is good at distinguishing $V(p)$ from $V(a + p)$, but
- V is poor at distinguishing $V(p)$ from $V(ap)$ ($a \in \mathbf{R} \setminus \{0\}$).
- V^* is good at distinguishing $V^*(p)$ from $V^*(ap)$
 - ($V^*(ap)$ is in obvious sense “finer” than $V^*(p)$ if $|a| \geq 1$, and “coarser” if $|a| \leq 1$), but
- V^* is less good at distinguishing $V^*(p)$ from $V^*(a + p)$
 - ($V^*(a + p) = V^*(a' + p)$ if $a - a' \in 2\mathbf{Z}$).

At the “nonconstant polynomial level” we have, for relatively prime polynomials $p, q \in \mathbf{R}[X_1, X_2] \setminus \mathbf{R}$:

- V^* is good at expressing $V^*(p + q)$ in terms of $V^*(p)$ and $V^*(q)$, but
- V^* is poor at expressing $V^*(pq)$ in terms of $V^*(p)$ and $V^*(q)$.
- V is good at expressing $V(pq)$ in terms of $V(q)$, but
- V is poor at expressing $V(p + q)$ in terms of $V(p)$ and $V(q)$.

Both V and V^* have equal claim to producing varieties from ideals, in the sense that

$$V(\mathfrak{a}) = V(p) = V^*(p) \stackrel{\text{def}}{=} V^*(\mathfrak{a}).$$

This whole idea of moiré fringes arising from superimposing diffraction gratings has been recognized and used in various physical sciences for some time. In 1926, for instance, Ronchi ([4]) succeeded in using the moiré effect to test lens systems for aberrations. Until the early 1950's, further progress was somewhat slow in coming due to the difficulty in creating sufficiently fine gratings. After that, there followed a great many applications of the moiré effect for very precise measurement. (See [3] and [5] for bibliographies.) More recently, some exciting applications of the moiré effect have been found. As one example, moiré fringes have been observed in electron micrographs of thin, crossed crystals. Such fringes supply far more information and detail than available from ordinary electron micrographs of single crystals. For instance, crystal dislocations of less than the diameter of one atom ($\cong 1$ angstrom) can be detected, which represents a resolving power of some 100 to 1000 times greater than that of the electron microscope itself.

Relative to the question raised at the beginning of this paper,

we have, in rough terms, the following: Given relatively prime polynomials $p, q \in \mathbf{R}[X_1, \dots, X_n] \setminus \mathbf{R}$, $V^*(p) \cup V^*(q)$ displays different systems of fringes, some obvious and others, fainter and not so obvious. These fringes correspond to rational linear combinations of p and q ; the fringes alternate between light and dark—in fact, the density “tries” to oscillate about $3/4$ in a piecewise linear way. The amplitudes of these oscillations are “arithmetic in character”—that is, in a linear combination $c_1 p + c_2 q$, if c_2/c_1 is rational, then the limit of the amplitudes as $b \rightarrow \infty$ depends on the prime factors of c_2/c_1 . If the prime factors are large and odd, this limit is small; if c_2/c_1 is irrational or a prime factor is even, this limit is zero.

These ideas have been worked out in [2]; in §2 next, we briefly summarize what we need of [2] in the present paper.

Everything we have mentioned so far deals with *polynomials*. However, one can go further and consider $V^*(r) = r^{-1}(A)$, where $r = r(X_1, \dots, X_n) \in \mathbf{R}(X_1, \dots, X_n) \setminus \mathbf{R}$ is a *rational function*. Such “diffraction rulings” now may possess “singular points”, where the bands flow together. This is in contrast to $V^*(p)$, where all the bands are disjoint.

This is not solely of mathematical interest; it arises in physical applications, just as the varieties $V^*(p)$ do. For example, this behavior is approximated in stress-strain problems when the material under consideration is subjected at certain points to very high stress or strain; stress-strain lines flow together at such points. We will see that points where bands of $V^*(r)$ flow together are points of indeterminacy of the rational function r .

The present paper is devoted to extending [2] to the rational case. This happens also to extend the affine study in [2] to a projective study, since polynomials, when recentered at a point at infinity, become no worse than rational functions.

2. Some basic results in the polynomial case.

DEFINITION 2.1. The subset $A = [0, 1] + 2Z$ of \mathbf{R} is called the *basic linear alternation on \mathbf{R}* .

DEFINITION 2.2. Let $p \in \mathbf{R}[X_1, X_2] \setminus \mathbf{R}$. Then $p^{-1}(A) \subseteq \mathbf{R}_{x_1 x_2}$ is called a *polynomial alternation in $\mathbf{R}_{x_1 x_2}$* , and is denoted by $V^*(p)$.

Let $p, q \in \mathbf{R}[X_1, X_2] \setminus \mathbf{R}$, with $(p, q) = 1$. For any nonzero linear combination $c_1 p + c_2 q$ ($c_i \in \mathbf{R}$), we shall look for fringes in $V^*(p) \cup V^*(q)$ running along the level varieties defined by $c_1 p + c_2 q$. Because it will simplify the exposition somewhat, *in the rest of this paper we write m in place of $-(c_2/c_1)$* . (The letter m reminds us of “slope”). Thus we will consider the varieties $q = mp + b$ ($b \in \mathbf{R}, m \in \mathbf{R} \cup \{\infty\}$). We

make the convention that $m = \infty$ in $q = mp + b$ means $p + b = 0$.

Since we will be testing for density along real curves defined by $q = mp + b$, we need the following definition.

DEFINITION 2.3. Let $p \in \mathbf{R}[X_1, X_2] \setminus \mathbf{R}$. Then $V(p) \subseteq \mathbf{R}_{x_1x_2}$ consists of infinitely many points, or only finitely many points, or else the set is \emptyset . $V(p)$ has dimension 1, 0 or -1 , respectively. In the first $V(p)$ is called a *real curve* in $\mathbf{R}_{x_1x_2}$, and we sometimes use the suggestive letter C for $V(p)$. If P is not isolated in C , we say that C has dimension 1 at P , and write $\dim_P C = 1$. Similarly, if P is isolated in C , we write $\dim_P C = 0$; if $P \notin C$, $\dim_P C = -1$.

REMARK 2.4. Let $p, q \in \mathbf{R}[X_1, X_2] \setminus \mathbf{R}$ with $(p, q) = 1$; let $b \in \mathbf{R}$, and let m be fixed in $\mathbf{R} \cup \{\infty\}$. Then the variety $V(q - mp - b)$, which we denote by $V_b(q - mp)$, is a *curve* either for all $b \in \mathbf{R}$, or for all b in a half line of \mathbf{R} . (See [2], Lemma 5.6.) When we use a phrase like “the curve $V_b(q - mp)$ ”, we shall be assuming that $V_b(q - mp)$ is a real curve. As b runs through \mathbf{R} in the first case, the collection of curves clearly covers $\mathbf{R}_{x_1x_2}$. As b runs through the half line in the second case, it is easy to check that the collection of these curves covers all points of $\mathbf{R}_{x_1x_2}$ except possibly on some real curve of $\mathbf{R}_{x_1x_2}$.

Let $\{V_b\}$ be the collection of real curves chosen from $V_b(q - mp)$, as b runs through \mathbf{R} . For fixed m , $\{V_b\}$ reveals “ m th order moiré behavior” if the density of $V^*(p) \cup V^*(q)$ within the curves of $\{V_b\}$ oscillates in b . (We still need to define “density”.) We refer to these testing curves as *m th order (moiré) fringes*, or simply (*moiré fringes*); the collection of these curves is called the *m th order fringe family*.

DEFINITION 2.5. Let S be a countable union of intervals in \mathbf{R} . Then S has density δ in \mathbf{R} if for any $\varepsilon > 0$, there is a decomposition of \mathbf{R} into a disjoint union of intervals $T_i \subseteq \mathbf{R}$, which are uniformly bounded in length, so that

$$(2.5.1) \quad \left| \delta - \frac{\text{length}(S \cap T_i)}{\text{length } T_i} \right| < \varepsilon$$

for all but possibly finitely many T_i .

Notation 2.6. $\Delta(X): \mathbf{R} \rightarrow \mathbf{R}$ denotes the periodic “triangle function” defined in the interval of periodicity $[-1, 1)$ by $\Delta(X) = 1 - 2|X|$.

If p and q are linear, we have the following basic result. (See [2], §3 for the proof.)

THEOREM 2.7. Let $(X_1, X_2) = X$, let $p(X) = a \cdot X$, and let $q(X) = a' \cdot X$, where a and a' are linearly independent in \mathbf{R}_X . Then any line in \mathbf{R}_X can be written as

$$L_b: q = mp + b$$

for some $m \in \mathbf{R} \cup \{\infty\}$ and $b \in \mathbf{R}$. Let $\mathfrak{d}(L_b)$ denote the density of $[V^*(p) \cup V^*(q)] \cap L_b$ in L_b . Then:

$$(2.7.1) \quad \left\{ \begin{array}{l} \mathfrak{d}(L_b) = 1 \quad (b \in A) \\ \mathfrak{d}(L_b) = \frac{1}{2} \quad (b \notin A) \end{array} \right\} \quad \text{if } m = 0 \quad \text{or} \quad \infty ;$$

$$(2.7.2) \quad \left\{ \mathfrak{d}(L_b) = \frac{3}{4} - \frac{\Delta(n_1 b)}{4n_1 n_2} \right\}$$

if $m = \frac{n_1}{n_2} \in \mathbf{Q} \setminus \{0\}$, where $(n_1, n_2) = 1$, and $n_1 n_2$ is odd.

$$(2.7.3) \quad \mathfrak{d}(L_b) = \frac{3}{4} \quad \text{for all other } m.$$

The L_b in (2.7.2) singles out those directions in which $V^*(p) \cup V^*(q)$ displays moiré behavior; in these directions the density oscillates about $3/4$ in a piecewise linear way. The amplitude is small when at least one of n_1, n_2 is large. (2.7.3) includes the limiting case in which the fraction becomes irrational ($n_1, n_2 \rightarrow \infty$), the oscillation about $3/4$ then becoming zero.

In place of the above linear polynomials p and q , we now consider arbitrary relatively prime $p, q \in \mathbf{R}[X_1, X_2] \setminus \mathbf{R}$. The following notion of “refinement” essentially makes the underlying rulings $V^*(p)$ and $V^*(q)$ very fine relative to the fringe bands in $V^*(p) \cup V^*(q)$ and one can then approximate each of the curves $q = mp + b$ by a piecewise linear curve to get a more general version of Theorem 2.7.

DEFINITION 2.8. Let $p, q \in \mathbf{R}[X_1, X_2] \setminus \mathbf{R}$, where $(p, q) = 1$. For any positive integer k and any $m \in \mathbf{R} \cup \{\infty\}$, define (s_m, t_m) by

$$\begin{cases} (kp, mkp + (q - mp)) & \text{if } m \in \mathbf{R} \\ (p, kq) & \text{if } m = \infty. \end{cases}$$

Then $V^*(s_m) \cup V^*(t_m)$ is called the k -refinement of $V^*(p) \cup V^*(q)$ along the m th order fringe family.

DEFINITION 2.9. Let $p, q \in \mathbf{R}[X_1, X_2] \setminus \mathbf{R}$, where $(p, q) = 1$; let $P \in \mathbf{R}_{X_1 X_2}$, and let $m \in \mathbf{R} \cup \{\infty\}$. Then $V^*(p) \cup V^*(q)$ has limit density \mathfrak{d}_P at P with respect to m if the following holds: For every $\varepsilon > 0$ there

is a $\delta > 0$ such that for all positive ρ less than δ , there is a $k_\rho > 0$ so that for all $k > k_\rho$,

$$\left| \mathfrak{d}_P - \frac{\text{area}[(V^*(s_m) \cup V^*(t_m)) \cap B(P, \rho)]}{\text{area}[B(P, \rho)]} \right| < \varepsilon.$$

Here $B(P, \rho)$ denotes the open disk of radius ρ , centered at P .

One can show that for fixed $m \in \mathbf{R} \cup \{\infty\}$, the above \mathfrak{d}_P is constant along any fringe curve $V_b(q - mp)$. With this \mathfrak{d}_P , Theorem 2.5 then holds for arbitrary relatively prime polynomials $p, q \in \mathbf{R}[X_1, X_2] \setminus \mathbf{R}$. See [2], §6 for details and proofs.

3. Some qualitative results about rational alternations and moirés.

We now begin our generalization to the “rational” level. We start with some preliminary definitions and notations.

DEFINITION 3.1. For any $r \in \mathbf{R}(X_1, X_2) \setminus \mathbf{R}$, $r^{-1}(\mathcal{A}) (\subseteq \mathbf{R}_{X_1 X_2})$ is called a *rational alternation on $\mathbf{R}_{X_1 X_2}$* ; we denote it by $V^*(r)$.

Notation 3.2. For any $r \in \mathbf{R}(X_1, X_2) \setminus \mathbf{R}$, write $r = p/q$, where $(p, q) = 1$. For $b \in \mathbf{R}$, let $V_b(r)$ denote $V(p - bq)$; let $V_\infty(r)$ denote $V(q)$. If r is clear from context, we may write simply V_b or V_∞ . In each case the variety may be a curve, consist of only finitely many points, or be \emptyset . If in a particular situation we wish to assume that any of these is actually a curve, we will use a phrase like “the curve V_b ”. Note that $V_0(r)$ and $V_\infty(r)$ share no one-dimensional components.

Let $r_1, r_2 \in \mathbf{R}(X_1, X_2) \setminus \mathbf{R}$, and write for $i = 1, 2$, $r_i = p_i/q_i$, where $(p_i, q_i) = 1$. Assume that $(r_1, r_2) = 1$ —that is, that no two of p_1, p_2, q_1, q_2 share a nonconstant factor, and let us agree that $V_b(r_2 - mr_1)$ means $V_b(r_1)$ when $m = \infty$. Then for fixed $m \in \mathbf{R} \cup \{\infty\}$ and $b \in \mathbf{R} \cup \{\infty\}$, any curve $V_b(r_2 - mr_1)$ is called an *mth order (moiré) fringe*, or simply a *(moiré) fringe*. One can then define the *k-refinement of $V^*(r_1) \cup V^*(r_2)$ along the mth order fringe family* just as in Definition 2.8, replacing p and q there by r_1 and r_2 , respectively. Also, at any $P \in \mathbf{R}_{X_1 X_2} \setminus (V_\infty(r_2 - mr_1))$, one can similarly generalize \mathfrak{d}_P in Definition 2.9 by replacing p and q there by r_1 and r_2 . One can then show, exactly as in the polynomial case (see [2], §6), that \mathfrak{d}_P is constant on each $V_b(r_2 - mr_1) \setminus V_\infty(r_2 - mr_1) (b \in \mathbf{R})$, and one can define $\mathfrak{d}(V_b(r_2 - mr_1))$ to be this constant value. With these definitions, the proof in [2], §6 of the polynomial version of Theorem 2.5, extends essentially *verbatim* to the r_1, r_2 -level.

We know at this stage that for r_1, r_2 as above, any two varieties $V_{b_1}(r_2 - mr_1)$ and $V_{b_2}(r_2 - mr_1) (b_1 \neq b_2)$ are disjoint in $\mathbf{R}_{X_1 X_2} \setminus V_\infty(r)$, and

we have determined the density induced on curves $V_b(r_2 - mr_1)$ ($b \in \mathbf{R}$) by $V^*(r_1) \cup V^*(r_2)$. To extend our results to all of $\mathbf{R}_{X_1 X_2}$, we next study the behavior of curves $V_b(r_2 - mr_1)$ near points of $V_\infty(r_2 - mr_1)$. The remainder of this paper is devoted to this study. We work out some basic qualitative results in this section. In §4 we find a density function.

We begin this study by determining, for a given $r \in \mathbf{R}(X_1, X_2) \setminus \mathbf{R}$, the “fixed points” in the family $\{V_b(r) \mid b \in \mathbf{R} \cup \{\infty\}\}$. First, since any two different $V_{b_1}(r)$, $V_{b_2}(r)$ are disjoint in $\mathbf{R}_{X_1 X_2} \setminus V_\infty(r)$, we know $V_{b_1}(r) \cap V_{b_2}(r) \subseteq V_\infty(r)$.

LEMMA 3.3. *Let $r \in \mathbf{R}(X_1, X_2) \setminus \mathbf{R}$, and let P be a point of $\mathbf{R}_{X_1 X_2}$. If there exist $b_1, b_2 \in \mathbf{R} \cup \{\infty\}$ ($b_1 \neq b_2$) such that $P \in V_{b_1}(r) \cap V_{b_2}(r)$, then P is contained in every $V_b(r)$ ($b \in \mathbf{R} \cup \{\infty\}$).*

Notation 3.4. The set of all P satisfying the conclusion in Lemma 3.3 is denoted by $W(r)$, or by simply W if r is clear from context.

REMARK 3.5. If r is a polynomial, then $W(r) = \emptyset$.

Proof of Lemma 3.3. Write $r = p_1/p_2$, $p_i \in \mathbf{R}[X_1, X_2]$, $(p_1, p_2) = 1$, and rewrite $p_1/p_2 = b$ as $b_{i1}p_1 + b_{i2}p_2 = 0$ ($b_{ij} \in \mathbf{R}$), where $\{b_{i1} = 0, b_{i2} \neq 0\}$ corresponds to $b_i = 0$, $\{b_{i1} \neq 0, b_{i2} = 0\}$ corresponds to $b_i = \infty$, and $\{b_{i1} \neq 0, b_{i2} \neq 0\}$ corresponds to $b_i \in \mathbf{R} \setminus \{0\}$. Then the condition $P \in V_{b_1}(r) \cap V_{b_2}(r)$ in the lemma becomes

$$(3.6) \quad \begin{aligned} b_{11}p_1(P) + b_{12}p_2(P) &= 0, \\ b_{21}p_1(P) + b_{22}p_2(P) &= 0. \end{aligned}$$

Now $b_1 \neq b_2$ implies that (3.6) has rank two; therefore $p_1(P) = p_2(P) = 0$. Thus for any $b = c_1/c_2 \in \mathbf{R} \cup \{\infty\}$, we have $c_1p_1(P) - c_2p_2(P) = 0$ —that is, $P \in V_b(r)$.

Here is an example of Lemma 3.3.

EXAMPLE 3.7. Let $r = (X_1^2 + X_2^2)/X_1X_2$. Then for all $|b| \leq 2$, $V_b = \{(0, 0)\} \subseteq \mathbf{R}_{X_1 X_2}$. $V_2(r)$ is the line $X_1 - X_2 = 0$, and $V_{-2}(r)$ is the line $X_1 + X_2 = 0$. For $|b| \geq 2$, V_b splits up into two lines, one approaching \mathbf{R}_{X_1} and the other approaching \mathbf{R}_{X_2} , as $b \rightarrow \infty$. $V_\infty = \mathbf{R}_{X_1} \cup \mathbf{R}_{X_2}$. Note that Lemma 3.3 holds (P is $(0, 0)$), even though V_b is not always a curve.

The next result relates $V_\infty(r_2 - mr_1)$ with $V_\infty(r_1)$ and $V_\infty(r_2)$.

LEMMA 3.8.

$$(3.8.1) \quad \text{For any } m \in \mathbf{R} \setminus \{0\}, V_\infty(r_2 - mr_1) \subseteq V_\infty(r_1) \cup V_\infty(r_2).$$

If $r_i = p_i/q_i$, where $(p_i, q_i) = 1$, and if $(q_1, q_2) = 1$, then equality holds in (3.8.1).

Proof. \subseteq : Obvious.

$=$: Follows at once from the following lemma:

LEMMA 3.9. Let $p_i, q_i \in \mathbf{R}[X_1, X_2]$, where $(p_1, q_1) = (p_2, q_2) = 1$, and let $m \in \mathbf{R} \setminus \{0\}$. If $p_2q_1 - mp_1q_2 \neq 0$, then

$$(p_2q_1 - mp_1q_2, q_1q_2) = 1 \iff (q_1, q_2) = 1.$$

Proof. $=$: Suppose that $(q_1, q_2) = s \in \mathbf{R}[X_1, X_2] \setminus \mathbf{R}$. Then s is a factor of q_1q_2 , and also a factor of $p_2q_1 - mp_1q_2$.

\Leftarrow : Suppose $(q_1, q_2) = 1$. Let t be any prime factor of q_1q_2 . We assume without loss of generality that $t|q_1$. Then tp_2q_1 . Therefore $t|p_2q_1 - mp_1q_2$ iff $t|p_1q_2$. However $(p_1, q_1) = 1$ and $(q_1, q_2) = 1$ imply, respectively, that $t \nmid p_1$ and $t \nmid q_2$. Thus $t \nmid p_1q_2$, and therefore $t \nmid p_2q_1 - mp_1q_2$. Thus $(p_2q_1 - mp_1q_2, q_1q_2) = 1$, which proves " \Leftarrow ".

The following ties in the "fixed points" of a moiré with those of the underlying alternations.

LEMMA 3.10. Let $m \in \mathbf{R} \setminus \{0\}$, and let $r_1 = p_1/q_1$, $r_2 = p_2/q_2$ and $r_2 - mr_1$ be elements of $\mathbf{R}(X_1, X_2) \setminus \mathbf{R}$. If $(q_1, q_2) = 1$, then

$$(3.10.1) \quad W(r_2 - mr_1) = W(r_1) \cup W(r_2) \cup (V_\infty(r) \cap V_\infty(r_2)).$$

Proof. First, $r_2 - mr_1 = (p_2q_1 - mp_1q_2)/q_1q_2$. Now $W(r_2 - mr_1)$ consists of those points of $\mathbf{R}_{X_1X_2}$ where q_1q_2 and $p_2q_1 - mp_1q_2$ are simultaneously zero. If both q_1 and q_2 vanish at $P \in W(r_2 - mr_1)$, then $P \in V_\infty(r_1) \cap V_\infty(r_2)$. If $q_1(P) = 0$ but $q_2(P) \neq 0$, then $p_2q_1 - mp_1q_2 = 0$ implies that $p_1(P) = 0$, so $P \in W(r_1)$. Similarly, if $q_2(P) = 0$ and $q_1(P) \neq 0$, then $P \in W(r_2)$.

REMARK 3.11. If q_1 and q_2 are not relatively prime, then (3.10.1) may not hold. For example if $r_1 = (X_1 + 1)/X_2$, $r_2 = (X_1 - 1)/X_2$, then $W(r_1 + r_2) = \{(0, 0)\}$, while (3.10.1) in this case is \mathbf{R}_{X_1} .

Moiré fringes, as well as the underlying alternations, display a wide range of behavior at points of W . For example let $r_1 = X_1$, and consider the projective completion in $\mathbf{P}^2(\mathbf{R})$ of $V^*(r_1) \subseteq \mathbf{R}_{X_1X_2}$. The boundaries of the bands all pass through a single, common point on the line at infinity, and any two such boundaries intersect there transversally. If $r_2 = X_2$, then the same holds for the boundaries of the completion of $V^*(r_2)$ at a different point at infinity; similarly for all the curves $V_i(X_2 - mX_1)$ (for any fixed $m \in \mathbf{R} \cup \{\infty\}$). However,

it may happen that each of the underlying alternations of a moiré has the above kind of transversally intersecting boundaries at infinity, but that for some m , the m th order moiré fringes may fail to have this property. For example, the boundaries of $V^*(X_1/X_2)$ intersect transversally at $(0, 0)$; likewise for $V^*(X_1 + X_1^2/X_2)$. However, any two fringes corresponding to $m = 1$ approach $(0, 0)$ in a mutually tangent manner. (We give definitions in a moment.) Just the opposite can happen, too. For example, any two boundaries of $V^*(X_1/X_2^2)$ are tangent at $(0, 0)$; similarly for the boundaries of $V^*(X_1 + X_1^2/X_2^2)$. However, any two fringes corresponding to $m = 1$ intersect *transversally* at $(0, 0)$. These are only two of a number of cases.

There exists a nine-way classification which covers all such possibilities. For this, we will need a few preliminary definitions and results.

First, throughout the rest of this section P will denote an arbitrary, fixed point in $\mathbf{R}_{X_1X_2}$. We will use the notion of *germ at P of a real curve C* . If $P \notin C$ or if P is isolated in C (i.e., if $\dim_P C = -1$ or 0), the usual germ can be identified with \emptyset and P , respectively. However for our purposes, it will simplify the exposition to agree that the notion *germ of C at P* carries with it the assumption that $\dim_P C = 1$. Since P is fixed throughout, if $\dim_P C = 1$ then the germ of C at P will be denoted by C^\sim .

We next consider the notion of *tangent cone to a germ at P* ; this generalizes "tangent space" at a point of a differentiable manifold. This will be useful when considering transversality and tangency.

DEFINITION 3.12. Suppose C^\sim exists, and let C be a representative of C^\sim in a $\mathbf{R}_{X_1X_2}$ -open neighborhood of P . The *tangent cone to C^\sim* , denoted by $T(C^\sim)$, is $\{P\}$ together with the set of all vectors v based at P satisfying this condition: there is a sequence of points $\{P_i\}$, $P_i \in C \setminus P$, $P_i \rightarrow P$, and a sequence of numbers $\{r_i\}$ such that $v = \lim_{i \rightarrow \infty} r_i(P_i - P)$.

EXAMPLE 3.13. Suppose $P = (0, 0) \in \mathbf{R}_{X_1X_2}$. For $C \subseteq \mathbf{R}_{X_1X_2}$ defined by $X_2^2 - X_1^3$, $T(C^\sim) = \mathbf{R}_{X_1}$; for $C \subseteq \mathbf{R}_{X_1X_2}$ defined by $X_2^2 - X_1(X_1^2 - 1)$, $T(C^\sim)$ consists of the two lines $X_1 + X_2 = 0$ and $X_1 - X_2 = 0$.

The following algebraic characterization of $T(C^\sim)$ is important in the sequel.

THEOREM 3.14. Let $C \subseteq \mathbf{R}_{X_1X_2}$ be defined by a polynomial $p \in \mathbf{R}[X_1, X_2] \setminus \mathbf{R}$, let $\dim_P C = 1$, and let p^\dagger denote the initial part of p is expanded about P . Then $T(C^\sim)$ is the zero set in $\mathbf{R}_{X_1X_2}$ of p^\dagger .

Proof. Theorem 3.14 is a special case of Theorem 2.8 in [1]

(which, in fact, includes the case when $\dim_P C = -1$ or 0 , though we don't need this here).

REMARK 3.15. $T(C^\sim)$ in Theorem 3.13 consists of finitely many lines through P . Thus both $T(C^\sim)$ and p^\dagger are homogeneous with respect to P . If p has no nonconstant repeated factors, and if the Jacobian (p_{x_1}, p_{x_2}) has rank one at P , then the zero set of p in $C_{x_1x_2}$ is a real differentiable manifold in a C^2 -neighborhood of P , $C \subseteq \mathbf{R}_{x_1x_2}$ is smooth at P , and $T(C^\sim)$ coincides with the usual (real) tangent line to C at P .

EXAMPLE 3.16. Let $P = (0, 0) \in \mathbf{R}_{x_1x_2}$. For $C \subseteq \mathbf{R}_{x_1x_2}$ defined by $X_2^2 - X_1^2$, $T(C^\sim)$ is defined by $X_2^2 = 0$ — that is, $T(C^\sim) = \mathbf{R}_{x_1}$; for C defined by $X_2^2 - X_1(X_1^2 - 1)$, $T(C^\sim)$ is defined by $X_2^2 - X_1^2 = 0$ — that is, it consists of the two lines $X_1 + X_2 = 0$ and $X_1 - X_2 = 0$. (Cf. Example 3.12.)

DEFINITION 3.17. Let $P \in \mathbf{R}_{x_1x_2}$, and suppose curves C_1 and C_2 in $\mathbf{R}_{x_1x_2}$ both have dimension one at P . Then C_1^\sim and C_2^\sim are said to be *tangent along (precisely) $T(C_1^\sim) \cap T(C_2^\sim)$* . If $T(C_1^\sim) = T(C_2^\sim)$, we say C_1^\sim and C_2^\sim are *fully tangent*. If $T(C_1^\sim) \cap T(C_2^\sim) = P$, we say that C_1^\sim and C_2^\sim *intersect transversally*.

REMARK 3.18. Let the curves C_i in Definition 3.16 be defined by $p_i \in \mathbf{R}[X_1, X_2] \setminus \mathbf{R}$. Then C_1^\sim and C_2^\sim intersect transversally iff p_1^\dagger and p_2^\dagger have no common real linear factors.

We now look at the behavior at points of W_∞ of a given rational alternation $V^*(r)$ ($r \in \mathbf{R}(X_1, X_2) \setminus \mathbf{R}$). Write $r = p/q$, where $(p, q) = 1$.

THEOREM 3.19. *With p, q and r as above, let P be a fixed, arbitrary point in $W(r)$, and let p^\dagger and q^\dagger be the initial forms of p and q at P .*

3.19.1. *If $\deg p^\dagger = \deg q^\dagger$, then for all but at most one value b_0 of $b \in \mathbf{R}$, any two $V_{b_1}^\sim(r), V_{b_2}^\sim(r)$ ($b_1 \neq b_2, b_1, b_2 \in \mathbf{R}$) are tangent at P along precisely $V(p^\dagger, q^\dagger)$. If there is a b_0 satisfying $p^\dagger = b_0 q^\dagger$, then $V_{b_0}^\sim(r)$, if it exists, is tangent at P to every other $V_b^\sim(r)$ iff each real linear factor of (p^\dagger, q^\dagger) is a factor of $(p - b_0 q)^\dagger$.*

3.19.2. *If $\deg p^\dagger \leq \deg q^\dagger$, then any two $V_{b_1}^\sim(r), V_{b_2}^\sim(r)$ ($b_1 \neq b_2, b_1, b_2 \in \mathbf{R}$) are fully tangent, with common tangent cone $V(p^\dagger)$. $V_\infty^\sim(r)$ is tangent to $V_b^\sim(r)$ ($b \in \mathbf{R}$) along precisely $V(p^\dagger, q^\dagger)$.*

3.19.3. If $\deg p^\dagger \geq \deg q^\dagger$, then any two $V_{b_1}^\sim(r)$, $V_{b_2}^\sim(r)$ ($b_1 \neq b_2$, $b_1, b_2 \in (\mathbf{R} \setminus \{0\}) \cup \{\infty\}$) are fully tangent, the common tangent cone being $V(q^\dagger)$. $V_b^\sim(r)$ is tangent to $V_b^\sim(r)$ ($b \in (\mathbf{R} \setminus \{0\}) \cup \{\infty\}$) along precisely $V(p^\dagger, q^\dagger)$.

Proof.

3.19.1. If $p^\dagger \neq b_i q^\dagger$, then $(p - b_i q)^\dagger = p^\dagger - b_i q^\dagger$ ($i = 1, 2$). Then 3.19.1 follows at once from this fact:

Suppose s and t are homogeneous polynomials in $\mathbf{R}[X_1, X_2] \setminus \mathbf{R}$ having the same degree, and let b_1, b_2 ($b_1 \neq b_2$) be two arbitrary elements of \mathbf{R} . If $s + b_1 t$ and $s + b_2 t$ are nonzero, then (up to a nonzero constant factor) $(s, t) = (s + b_1 t, s + b_2 t)$.

We prove this as follows: First, clearly $(s, t) | (s + b_1 t, s + b_2 t)$. We wish to show that $(s + b_1 t, s + b_2 t) | (s, t)$. For this, suppose $u | (s + b_1 t, s + b_2 t)$ —say

$$\left. \begin{aligned} s + b_1 t &= uv_1 \\ s + b_2 t &= uv_2 \end{aligned} \right\} (u, v_i \in \mathbf{R}[X_1, X_2]) .$$

Solving these two equations for s and t gives

$$\begin{aligned} s &= u(v_1 - b(v_2 - v_1)/b_2 - b_1) \\ t &= u(v_2 - v_1)/b_2 - b_1 . \end{aligned}$$

Thus $u | (s, t)$, so $(s + b_1 t, s + b_2 t) | (s, t)$.

3.19.2. For any $b \in \mathbf{R}$ we have $(p - bq)^\dagger = p^\dagger$, and the first sentence of 3.19.2 follows. Now when $b = \infty$, instead of $p - bq$ we may write $a_1 p - a_2 q$, where $a_1 = 0$ and $a_2 \neq 0$; this gives $(p - bq)^\dagger = q^\dagger$. Since $(p - bq)^\dagger = p^\dagger$ for any other b , the second sentence of 3.19.2 follows.

3.19.3. For any $b \in (\mathbf{R} \setminus \{0\}) \cup \{\infty\}$, we have $(p - bq)^\dagger = q^\dagger$, giving the first sentence of 3.19.3. When $b = 0$, $(p - bq)^\dagger = p^\dagger$, and the second sentence follows.

We next turn our attention to the question of how at a point P , the geometric behavior of rational alternations $r_1^{-1}(A), r_2^{-1}(A) \subseteq \mathbf{R}_{x_1 x_2}$ influences the geometry at P of the m th-order fringes of $r_1^{-1}(A) \cup r_2^{-1}(A)$. The next theorem is the promised 3×3 classification of local behavior of moiré fringes in the neighborhood P .

Before stating it, suppose that $r_i = p_i/q_i$, where $(p_i, q_i) = 1$ ($i = 1, 2$); now if, for example, $p_1 = p_2$ and $q_1 = q_2$, then $V^*(r_1) \cup V^*(r_2) =$

$V^*(r_1) = V^*(r_2)$. The resulting moiré fringes $V_b(r_2 - mr_1) = V_b((p_2q_1 - mp_1q_2)/q_1q_2)$ are in a sense "trivial". However by Lemma 3.9, it is clear that assuming $(q_1, q_2) = 1$ avoids this behavior. We make this assumption in Theorem 3.20, next.

THEOREM 3.20. *Let $r_1 = p_1/q_1, r_2 = p_2/q_2$, where $r_i \in \mathbf{R}(X_1, X_2) \setminus \mathbf{R}$. Let p_i^\dagger, q_i^\dagger be the respective parts at P of p_i, q_i ($i = 1, 2$), and assume that $(p_1, q_1) = (p_2, q_2) = (q_1, q_2) = 1$. We then have the following classification where, for any $m \in \mathbf{R} \setminus \{0\}$, V_b^\sim denotes $V_b^\sim(r_2 - mr_1)$, and where the italicized statement under (i, j) ($i = \text{I, II, III}; j = 1, 2, 3$) is the conclusion obtained from assuming i and j .*

I. $\text{ord}_P(p_1/q_1) = \text{ord}_P(p_2/q_2)$

1. $\text{ord}_P(p_1) = \text{ord}_P(q_1)$

(a) If $p_2^\dagger q_1^\dagger - mp_1^\dagger q_2^\dagger \neq 0$, then any two different germs $V_{b_1}^\sim$ and $V_{b_2}^\sim$ ($b_1, b_2 \in \mathbf{R} \cup \{\infty\}$) are tangent along precisely $V(p_2^\dagger q_1^\dagger - mp_1^\dagger q_2^\dagger, q_1^\dagger q_2^\dagger)$.

(b) If $p_2^\dagger q_1^\dagger - mp_1^\dagger q_2^\dagger = 0$, then any two germs $V_{b_1}^\sim$ and $V_{b_2}^\sim$ ($b_1, b_2 \in (\mathbf{R} \setminus \{0\}) \cup \{\infty\}$) are fully tangent, the tangent cone being $V(q_1^\dagger) \cup V(q_2^\dagger)$; and V_∞^\sim is tangent to each V_b^\sim ($b \in (\mathbf{R} \setminus \{0\}) \cup \{\infty\}$) along precisely $V(p_2^\dagger q_1^\dagger - mp_1^\dagger q_2^\dagger, q_1^\dagger q_2^\dagger)$.

2. $\text{ord}_P(p_1) \leq \text{ord}_P(q_1)$

(a) If $p_2^\dagger q_1^\dagger - mp_1^\dagger q_2^\dagger \neq 0$, then any two germs $V_{b_1}^\sim$ and $V_{b_2}^\sim$ ($b_1, b_2 \in \mathbf{R}$) are fully tangent, the tangent cone being $V(p_2^\dagger q_1^\dagger - mp_1^\dagger q_2^\dagger)$. V_∞^\sim is tangent to each V_b^\sim ($b \in \mathbf{R}$) along precisely $V(p_2^\dagger q_1^\dagger - mp_1^\dagger q_2^\dagger, q_1^\dagger q_2^\dagger)$.

(b) If $p_2^\dagger q_1^\dagger - mp_1^\dagger q_2^\dagger = 0$, then any of $>$, $=$, or $<$ may hold between $\deg((p_2q_1 - mp_1q_2)^\dagger)$ and $\deg((q_1q_2)^\dagger)$, respectively. For " $>$ ", conclusion (I, 1(b)) holds; for " $=$ ", conclusion (I, 1(a)) holds; for " $<$ ", conclusion (I, 2(a)) holds.

3. $\text{ord}_P(p_1) \geq \text{ord}_P(q_1)$

Conclusion is same as that of (I, 1(b)).

II. $\text{ord}_P(p_1/q_1) \leq \text{ord}_P(p_2/q_2)$

1. $\text{ord}_P(p_1) = \text{ord}_P(q_1)$

Any two different germs $V_{b_1}^\sim$ and $V_{b_2}^\sim$ ($b_1, b_2 \in \mathbf{R} \cup \{\infty\}$) are tangent along precisely $V(q_2^\dagger) \cup V(p_1^\dagger, q_1^\dagger)$.

2. $\text{ord}_P(p_1) \leq \text{ord}_P(q_1)$

Any two germs $V_{b_1}^\sim$ and $V_{b_2}^\sim$ ($b_1, b_2 \in \mathbf{R}$) are fully tangent, the tangent cone being $V(p_1^\dagger) \cup V(q_2^\dagger)$. V_∞^\sim is in tangent to each V_b^\sim ($b \in \mathbf{R}$) along precisely $V(q_2^\dagger) \cup V(p_1^\dagger, q_1^\dagger)$.

3. $\text{ord}_P(p_1) \geq \text{ord}_P(q_1)$

Any two germs $V_{b_1}^\sim$ and $V_{b_2}^\sim$ ($b_1, b_2 \in (\mathbf{R} \setminus \{0\}) \cup \{\infty\}$) are fully tangent, with tangent cone $V(q_1^\dagger) \cup V(q_2^\dagger)$. V_∞^\sim is tangent to each V_b^\sim ($b \in (\mathbf{R} \setminus \{0\}) \cup \{\infty\}$) along precisely $V(q_2^\dagger) \cup V(p_1^\dagger, q_1^\dagger)$.

III. $\text{ord}_P(p_1/q_1) \not\cong \text{ord}_P(p_2/q_2)$

Same as in II, except that subscripts "1" and "2" of p and q are everywhere reversed.

REMARK 3.21. Among the polynomials p_i, q_i ($i = 1, 2$) satisfying the condition stated at the beginning of Theorem 3.20, almost all of them also satisfy

$$(3.21.1) \quad (p_1^\dagger, q_1^\dagger) = (p_2^\dagger, q_2^\dagger) = (q_1^\dagger, q_2^\dagger) = 1 .$$

More precisely, for a fixed positive integer d , there is in the vector space Y_1 (resp. Y_2, Z_1, Z_2) of polynomials p_1 (resp. p_2, q_1, q_2) of degree $\leq d$, a proper algebraic subvariety A_1 (resp. A_2, B_1, B_2) such that for polynomials $p_1 \in Y_1 \setminus A_1, p_2 \in Y_2 \setminus A_2, q_1 \in Z_1 \setminus B_1$ and $q_2 \in Z_2 \setminus B_2$, 3.21.1 holds. In this sense, the conclusion of the following corollary of Theorem 3.20 "almost always" holds.

COROLLARY 3.22. *If in Theorem 3.20, 3.21.1 holds, then in Theorem 3.20, any variety written in the form $V(\cdot, \cdot)$, is P . In each such case, the corresponding germs V_{b_1}, V_{b_2} intersect transversally.*

Proof of Corollary 3.22. Cases II and III are clear, since $(p_i^\dagger, q_i^\dagger) = 1$. For case I, it suffices to show that $(p_2^\dagger q_1^\dagger - m p_1^\dagger q_2^\dagger, q_1^\dagger q_2^\dagger) = 1$. But this follows at once from Lemma 3.9, with p_i, q_i replaced by p_i^\dagger, q_i^\dagger ($i = 1, 2$).

Proof of Theorem 3.20. In each case we compare $\text{deg}((p_2 q_1 - m p_1 q_2)^\dagger)$ with $\text{deg}((q_1 q_2)^\dagger)$ and apply Theorem 3.19.

I. Note that $\text{ord}_P(p_1/q_1) = \text{ord}_P(p_2/q_2)$ implies that $\text{ord}_P(p_1 q_2) = \text{ord}_P(p_2 q_1)$. Now

$$(3.23) \quad p_2^\dagger q_1^\dagger - m p_1^\dagger q_2^\dagger \neq 0 \implies (p_2 q_1 - m p_1 q_2)^\dagger = p_2^\dagger q_1^\dagger - m p_1^\dagger q_2^\dagger ;$$

and

$$(3.24) \quad p_2^\dagger q_1^\dagger - m p_1^\dagger q_2^\dagger = 0 \implies \text{deg}((p_2 q_1 - m_1 p_2)^\dagger) \cong \text{deg } p_1^\dagger q_2^\dagger \quad (= \text{deg } p_2^\dagger q_1^\dagger) .$$

1. $\text{ord}_P(p_1/q_1) = \text{ord}_P(p_2/q_2)$ and $\text{ord}_P(p_1) = \text{ord}_P(q_1)$ imply that $\text{ord}_P(p_2) = \text{ord}_P(q_2)$. Thus in this case $\text{deg } p_i^\dagger = \text{deg } q_i^\dagger$ ($i = 1, 2$). Therefore if $p_2^\dagger q_1^\dagger - m p_1^\dagger q_2^\dagger \neq 0$, then $\text{deg}(p_2^\dagger q_1^\dagger - m p_1^\dagger q_2^\dagger) = \text{deg } q_1^\dagger q_2^\dagger$. This, together with 3.23 and Theorem 3.19.1 establishes (I, 1(a)). Next, suppose that $p_2^\dagger q_1^\dagger - m p_1^\dagger q_2^\dagger = 0$. Then $\text{deg}((p_2 q_1 - m p_1 q_2)^\dagger) \cong \text{deg } q_1^\dagger q_2^\dagger$; this together with Theorem 3.18.3 establishes (I, 1(b)).

2(a). $\text{ord}_P(p_1/q_1) = \text{ord}_P(p_2/q_2)$ together with $\text{ord}_P(p_1) \not\cong \text{ord}_P(q_1)$ imply that $\text{ord}_P(p_2) \not\cong \text{ord}_P(q_2)$. $p_2^\dagger q_1^\dagger - m p_1^\dagger q_2^\dagger \neq 0$ then implies that

$\deg(p_2^*q_1^* - mp_1^*q_2^*) \leq \deg q_1^*q_2^*$. This, with 3.23 and Theorem 3.19.2 proves (I, 2(a)). (I, 2(b)) is clear.

3. $\text{ord}_P(p_1/q_1) = \text{ord}_P(p_2/q_2)$ together with $\text{ord}_P(p_1) \geq \text{ord}_P(q_1)$ imply that $\deg(p_2^*q_1^* - mp_1^*q_2^*) \geq \deg(q_1^*q_2^*)$; if $p_2^*q_1^* - mp_1^*q_2^* \neq 0$, then 3.23 implies

$$(3.25) \quad \deg((p_2q_1 - mp_1q_2)^\dagger) \geq \deg(q_1^*q_2^*) .$$

If $p_2^*q_1^* - mp_1^*q_2^* = 0$, then (3.24) shows that (3.25) holds *a fortiori*. Now applying Theorem 3.19.3 establishes (I, 3).

II. Note that $\text{ord}_P(p_1/q_1) \leq \text{ord}_P(p_2/q_2)$ implies that $\text{ord}_P(p, q_2) \leq \text{ord}_P(p, q_1)$; hence in this case

$$(3.26) \quad (p_2q_1 - mp_1q_2)^\dagger = -mp_2^*q_1^* .$$

1. $\text{ord}_P(p_1) = \text{ord}_P(q_1)$ (that is, $\deg p_1^* = \deg q_1^*$) and (3.26) imply that $\deg((p_2q_1 - mp_1q_2)^\dagger) = \deg(q_1^*q_2^*)$. Theorem 3.19.1 now establishes (II, 1).

2. $\text{ord}_P(p_1) \leq \text{ord}_P(q_1)$ and (3.26) show that $\deg((p_2q_1 - mp_1q_2)^\dagger) \leq \deg(q_1^*q_2^*)$. Then Theorem 3.19.2 establishes (II, 2).

3. In analogy to the above, we have $\deg((p_2q_1 - mp_1q_2)^\dagger) \geq \deg(q_1^*q_2^*)$, and Theorem 3.19.3 establishes (II, 3).

III. The proofs are the same as those in II, except that the subscripts "1" and "2" of p and q are everywhere reversed.

Here is an example of Theorem 3.20.

EXAMPLE 3.23. Let $\varepsilon > 0$, and in the notation of Theorem 3.20 let $r_1 = p_1/q_1 = 1/(Y + \varepsilon X)$, $r_2 = p_2/q_2 = 1/(Y - \varepsilon X)$. We consider fringes in $V^*(1/(Y + \varepsilon X)) \cup V^*(1/(Y - \varepsilon X))$, corresponding to $m = 1$, passing through singular points—that is, through points of $W_2(r_2 - mr_1)$. Now from Lemma 3.10 we see that $W(r_2 - mr_1) = W(r_1) \cup W(r_2) \cup (V_\infty(r_1) \cap V_\infty(r_2))$. In this example, any two different $V_i(r_1)$ are disjoint (they are the mutually parallel lines $Y + \varepsilon X = 1/b$); similarly for $V_i(r_2)$. Hence $W(r_1) = W(r_2) = \emptyset$. But $V_\infty(r_1) = V(Y + \varepsilon X)$ and $V_\infty(r_2) = V(Y - \varepsilon X)$, so $V_\infty(r_1) \cap V_\infty(r_2) = \{(0, 0)\}$. Hence $W(r_2 - mr_1)$ consists of the single point $W(r_1) \cup W(r_2) \cup (V_\infty(r_1) \cap V_\infty(r_2)) = \{(0, 0)\}$ —that is, each 1-fringe passes through $(0, 0)$. This can be seen in Figure 1, where the obvious fringes correspond to $m = 1$. $V_i(r_2 - mr_1)$ is given by $1/(Y + \varepsilon X) - 1/(Y - \varepsilon X) = b$, which reduces to

$$(3.23.1) \quad \frac{\left(X - \frac{1}{\varepsilon b}\right)^2}{\left(\frac{1}{\varepsilon b}\right)^2} - \frac{Y^2}{\left(\frac{1}{b}\right)^2} = 1 .$$

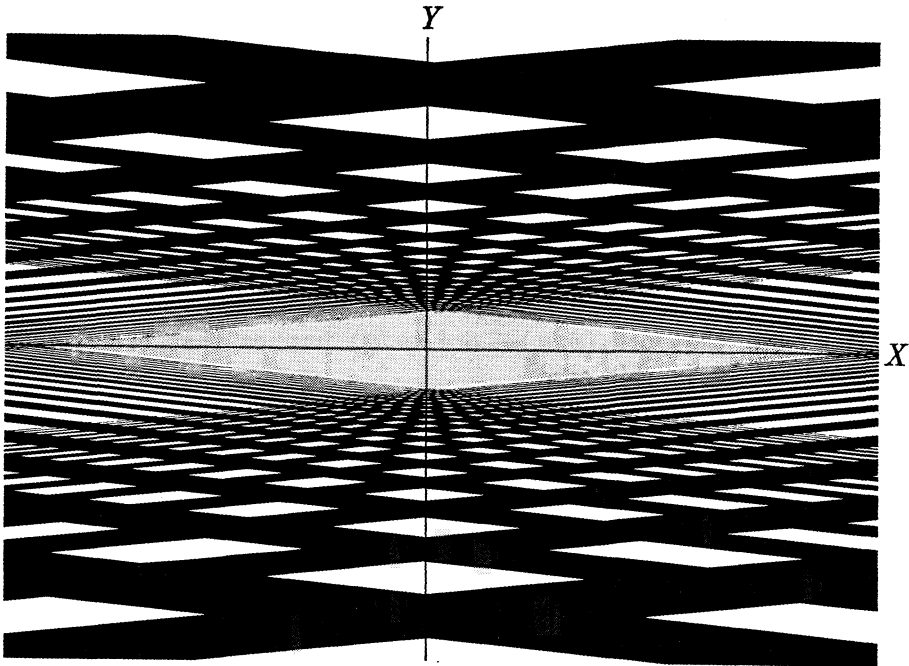


FIGURE 1

For each $b \in \mathbf{R}$, one branch of the above hyperbola passes through $(0, 0)$.

Let $P \doteq (0, 0)$. Then $\text{ord}_P(p_1/q_1) = \text{ord}_P(p_2/q_2) = -1$. We see that (I, 2(a)) of Theorem 3.20 applies, since also $\text{ord}_P(p_1) = 0 \not\cong \text{ord}_P(q_1) = 1$, and $p_2^*q_1^* - mp_1^*q_2^* = 1 \cdot (Y + \varepsilon X) - 1 \cdot (Y - \varepsilon X) = 2\varepsilon X \neq 0$. The theorem then tells us that any two germs $V_{b_1}^{\sim}(r_2 - mr_1)$ and $V_{b_2}^{\sim}(r_2 - mr_1)$ ($b_1, b_2 \in \mathbf{R}$) are fully tangent, the tangent cone being $V(2\varepsilon X) = \mathbf{R}_Y$. This is also evident in Fig. 1. For $b = \infty$, (I, 2(a)) tells us that $V_{\infty}^{\sim}(r_2 - mr_1)$ is tangent to each $V_b^{\sim}(r_2 - mr_1)$ ($b \in \mathbf{R}$) along $V(p_1^*q_1^* - mp_1^*q_2^*, q_1^*q_2^*) = V(2\varepsilon X, (Y + \varepsilon X)(Y - \varepsilon X)) = \{(0, 0)\}$ —that is, that $V_{\infty}^{\sim}(r_2 - mr_1)$ is transverse to each $V_b^{\sim}(r_2 - mr_1)$ ($b \in \mathbf{R}$) at $(0, 0)$. This is in fact so, since a representative of $V_{\infty}^{\sim}(r_2 - mr_1)$ is \mathbf{R}_X , which is transverse to \mathbf{R}_Y at $(0, 0)$.

Note. The fringes running along the other branches of the hyperbola in (3.23.1) cross \mathbf{R}_X at $1/\varepsilon b$. These fringes are especially obvious only for smaller values of b —that is, when $1/\varepsilon b$ is large—and therefore do not appear in Figure 1.

Incidentally, one can also verify Lemma 3.8 in this case, too. $V_{\infty}(r_1) = V(Y + \varepsilon X)$, which is the limit of the set of parallel lines $V_b(r_1)$ ($b \in \mathbf{R}$). Similarly, $V_{\infty}(r_2) = V(Y - \varepsilon X)$. Since $(q_1, q_2) = (Y + \varepsilon X, Y - \varepsilon X) = 1$, we see that equality holds in (3.8.1), so the hyperbolic fringes in Fig. 1 should approach the limit $V(Y + \varepsilon X) \cup V(Y - \varepsilon X)$

as $b \rightarrow \infty$. Of course, this union is the set of two asymptotes of the hyperbolic branches through $(0, 0)$. As $b \rightarrow \infty$, these branches do indeed approach the two lines $Y = \pm \varepsilon X$.

4. **Density of rational alternations and moirés at V_∞ .** In this section we extend density results in $\mathbf{R}_{x_1x_2} \setminus V_\infty$ noted at the beginning of §3, to points of V_∞ . More precisely, let r_1, r_2 be relatively prime in $\mathbf{R}(X_1, X_2) \setminus \mathbf{R}$ (i.e., if $r_i = p_i/q_i$, $(p_i, q_i) = 1$, then no two of p_1, p_2, q_1, q_2 share a nonconstant factor). We wish to study the density at points of $V_\infty(r_i)$ of $V^*(r_i)$, and at points of $V_\infty(r_2 - mr_1)$ of $V^*(r_1) \cup V^*(r_2)$. That is, let P be a point in $V_\infty(r_2 - mr_1)$, let \mathcal{S} be a representative of a real place \mathcal{S} centered at P (as in (4.3)), and let $B(P; \rho)$ be a disk of radius ρ centered at P . The problem is then to evaluate

$$(4.1) \quad \lim_{\rho \rightarrow 0} \frac{\text{length} [(V^*(r_1) \cup V^*(r_2)) \cap C \cap B(P; \rho)]}{\text{length} [C \cap B(P; \rho)]} .$$

It turns out that this limit always exists; it is independent of \mathcal{S} if $P \in V_\infty(r_2 - mr_1) \setminus W(r_2 - mr_1)$, but depends strongly on \mathcal{S} if $P \in W(r_2 - mr_1)$. We begin by looking at the underlying alternations (corresponding to $m = 0$ or ∞), and then turn to moiré fringes.

In evaluating limits like (4.1), the following result is useful.

LEMMA 4.2. *Let α be fixed in $\mathbf{R} \setminus \{0\}$; let $z \in \mathbf{R}$. Then*

$$(4.2.1) \quad \lim_{z \rightarrow \infty} \left(\frac{(z + 1)^\alpha - (z + 2)^\alpha}{z^\alpha - (z + 1)^\alpha} \right) = 1 .$$

Proof. We may assume without loss of generality that $\alpha \geq 0$, for

$$\frac{(z + 1)^{-\alpha} - (z + 2)^{-\alpha}}{z^{-\alpha} - (z + 1)^{-\alpha}} = \left(\frac{z}{z + 2} \right)^\alpha \left(\frac{(z + 1)^\alpha - (z + 2)^\alpha}{z^\alpha - (z + 1)^\alpha} \right) .$$

Divide numerator and denominator of the quotient in (4.2.1) by z^α and set $1/z = s$; this gives us

$$(4.2.2) \quad \frac{(1 + s)^\alpha - (1 + 2s)^\alpha}{1 - (1 + s)^\alpha} \quad (\alpha \geq 0) .$$

The binomial series then allows us to rewrite (4.2.2) as

$$(4.2.3) \quad \frac{-\alpha s + s^2 \cdot [\text{bounded}]}{-\alpha s + s^2 \cdot [\text{bounded}]} .$$

As $z \rightarrow \infty$ (i.e., as $s \rightarrow 0$), (4.2.3) approaches 1.

Let $r \in \mathbf{R}(X_1, X_2) \setminus \mathbf{R}$. We now consider the density of $V^*(r)$ at points of $V_\infty(r)$. We consider separately the two cases $P \in V_\infty(r) \setminus W(r)$, and $P \in W(r)$.

First, let $P \in W_\infty(r) \setminus W(r)$. We wish to show that with respect to any real place \mathcal{P} centered at P and not contained in V_∞ , the density $\delta_{\mathcal{P}}(\mathcal{P})$ at P is $1/2$. For this, suppose without loss of generality that $P = (0, 0)$; then we may write $r = p(X_1, X_2)/q(X_1, X_2)$, where $p(0, 0) = 0, q(0, 0) \neq 0$. Let \mathcal{P} be a place centered at $(0, 0)$, with representative $\bar{\mathcal{P}}$, defined near $(0, 0)$ by, say,

$$(4.3) \quad \begin{aligned} X_1 &= t \\ X_2 &= f(t) \quad (f \text{ real analytic at } t = 0). \end{aligned}$$

We may assume without loss of generality that \mathbf{R}_{X_1} is tangent to $\bar{\mathcal{P}}$. We are interested in the intersections near $(0, 0)$ of $\bar{\mathcal{P}}$ with the curves defined by $r = n$ ($n \in \mathbf{Z}$). Consider, therefore,

$$(4.4) \quad \frac{p(t^\alpha, f(t))}{q(t^\alpha, f(t))} = n \quad (|n| \text{ large}).$$

Since $\text{ord}_{(0,0)} p = 0$ and $\text{ord}_{(0,0)} q \geq 0$, the order k in t of $q(t^\alpha, f(t))/p(t^\alpha, f(t))$ is $k \geq 1$. Then for $|n|$ sufficiently large, (4.4) has approximate solutions near $t = 0$ given by

$$(4.5) \quad at^k = \frac{1}{n} \quad (a \in \mathbf{R} \setminus (0) \text{ independent of } t).$$

Therefore we have $t \cong (1/an)^{1/k}$, i.e., $t^\alpha \cong (1/an)^{\alpha/k}$, or

$$(4.6) \quad X_1 \cong \left(\frac{1}{an}\right)^{\alpha/k}.$$

Since \mathbf{R}_{X_1} is tangent to $\bar{\mathcal{P}}$ at $(0, 0)$, we see that for any large n , the distance from $(0, 0)$ to any intersection of $r(X_1, X_2) = n$ with $\bar{\mathcal{P}}$ is well-approximated by $|X_1|$ in (4.6). (Note that for a fixed, arbitrarily large n , there may be zero, one or two such intersections. There are no intersections corresponding to n iff there are two intersections corresponding to $-n$.) But (4.6) together with Lemma 4.2 shows that the distance measured along $\bar{\mathcal{P}}$ between successive points of intersection of $\bar{\mathcal{P}}$ with the curves $r = n$, become more and more nearly equal as $|n| \rightarrow \infty$. From Definition 2.3, one easily sees that in this case the density $\delta_{\mathcal{P}}(\mathcal{P})$ of $V^*(r)$ along \mathcal{P} is $1/2$.

We now look at the case when $P \in W(r)$. Let $\bar{\mathcal{P}}$ be as above. Now in contrast to the situation of $P \in V_\infty(r) \setminus W(r)$, there may be either infinitely many or only finitely many $n \in \mathbf{Z}$ such that the curve $V_n = V_n(r)$ intersects $\bar{\mathcal{P}}$ near $(0, 0)$. We consider these two possibilities separately. First, let $\bar{\mathcal{P}}^+$ be the subset of $\bar{\mathcal{P}}$ corresponding to $t \geq 0$,

and let $\bar{\mathcal{P}}^-$ be the subset of $\bar{\mathcal{P}}$ corresponding to $t \not\leq 0$.

We now look at the “infinite” case. In this case

(4.7) There are both infinitely many points of $V_n \cap \bar{\mathcal{P}}^+$ near $(0, 0)$,
and infinitely many points of $V_n \cap \bar{\mathcal{P}}^-$ near $(0, 0)$.

(4.7) easily follows from this fact: if $q(t^\alpha, f(t))/p(t^\alpha, f(t))$ is unbounded as $t \rightarrow 0$ from one side, then it is unbounded as $t \rightarrow 0$ from the other side. This fact also implies that $\text{ord}_0(p(t^\alpha, f(t))) \neq \text{ord}_0(q(t^\alpha, f(t)))$. Then k in (4.6) is replaced by $\text{ord}_0(q(t^\alpha, f(t))) - \text{ord}_0(p(t^\alpha, f(t)))$, and an argument like that in the case $P \in V_\infty(r) \setminus W(r)$ (using Lemma 4.2 and (4.7)) shows that on each of $\bar{\mathcal{P}}^+$ and $\bar{\mathcal{P}}^-$, the density $\text{d}_P(\mathcal{P})$ is $1/2$; hence in this case the density $\text{d}_P(\mathcal{P})$ of $V^*(r)$ along \mathcal{P} is $1/2$.

We now consider the other case, when there are not infinitely many $n \in \mathbb{Z}$ such that $V_n(r)$ intersects $\bar{\mathcal{P}}$ near $(0, 0)$. Then for a sufficiently small neighborhood U about $(0, 0)$, $\bar{\mathcal{P}}^+ \cap U$ is either contained in $V^*(r)$, or else is contained in $\mathbf{R}_{X_1 X_2} \setminus V^*(r)$. In the first instance, the density of $V^*(r) \cap \bar{\mathcal{P}}^+$ in $\bar{\mathcal{P}}^+$ is 1 and in the second, it is 0; similarly for $\bar{\mathcal{P}}^-$. Thus in this case

the density $\text{d}_P(\mathcal{P})$ of $V^*(r) \cap \bar{\mathcal{P}}$ along $\bar{\mathcal{P}}$ is 1, $1/2$ or 0 depending on whether both, exactly one, or neither $\bar{\mathcal{P}}^+ \cap U$, $\bar{\mathcal{P}}^- \cap U$ is contained in $V^*(r)$.

In the above “finite” case, all three possibilities can occur, as the following example shows.

EXAMPLE 4.8. Let $r = X_2/X_1 \in \mathbf{R}(X_1, X_2) \setminus \mathbf{R}$; let $P = (0, 0)$.

4.8.1. If \mathcal{P} is defined by

$$\begin{cases} X_1 = 2t \\ X_2 = t, \end{cases}$$

then $\bar{\mathcal{P}} \subseteq V^*(X_2/X_1)$, so $\text{d}_P(\mathcal{P}) = 1$.

4.8.2. If \mathcal{P} is defined by

$$\begin{cases} X_1 = -2t \\ X_2 = t \end{cases}$$

then $\bar{\mathcal{P}} \subseteq \mathbf{R}_{X_1 X_2} \setminus V^*(X_2/X_1)$, so $\text{d}_P(\mathcal{P}) = 0$.

4.8.3. If \mathcal{P} is defined by

$$\begin{cases} X_1 = t^2 \\ X_2 = t^3, \end{cases}$$

then for sufficiently small $\bar{\mathcal{P}}^+, \bar{\mathcal{P}}^- \subseteq V^*(X_2/X_1)$, and $\bar{\mathcal{P}}^- \subseteq \mathbf{R}_{X_1 X_2} \setminus$

$V^*(X_2/X_1)$. Thus in this case $\delta_P(\mathcal{S}) = 1/2$.

We now consider densities of moirés at points P of $V_\infty(r_2 - mr_1)$. Let $r_1, r_2 \in \mathbf{R}(X_1, X_2) \setminus \mathbf{R}$, $(r_1, r_2) = 1$, and let $m \in \mathbf{R}$. We look separately at the two cases $P \in V_\infty(r_2 - mr_1) \setminus W(r_2 - mr_1)$, and $P \in W(r_2 - mr_1)$.

First, suppose $P \in V_\infty(r_2 - mr_1) \setminus W(r_2 - mr_1)$. Let \mathcal{S} be any real place centered at P and not contained in $V_\infty(r_2 - mr_1)$; let \mathcal{S} be a representative of \mathcal{S} . As a function of b , the density $\delta(V_b)$ of $(V^*(r_1) \cup V^*(r_2)) \cap V_b$ ($b \in \mathbf{R}$) in V_b is either $3/4$, or else it oscillates about $3/4$, with formulas analogous to those in Theorem 2.7 (cf. §3). Using Lemma 4.2 and the piecewise linearity with respect to b of $\delta(V_b)$, we see that for any $\varepsilon > 0$, there exists a sufficiently large positive integer n_0 so that for any n ($|n| \geq n_0$), the density of $V^*(r_1) \cup V^*(r_2)$ measured over the part of \mathcal{S} corresponding to $1/(n+2) \leq t \leq 1/n$ is within ε of $3/4$. Thus for any $\varepsilon > 0$, the density of $V^*(r_1) \cup V^*(r_2)$ within the part of \mathcal{S} corresponding to all sufficiently large $|t|$, is within ε of $3/4$. Therefore at P itself ($P \in V_\infty(r_2 - mr_1) \setminus W(r_2 - mr_1)$), the density of $V^*(r_1) \cup V^*(r_2)$ with respect to \mathcal{S} , is $3/4$.

Now suppose $P \in W_\infty(r_2 - mr_1)$. Let $V_b = V_b(r_2 - mr_1)$ be a fixed fringe curve through P . Then for any $\varepsilon > 0$, there exists a sufficiently small neighborhood U_P about P such that the density of $(V^*(r_1) \cup V^*(r_2)) \cap V_b$ measured along $V_b \cap U_P$, is within ε of $\delta(V_b)$. This is because for any $Q \in V_b$ sufficiently close to P ($Q \neq P$), the set of points of $(V^*(r_1) \cup V^*(r_2)) \cap V_b$ near Q has a repeating pattern which is arbitrarily well-approximated by the pattern $R \cap V_b$ about any fixed point $\neq P$, where R is a sufficiently high m -refinement of $V^*(r_1) \cup V^*(r_2)$. Thus the density at $P \in W(r_2 - mr_1)$ along $V_b(r_2 - mr_1)$ ($b \in \mathbf{R}$) is just $\delta(V_b)$.

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