PACIFIC JOURNAL OF MATHEMATICS
Vol. 90, No. 2, 1980

ON BISIMPLE WEAKLY INVERSE SEMIGROUPS

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A regular semigroup S with a commutative subsemigroup of idempotents E is called weakly inverse if for any $a \in S$ the set E_a of inverses a' of a for which $a'a \in E$ is nonempty and for all, $a, b \in S, E_{ab} \subseteq E_b E_a$ and $E_a = E_b \Rightarrow a = b$. In this paper we show that in a weakly inverse semigroup S with partial identities the $\mathscr{R}$-class R which contains the partial identities is a right skew semigroup and conversely, every right skew semigroup R may be so represented. If R satisfies the condition that for every $a, b \in R$ there exists a $c \in R$ such that $Ra \cap Rb = Rc$, then our considerations lead to a construction of bisimple weakly inverse semigroup with partial identities.

The weakly inverse semigroups have been introduced and investigated by B. R. Srinivasan [5] and the results we have obtained generalize same results of Reilly [4] concerning bisimple inverse semigroups.

2. **Preliminaries.** We assume that the reader is familiar with some of the basic results of [2].

Let S be a semigroup. An idempotent e of S is called a *principal idempotent* of S if $fef = fe$ for every idempotent f of S An element a of S is called a principal element of S if there exists an inverse a' of S such that aa' is a principal idempotent of S. It is easy to show [5] that these two definitions are consistent. If a is any element of S, then an inverse a' of S will be called a *principal inverse* of a if $a'a$ is a principal idempotent of S. If $a \in S$, then E_a will denote the set of the principal inverses of a. Following [1] and [5], a semigroup S is called a *weakly inverse semigroup* if for every $a \in S, E_a \neq \square$, and for every $a, b \in S$ we have

(i) $E_{ab} \subseteqq E_b E_a$,

(ii) $E_a = E_b$ implies $a = b$.

The following lemma summarizes some of the results of [5].

LEMMA 2.1. *If S is a weakly inverse semigroup, then*

(i) *the principal idempotents of S form a semilattice,*

(ii) *$E_a a$ consists of a single idempotent e_a for every $a \in S$,*

(iii) *every principal left ideal of S has a unique principal idempotent generator,*

(iv) *the set* I *of the principal elements of* S *forms an inverse subsemigroup of* S;

(v) *an element* $a \in S$ *is a principal element of* S *if and only if* a *has a unique principal inverse*;

(vi) *for every* $a, b \in S$, *we have* $E_{ab} = E_b^a E_a$, *where*

$$E_b^a = \{b' \in E_b \colon e_a b b' e_a = e_a b b'\} .$$

If a is any element of the weakly inverse semigroup S, then $a', a_1', \cdots$ will denote principal inverses of a, whereas a'' will denote the unique principal inverse of $a' \in E_a$.

The semigroup $T(X)$ of the partial transformations on the set X is a weakly inverse semigroup. An element $\alpha \in T(X)$ is a principal element of $T(X)$ if and only if it is a one-to-one partial transformation on the set X. The Semigroup $T(X)$ will be called the *symmetric weakly inverse semigroup on the set* X [5]. Let us recall the main theorem of [5]:

LEMMA 2.2. *Let* S *be a weakly inverse semigroup. For any* $a \in S$ *let* ψ_a *be the partial transformation on* S *where* $\operatorname{dom} \psi_a = SE_a$, *and where for every* $x \in \operatorname{dom} \psi_a$, $x\psi_a = xa$. *The mapping* $S \to T(S)$, $a \to \psi_a$ *embeds* S *isomorphically into the symmetric weakly inverse semigroup* $T(S)$ *in such a way that an element* $a \in S$ *is principal in* S *if and only if* ψ_a *is principal in* $T(S)$.

With the notation of Lemma 2.2 we now have the following

LEMMA 2.3. *Let* S *be a weakly inverse semigroup, and let* a *and* b *be elements of* S. *The following conditions are equivalent*:

(i) $E_a b = \{e_a\}$,

(ii) *for every* $a' \in E_a$ *there exists a* $b' \in E_b$ *such that* $a' \leqq b'$ *in* I,

(iii) $\psi_a \subseteqq \psi_b$.

Proof. (i) $\Rightarrow$ (ii). Let a' be any element of E_a. By Lemma 2.1 (vi), there exists a $b' \in E_b$ such that $b'a'' \in E_{a'b}$. Since $a'b = e_a = a'a''$ we have $b'a'' = a'a''$, and so $a' \leqq b'$ in I.

(ii) $\Rightarrow$ (i). Ler a' be any element of E_a, and let b' be an element of E_b such that $a' \leqq b'$ in I. Then $a'b = a'b''b'b = a'b'' = a'a'' = e_a$. Therefore (i) holds.

(i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). Let x be any element of $\operatorname{dom} \psi_a$. Then there exists a $a' \in E_a$ such that $x = xa''a'$. Let b' be any element of E_b such that $a' \leqq b'$ in I. Then $x = xa''a' = xa''a'b''b' = xb''b' \in \operatorname{dom} \psi_b$; moreover $xb = xb''b'b = xb'' = xa''a'b'' = xa''a'a'' = xa''a'a = xa$. We conclude that $\psi_a \subseteqq \psi_b$.

(iii) $\Rightarrow$ (i) $\Leftrightarrow$ (ii). Let a' be any element of E_a. Since $E_a \subseteqq \operatorname{dom} \psi_a \subseteqq \operatorname{dom} \psi_b$, we have $e_a = a'a = a'\psi_a = a'\psi_b = a'b$. Hence $E_a b = \{e_a\}$.

It follows from Lemma 2.2 and Lemma 2.3 that the relation $\leqq$ on the weakly inverse semigroup S which is defined by $a \leqq b$ if and only if a and b satisfy the equivalent conditions of Lemma 2.3, must be a partial order on S which is compatible with the multiplication. We shall call this partial order the *natural partial order* on the weakly inverse semigroup S. The natural partial order $\leqq$ induces the usual natural partial order on the inverse subsemigroup I. However, $\leqq$ does not induce the usual natural partial order on the idempotents of S in the general case; indeed, if $f = f^2$ is an idempotent of S which is not principal in S, then $f \neq e_f$, $f \mathscr{L} e_f$ and $e_f \leqq f$, whereas e_f cannot be below f for the usual natural partial order on the set of idempotents of S. The above defined natural partial order on the weakly inverse semigroup S. The above defined natural partial order on the weakly inverse semigroup S will henceforth be denoted by $\leqq$.

LEMMA 2.4. *If S is any weakly inverse semigroup, then I is an order ideal of S, $\leqq$.*

Proof. Let b be any element of I, and suppose that $a \leqq b$ in I. Clearly $E_b = \{b'\}$ is a singletion. If a', a'_1 are any elements of E_a, then $a \leqq b$ implies that $a' \leqq b'$ and $a'_1 \leqq b'$ in I. Since $a' \mathscr{R} a'_1$ in the inverse semigroup I, we must have $a' = a'_1$. Hence E_a is a singleton, and by Lemma 2.1 (v) it follows that $a \in I$.

LEMMA 2.5. *If e is a principal idempotent of the weakly inverse semigroup S, and $a \in S$, then $ea \leqq a$ and $ae \leqq a$.*

Proof. Any element of E_{ea} is of the form $a'e$ for some element $a' \in E_a$ by Lemma 2.1(vi). Hence $(a'e)a = (a'e)(ea) = e_{ea}$. Thus $E_{ea} = \{e_{ea}\}$, and so $ea \leqq a$.

Any element of E_{ae} is of the form ea', where $a' \in E_a$ by Lemma 2.1. (vi). Then $ea'a = ea'ae = e_{ae}$, thus $E_{ae}a = \{e_{ae}\}$, and so $ae \leqq a$.

LEMMA 2.6. *Let S be a weakly inverse subsemigroup of the symmetric weakly inverse semigroup $T(X)$ on the set X. Let us suppose that for every $\alpha \in S$ and for every $x \in \operatorname{dom} \alpha$ there exists a principal inverse α' of α in S such that $x\alpha\alpha' = x$. Then the natural partial order on S coincides with the inclusion relation for partial transformations.*

Proof. Let α and β be any elements of S such that $\alpha \leqq \beta$, and let us suppose that $x \in \operatorname{dom} \alpha$. There exists $\alpha' \in E_\alpha$ such that $x\alpha\alpha' = x$. From $\alpha \leqq \beta$ it follows that $\alpha'\beta = \alpha'\alpha$, and so $x\alpha = x\alpha\alpha'\alpha = x\alpha\alpha'\beta = x\beta$. Hence $\alpha \subseteqq \beta$. Let us conversely suppose that α and β are elements of S such that $\alpha \subseteqq \beta$. Let α' be any element of E_α. Clearly $\operatorname{dom} \alpha'\alpha = \operatorname{dom} \alpha'\beta$. If $x \in \operatorname{dom} \alpha'\beta$, then $x \in \operatorname{dom} \alpha' = \operatorname{dom} \alpha'\alpha$, and so $\operatorname{dom} \alpha'\alpha = \operatorname{dom} \alpha'\beta$. From $\alpha \subseteqq \beta$ it now follows that $\alpha'\alpha = \alpha'\beta$. Hence $E_\alpha\alpha = \{e_\alpha\}$, and we conclude that $\alpha \subseteq \beta$.

The following alternative characterization of weakly inverse semigroups will be used later in this paper.

THEOREM 2.7. *For a regular semigroup S the following conditions are equivalent*:

(i) *S is a weakly inverse smigroup.*

(ii) *There exists a commutative subsemigroup E of idempotents of S such that*

(a) *for every $a \in S$ the set C_a of inverses a' of a for which $a'a \in E$ is nonempty,*

(b) $C_{ab} \subseteqq C_b C_a$ *for all* $a, b \in S$,

(c) $C_a = C_b$ *implies* $a = b$ *for all* $a, b \in S$.

Proof. That (i) implies (ii) is immediate. Let us now suppose that (ii) holds. Let e be any element of E, let $f = f^2$ be any idempotent of S, and suppose that $f' \in C_f$. Then

$$fef = f(f'f)ef = fe(f'f)f = fe(f'f) = f(f'f)e = fe\ ,$$

and so e is a principal idempotent of S. Let $f = f^2$ be any principal idempotent of S, and suppose that $f' \in C_f$. Then $f'f$ is the idempotent which belongs to E, and which is $\mathscr{L}$-related to f. Using the fact that f is principal we have

$$f' = f'(ff') = f'(ff')f(ff') = f'(ff')f = f'f\ .$$

Thus C_f is the singleton which consists of the element $f'f = f'$ which is $\mathscr{L}$-related to f; clearly $C_{f'} = \{f'\}$ and so $C_f = C_{f'}$. Hence $f = f' \in E$. We conclude that E is precisely the set of principal idempotents of S. Consequently, S is a weakly inverse semigroup.

3. **Right skew semigroups.** A semigroup R is called a *right skew semigroup* if for all $x, y, a \in R$, $xa = ya$ implies that there exists a left identity e of R such that $x = ye$.

If a is any element of the right skew semigroup R, then $a^2 = a^2$ implies that $a = ae$ for some left identity e of R. This already indicates that the set of left identities of R is nonempty. If f is any idempotent of R, and e any left identity of R, then $ef = f$ implies that there exists a left identity g of R such that $f = eg = g$. We conclude that the set of idempotents of R coincides with the set of left identities of R. It is then obvious that the set of idempotents of R forms a right zero semigroup.

We shall now provide an example of a right skew semigroup. Let X be a set, and μ an equivalence relation of X. Let $\mathscr{T}_\mu(X)$ be the set of transformations of the set X where

(i) $\operatorname{Ker} \alpha = \mu$,

(ii) $(x\alpha, y\alpha) \in \mu$ implies $(x, y) \in \mu$ for all $x, y \in X$.

In the terminology of [4] $\mathscr{T}_\mu(X)$ is the *semigroup of all μ-transformations with domain X.*

THEOREM 3.1. *$\mathscr{T}_\mu(X)$ is a subsemigroup of the full transformation semigroup on the set X which is a right skew semigroup. Every right skew semigroup R can be represented faithfully by a semigroup of μ-transformation with domain R.*

Proof. It follows from [4] that $\mathscr{T}_\mu(X)$ is a subsemigroup of the full transformation semigroup on the set X. Let us now suppose that $\varphi\alpha = \psi\alpha$ for some $\varphi, \psi, \alpha \in \mathscr{T}_\mu(X)$. Since $X\psi$ intersects every μ-class in at most one element we can choose an idempotent $\varepsilon \in \mathscr{T}_\mu(X)$ such that $X\psi \subseteq X\varepsilon$. From $\operatorname{Ker} \varepsilon = \operatorname{Ker} \alpha = \mu$ it follows that ε and α are $\mathscr{R}$-related in the full transformation semigroup on the set X. Therefore $\varphi\alpha = \psi\alpha$ implies $\varphi\varepsilon = \psi\varepsilon$, where $\psi\varepsilon = \psi$ since $X\psi \subseteq X\varepsilon$. Obviously ε is a left identity of $\mathscr{T}_\mu(X)$. We conclude that $\mathscr{T}_\mu(X)$ is a right skew semigroup.

If R is a right skew semigroup, then

$$\begin{aligned}\mu &= \{(x, y) \in R \times R\colon\ xa = ya \quad \text{for some} \quad a \in R\} \\ &= \{(x, y) \in R \times R\colon\ xa = ya \quad \text{for all} \quad a \in R\}\end{aligned}$$

is a congruence relation on R, and the right regular representation of R provides a representation of R by a subsemigroup of $\mathscr{T}_\mu(R)$. Since R contains left identities, the right regular representation of R is faithful.

A right zero subsemigroup E of a weakly inverse semigroup S will be called a *system of partial identities of S* if the following conditions are satisfied.

(i) If a in any nonprincipal element of S, and $e \in E$, then $ea = a$.

(ii) If $f = f^2$ is any idempotent of S, then there exists an $e \in E$ such that $f \leqq e$.

We remark that $\leqq$ always denotes the natural partial order on the weakly inverse semigroup S, as defined in § 2. If S is an inverse semigroup, then E must be a singleton. Conversely, if E is a singleton, then $E = \{e\}$, and E_e must be a singleton; by Lemma 2.1 (v) it then follows that e is a principal idempotent; since $f \leqq e$ for every idempotent $f \in S$, we conclude that f must be principal by virtue of Lemma 2.4; Hence S is an inverse semigroup with identity e. Consequently, a weakly inverse semigroup S with a system E of partial identities is an inverse monoid if and only if E is a singleton.

THEOREM 3.2. *If S is a weakly inverse semigroup with a system E of partial identities, then the $\mathscr{R}$-class R of S which contains the partial identities is a right skew subsemigroup of S.*

Proof. Let a and b be any elements of R, and let e_a be the principal idempotent which is $\mathscr{L}$-related to a. There exists an $e \in E$ such that $e_a \leqq e$. This condition implies that $E_{e_a}e = \{e_a\}$ or, $e_a e = e_a$. Consequently $ae = a$. Since e is $\mathscr{R}$-related to b, there exists a $b' \in E_b$ such that $bb' = e$. Then $abb' = ae = a$ implies that ab is $\mathscr{R}$-related to a, hence $ab \in R$. We conclude that R is a subsemigroup of S. Let c be any other element of R, and suppose that $ac = bc$. Let $c' \in E_c$, where $cc' = e$. Then $ac = bc$ implies that $be = ae = a$, where $e \in R$ is a left identity of R. Thus, R is a right skew subsemigroup of S.

We now proceed to show the converse for Theorem 3.2. We shall show that, given any right skew semigroup R, we can construct a weakly inverse semigroup with a system of partial identities in such a way that the $\mathscr{R}$-class which contains the partial identities is a right skew semigroup which is isomorphic to R.

In the remainder of this section R will denote a right skew semigroup, and E the set of idempotents of R. We know from Theorem 3.1, that the right regular representation ρ maps R isomorphically into the symmetric weakly inverse semigroup $T(R)$. For any $\alpha \in T(R)$, let E_α denote the set of principal inverses of α in $T(R)$. Define

$$(R\rho)' = \{\alpha' \in E_\alpha : \alpha \in R\rho \quad \text{and} \quad \alpha\alpha' \in R\rho\} ,$$

and let

$$(R\rho)'' = \{\alpha'' \in E_{\alpha'} : \alpha' \in (R\rho)'\} .$$

Let $\sum$ be the subsemigroup of $T(R)$ which is generated by the elements of $R\rho \cup (R\rho)' \cup (R\rho)''$. We shall show that the semigroup $\sum$ is a weakly inverse semigroup with a system of partial identities, and that $R\rho$ is the $\mathscr{R}$-class of $\sum$ which contains the partial identities.

LEMMA 3.3. *For every $\alpha \in R\rho$ and every $\varepsilon = \varepsilon^2 \in R\rho$ there exists an $\alpha' \in E_\alpha \cap (R\rho)'$ such that $\alpha\alpha' = \varepsilon \cdot R\rho$ is an $\mathscr{R}$-class of $\sum$.*

Proof. Let $\alpha \in R\rho$, and $\varepsilon = \varepsilon^2 \in R\rho$. Then $\alpha = a\rho$ and $\varepsilon = e\rho$ for some $a, e = e^2 \in R$. The mapping α': $Ra \to Re$, $xa \to xe$ is a well-defined one-to-one partial transformation on the set R, and it is easy to see that $\alpha' \in E_\alpha \cap (R\rho)'$ and $\alpha\alpha' = \varepsilon$. This already indicates that $R\rho$ is contained in an $\mathscr{R}$-Class of $\sum$.

If $\alpha \in R\rho$ then obviously $\operatorname{dom} \alpha = R$, and α is a right translation of R. Let α be any element of $R\rho$, and let $\alpha' \in E_\alpha$, where $\alpha\alpha' \in R\rho$. Let $s \in \operatorname{dom} \alpha$ and $s\alpha' = q$. Since $\alpha'\alpha$ is the restriction to $\operatorname{dom} \alpha'$ of the identity mapping, we have $s\alpha'\alpha = q\alpha = s$. For any $r \in R$, $(rq)\alpha = r(q\alpha) = rs$, and so $rs \in \operatorname{dom} \alpha'$. Moreover, $(rs)\alpha' = (rq)\alpha\alpha' = r(q\alpha\alpha') = r(s\alpha')$ and so we may conclude that, whenever $s \in \operatorname{dom} \alpha'$, then $rs \in \operatorname{dom} \alpha'$ for all $r \in R$, and $(rs)\alpha' = r(s\alpha')$. In other words, α' is a partial right translation for all $\alpha' \in (R\rho)'$. Let $\alpha'' \in (R\rho)''$, where $\alpha'' \in E_{\alpha'}$, with $\alpha' \in E_\alpha$ and $\alpha\alpha' \in R\rho$. Since $\alpha''\alpha' \in E_{\alpha\alpha'}$, where $\alpha\alpha' \in R\rho$ and $(\alpha\alpha')(\alpha''\alpha') \in R\rho$ it follows that $\alpha''\alpha' \in (R\rho)'$ is a partial right translation of R. Thus $\alpha'' = (\alpha''\alpha')\alpha$ being a composition of partial right translations of R must also be a partial right translation of R. We showed that every element of $R\rho \cup (R\rho)' \cap (R\rho)''$ must be a partial right translation of R. Thus, all elements of Σ are partial right translation of R. If ξ is any element in the $\mathscr{R}$-class which contains $R\rho$ as a subset, then $\operatorname{dom} \xi = R$, and so ξ must be a right translation of R. If ξ is any fixed left identity of R, then $f\rho$ is an idempotent of $R\rho$, and there exists a $\xi' \in \Sigma$ such that $\xi\xi' = f\rho$. If g is any left identity of R, then $g\xi = g\xi\xi'\xi = gf\xi = f\xi$. If r is any element of R, then there exists a left identity e of R such that $re = r$, and then $r\xi = (re)\xi = r(e\xi) = r(f\xi)$. We conclude that $\xi = (f\xi)\rho \in R\rho$. Thus $R\rho$ is an $\mathscr{R}$-class of Σ.

LEMMA 3.4. *If $\alpha \in R\rho$ and $\beta' \in (R\rho)'$, then $\beta'\alpha = \beta'\alpha''$, where $\alpha'' \in (R\rho)'' \cap E_{\alpha'}$ for some $\alpha' \in (R\rho)' \cap E_\alpha$ for which $\alpha\alpha' \in R\rho$. If $\beta'' \in (R\rho)''$, then $\beta''\alpha = \beta''\alpha_1''$, where $\alpha_1'' \in (R\rho)'' \cap E_{\alpha_1'}$ for some $\alpha_1' \in (R\rho)' \cap E_\alpha$ for which $\alpha\alpha_1' \in R\rho$.*

Proof. There exists a $\beta \in R\rho$ such that $\beta' \in (R\rho)' \cap E_\beta$ and $\{\beta''\} = E_{\beta'}$. By Lemma 3.3 there exists $a\alpha'$ in $E_\alpha \cap (R\rho)'$ such that $\alpha\alpha' = \beta\beta'$. Let α'' be the unique element of E_α. Clearly $\alpha'' \in (R\rho)''$. From $\alpha\alpha' = \beta\beta'$ it follows that $\beta' \mathscr{L} \alpha' \mathscr{L} \alpha''\alpha'$, and so $\beta'\alpha = \beta'\alpha''\alpha'\alpha' = \beta'\alpha''$.

Since $R\rho$ is a right skew semigroup, there exists a left identity ε of $R\rho$ such that $\beta = \beta\varepsilon$. By Lemma 3.3, there exists $a\alpha_1'$ in $E_\alpha \cap (R\rho)'$ such that $\alpha\alpha_1' = \varepsilon$. Let α_1'' be the unique element of $E_{\alpha_1'}$.

Clearly $\alpha_1'' \in (R\rho)''$ and $\beta''\alpha = \beta''\alpha\alpha_1'\alpha = \beta''\alpha\alpha_1'\alpha_1'' = \beta''\varepsilon\alpha_1'' = \beta''\beta'\beta\varepsilon\alpha_1'' = \beta''\beta'\beta\alpha_1'' = \beta''\alpha_1''$.

LEMMA 3.5. *Let I be the subsemigroup of Σ which is generated by the elements of $(R\rho)' \cup (R\rho)''$. Then I is an inverse subsemigroup of Σ, and all the elements of I are principal in Σ. Moreover $\Sigma = (R\rho)I \cup I$.*

Proof. It is clear that I consists of elements which are principal in $T(R)$, and so must be a subsemigroup of the symmetric inverse semigroup on the set R, i.e., the semigroup of all one-to-one partial transformations on the set R. Since I is generated by a set of elements together with their inverses, I must be an inverse subsemigroup of the symmetric inverse semigroup on the set R. Since all the idempotents of I are principal in $T(R)$ we must have all the elements of I are principal in Σ. That $\Sigma = (R\rho)I \cup I$ follows immediately from Lemma 3.4.

LEMMA 3.6. *For any $\xi \in \Sigma$, let G_ξ denote the set of inverses ξ' of ξ in Σ such that $\xi'\xi \in I$. Then $G_\xi = E_\xi \cap \Sigma \neq \square$. For every $\alpha \in R\rho$ and every $\zeta \in I$ we have $G_{\alpha\zeta} = G_\zeta G_\alpha$.*

Proof. If $\xi \in \Sigma$, then $\xi \in I$ or $\xi \in (R\rho)I$. If $\xi \in I$, then $G_\xi = E_\xi = E_\xi \cap \Sigma$ is the singleton $\{\xi'\}$ where ξ' is the unique inverse of ξ in I. Let us now suppose that $\xi = \alpha\zeta$, where $\alpha \in R\rho$ and $\zeta \in I$. By Lemma 3.3 $G_\alpha \neq \square$. If ζ' is the unique inverse of ζ in I and $\alpha' \in G_\alpha \subseteq E_\alpha \cap \Sigma$, then $\zeta'\alpha'$ is an element of I which is an inverse of $\alpha\zeta$, where $\zeta'\alpha'\alpha\zeta$ is an idempotent of I. Consequently $\square \neq G_\zeta G_\alpha \subseteq G_{\alpha\zeta} \subseteq E_{\alpha\zeta} \cap \Sigma$. Let us now suppose that $(\alpha\zeta)'$ is any element of $E_{\alpha\zeta} \cap \Sigma$. Since $E_{\alpha\zeta} \subseteq E_\zeta E_\alpha = \zeta' E_\alpha$, where ζ' is the unique inverse of ζ in I, we must have that $(\alpha\zeta)'$ is of the form $\zeta'\alpha_1'$ for some $\alpha_1' \in E_\alpha$. Obviously $(\alpha\zeta)(\zeta'\alpha_1') \in \Sigma$, and so $\alpha\zeta\zeta'\alpha_1' = \beta_1 \cdots \beta_n$, where $\beta_i \in R\rho \cup (R\rho)' \cup (R\rho)''$ for all $i = 1, \cdots, n$. Since $\alpha\zeta\zeta'\alpha' \in (R\rho)I \cup I$ we may suppose that $\beta_n \in (R\rho)'$ or $\beta_n \in (R\rho)''$. In both cases $\beta_n \mathscr{L} \beta$ for some $\beta \in R\rho$. There exists a left identity ε of $R\rho$ such that $\beta\varepsilon = \beta$, and then $\alpha\zeta\zeta'\alpha_1'\varepsilon = \alpha\zeta\zeta'\alpha_1'$. Let α_2' be any element of $(R\rho)' \cap E_\alpha$ such that $\alpha\alpha_2' = \varepsilon$. Clearly $\alpha_2' \in G_\alpha$ and $\alpha\zeta\zeta'\alpha_1' = \alpha\zeta\zeta'\alpha_1'\varepsilon = \alpha\zeta\zeta'\alpha_1'\alpha\alpha_2' = \alpha\zeta\zeta'\alpha_2'$. Since also $\zeta'\alpha_2' \in E_{\alpha\zeta}$: we have $\zeta'\alpha_1'\alpha\zeta = \zeta'\alpha_2'\alpha\zeta$, and we conclude that $(\alpha\zeta)' = \zeta'\alpha_1' = \zeta'\alpha_2' \in G_\zeta G_\alpha$. Thus $\square \neq G_\zeta G_\alpha = G_{\alpha\zeta} = E_{\alpha\zeta} \cap \Sigma$.

LEMMA 3.7. *Σ is a weakly inverse semigroup.*

Proof. Let ξ, η be any elements of Σ. If ξ, $\eta \in I$, then it is clear that $G_{\xi\mu} = G_\mu G_\xi$. If ξ, $\eta \in (R\rho)I$, then $\xi = \alpha\zeta$ and $\eta = \beta\theta$ for some α, $\beta \in R\rho$ and $\zeta, \theta \in I$ by Lemma 3.4, there exists a $\beta'' \in I$, with $G_{\beta''} \subseteq G_\beta$, such that $\zeta\beta = \zeta\beta''$, and so

$$G_{\xi\eta} = G_{\alpha\zeta\beta\theta} = G_{\alpha\zeta\beta''\theta} = G_{\zeta\beta''\theta}G_\alpha = G_{\beta''\theta}G_\zeta G_\alpha$$
$$= G_\theta G_{\beta''}G_{\alpha\zeta} \subseteq G_\theta G_\beta G_{\alpha\zeta} = G_{\beta\theta}G_{\alpha\zeta} = G_\eta G_\xi \ ,$$

by Lemma 3.6. The two other cases may be dealt with in a similar way, hence it follows from $\Sigma = (R\rho)I \cup I$ that $G_{\xi\eta} \subseteq G_\eta G_\xi$ for all $\xi, \mu \in \Sigma$.

Let $\xi = \alpha\zeta, \alpha \in R\rho, \zeta \in I$, be any element of $(R\rho)I$, and let us suppose that G_ξ is a singleton. If $x\alpha\zeta = y\alpha\zeta$ for some $x, y \in R$, then $x\alpha = y\alpha$ since ζ is a one-to-one partial transformation. Putting $\alpha = a\rho$, we then have $xa = ya$, and since R is right skew this implies $x = ye$ for some left identity e of R. If $\varepsilon = e\rho$, then Lemma 3.3 guarantees that there exists a $\alpha' \in G_\alpha$ such that $\alpha\alpha' = \varepsilon$. If ζ' is the unique element of G_ζ, then $\zeta'\alpha' \in G_{\alpha\zeta}$. If $u = y\alpha\zeta\zeta'\alpha'$, then $u\alpha\zeta\zeta'\alpha' = y\alpha\zeta\zeta'\alpha'$, hence $u\alpha\zeta = y\alpha\zeta$. Again we may conclude that $y = u\lambda$ for some left identity λ of $R\rho$, and that there exists a $\alpha_1' \in G_\alpha$ such that $\alpha\alpha_1' = \lambda$. Since both $\zeta'\alpha'$ and $\zeta'\alpha_1'$ belong to $G_{\alpha\zeta}$, and since $G_{\alpha\zeta}$ is a singleton, we must have $\zeta'\alpha' = \zeta'\alpha_1'$. Therefore

$$y = u\lambda = y\alpha\zeta\zeta'\alpha'\lambda = y\alpha\zeta\zeta'\alpha_1'\alpha\alpha_1' = y\alpha\zeta\zeta'\alpha_1' = y\alpha\zeta\zeta'\alpha' = u$$

and so

$$u = u\alpha\alpha' = ue = ye = x \ ,$$

from which we have that $x = y$. Thus $\xi = \alpha\zeta$ is a one-to-one partial transformation on R, which implies that ξ is a principal element of $T(R)$.

If ξ and η are any elements of Σ such that $G_\xi = G_\eta$, and if $\eta \in I$, then $G_\xi = G_\eta = E_\eta$ is a singleton. By the foregoing this implies that ζ must be principal in $T(R)$, hence $G_\xi = E_\xi$. Since $T(R)$ is a weakly inverse semigroup $E_\xi = E_\eta$ then implies that $\xi = \eta$.

Let us now suppose that $\xi = \alpha\zeta$ and $\eta = \beta\theta$, where $\alpha, \beta \in R\rho$ and $\zeta, \theta \in I$, and $G_\xi = G_\eta$. Every element of G_ξ is of the form $\zeta'\alpha' = \xi'$ with $\zeta' \in G_\zeta$, $\alpha' \in G_\alpha$. Then $\xi' \in G_\eta$, and so $\xi'\eta = \xi'\xi$. Since $\alpha\alpha'$ is a left identity for $R\rho$ we also have $\alpha\alpha'\eta = \alpha\alpha'\beta\theta = \beta\theta = \eta$. Since $\zeta'\zeta$ is the restriction of the identity transformation to $\operatorname{dom}\zeta'\zeta$ we have $\xi\xi' = \alpha\zeta\zeta'\alpha' \subseteq \alpha\alpha'$. Therefore

$$\xi = \xi\xi'\xi = \xi\xi'\eta \subseteqq \alpha\alpha'\eta = \eta \ .$$

One can show dually that $\eta \subseteqq \xi$, and thus $\xi = \eta$. Since $\Sigma = (R\rho)I \cup I$ we may conclude that $G_\xi = G_\eta$ implies $\xi = \eta$ for all $\xi, \eta \in \Sigma$.

By Theorem 2.7 and Lemma 3.6 we have that Σ is a weakly inverse semigroup.

We shall call Σ the *weakly inverse hull of the right skew semigroup R.*

LEMMA 3.8. *The set of idempotents of the $\mathscr{R}$-class $R\rho$ of form a system of partial identities of Σ.*

Proof. Let $\xi = \alpha\zeta$, $\alpha \in R\rho$, $\zeta \in I$, be any element of $(R\rho)I$, and let $x \in \operatorname{dom} \alpha\zeta$. If e is any left identity of R such that $x = xe.$, then there exists a $\alpha' \in G_\alpha$ such that $\alpha\alpha' = \varepsilon = e\rho$. If ζ' is the unique element of G_ζ, then $\zeta'\alpha' \in G_\xi$ and $x\alpha\zeta\zeta'\alpha' = x\alpha\alpha' = x\varepsilon' = xe = x$. Hence for every $\xi \in (R\rho)I$ and every $x \in \operatorname{dom} \xi$ there exists a principal inverse ξ' of ξ in Σ such that $x\xi\xi' = x$. Clearly if $\xi \in I$, and $x \in \operatorname{dom} \xi$, then also $x = x\xi\xi'$ where ξ' is the unique element of G_ζ. Since $\Sigma = (R\rho)I \cup I$ we conclude from Lemma 2.6 that the natural partial order on Σ coincides with the inclusion relation for partial transformations.

Since every idempotent of the $\mathscr{R}$-class $R\rho$ is a left identity for $R\rho$, it must also be a left identity for the elements of the set $(R\rho)$ I which contains all the nonprincipal elements of Σ.

Every idempotent of Σ is of the form $\xi\xi'$ where $\xi \in (R\rho)I$ or $\xi \in I$ and $\xi' \in G_\xi$. If $\xi = \alpha\zeta$ where $\alpha \in R\rho$ and $\zeta \in I$, then ξ' is of the form $\zeta'\alpha'$ where $\alpha' \in G_\alpha$ and $\zeta' \in G_\zeta$. Clearly

$$\xi\xi' = \alpha\zeta\zeta'\alpha' \subseteq \alpha\alpha' \in R\rho$$

in this case, and so $\xi\xi' \leqq \alpha\alpha' \in R\rho$. Let us now suppose that $\xi \in I$. Then $\xi\xi' \in I$, and $\xi\xi'$ is of the form $\xi\xi' = \beta_1, \cdots, \beta_n$, where $\beta_i \in (R\rho)' \cup (R\rho)''$, $i = 1, \cdots, n$. In all cases $\beta_n \mathscr{L} \beta$ for some $\beta \in R\rho$. Since $R\rho$ is a right skew semigroup there exists an idempotent ε in $R\rho$ such that $\beta\varepsilon = \beta$. Then $\xi\xi'\varepsilon = \xi\xi'$. Since $\xi\xi' \in I$ is the restriction of the identity transformation to $\operatorname{dom} \xi\xi'$, we must have $\xi\xi' = \xi\xi'\varepsilon \subseteq \varepsilon$, and so $\xi\xi' \leqq \varepsilon \in R\rho$.

We conclude that the set of idempotents of $R\rho$ forms a system of partial identities for Σ.

We summarize the results of Lemmas 3.3, 3.4, 3.5, 3.6, 3.7 and 3.8 in the following theorem.

THEOREM 3.9. *Let R be any right skew semigroup and let Σ be the weakly inverse hull of R. Then Σ is a weakly inverse semigroup which contains R as a subsemigroup and as an $\mathscr{R}$-class, and the set of idempotents of R forms a system of partial identities for Σ.*

4. Bisimple weakly inverse semigroups with partial identities. In this section we characterize the right skew semigroups whose weakly inverse hull is a bisimple weakly inverse semigroup.

THEOREM 4.1. *Let S be a bisimple weakly inverse semigroup with a system of partial identities. Then the $\mathscr{R}$-class R of S which contains the partial identities is a right skew subsemigroup of S, where for every $a, b \in R$ there exists a $c \in R$ such that $Rb \cap Rb = Rc$.*

Proof. It follows from Theorem 3.2 that R is a right skew subsemigroup of S. Let $a, b \in R$, and let $\{e_a\} = E_a a$, $\{e_b\} = E_b b$. The principal idempotents form a commutative subsemigroup of S, and so $Sa \cap Sb = Se_a \cap Se_b = Se_a e_b$. Since R is an $\mathscr{R}$-class of the bisimple semigroup S, there exists a $c \in R$ such that $Se_a e_b = Sc$ and thus $Sa \cap Sb = Sc$ for some $c \in R$.

Let $x \in R \cap Sa$. Then $x = sa$ for some $s \in S$. Since S is bisimple there exists a $t \in L_s \cap R$, and since R is a right skew semigroup, there exists an idempotent e of R such that $te = t$. Then $se = s$, with $e = e^2 \in R$. Let a' be any inverse of a in S such that $aa' = e$. Then $x = sa$ and $xa' = saa' = se = s$ imply that $s \in R$. Thus $x \in Ra$, and so $Sa \cap R \subseteq Ra$. From this follows that $Sa \cap R = Ra$. Similarly $Sb \cap R = Rb$ and $Sc \cap R = Rc$. Hence from $Sa \cap Sb = Sc$ we have $Ra \cap Rb = Rc$.

THEOREM 4.2. *Let R be a right skew semigroup such that for every $a, b \in R$, $Ra \cap Rb = Rc$ for some $c \in R$, and let Σ be the weakly inverse hull of R. Then Σ is a bisimple weakly inverse semigroup which contains R as a subsemigroup and as an $\mathscr{R}$-Class, and the set of idempotents of R forms a system of partial identities for Σ.*

Proof. From Theorem 3.9, it follows that we only need to show that Σ is a bisimple semigroup.

Let α and β be any elements of $R\rho$, and let $\beta' \in (R\rho)' \cap E_\beta$. Let γ be an element of $R\rho$ such that $(R\rho)\alpha \cap (R\rho)\beta = (R\rho)\gamma$. Putting $G_\alpha \alpha = \{e_\alpha\}$, $G_\beta \beta = \{e_\beta\}$ and $G_\gamma \gamma = \{e_\gamma\}$ the foregoing implies that $e_\alpha e_\beta = e_\gamma$ since then e_γ[resp. e_α, e_β] is the identity mapping on $R\gamma = R\alpha \cap R\beta$ [resp. $R\alpha$, $R\beta$]. If $(\alpha\beta'\beta)'$ is any element of $G_{\alpha\beta'\beta} = \beta'\beta G_\alpha$, then $(\alpha\beta'\beta)'\alpha\beta'\beta = e_\beta e_\alpha e_\beta = e_\gamma$. Therefore $\alpha\beta' \mathscr{R} \alpha\beta'\beta \mathscr{L} \gamma$, and so $\alpha\beta'$ belongs to the $\mathscr{D}$-class which contains $R\rho$ as an $\mathscr{R}$-class. Let α' be any element of $(R\rho)' \cap E_\alpha$ such that $\alpha\alpha' = \beta\beta'$; then $\beta' \mathscr{L} \alpha'$ and $\beta'\alpha \mathscr{L} \alpha'\alpha \mathscr{L} \alpha$, and so $\beta'\alpha$ belongs to the $\mathscr{D}$-class which contains $R\rho$. If α_1' is any element of $(R\rho)' \cap E_\alpha$, then $\alpha_1'\beta' \mathscr{L} (\alpha\alpha_1')\beta'$, where $\alpha\alpha_1' \in R\rho$, and by the foregoing we can again conclude that $\alpha_1'\beta'$

belongs to the $\mathscr{D}$-class which contains $R\rho$. We showed that the products of any two elements of $R\rho \cup (R\rho)'$ belongs to the $\mathscr{D}$-class which contains $R\rho$. Let ξ be any element of this $\mathscr{D}$-class, and let ζ be any element of $R\rho \cup (R\rho)'$. If γ is an element of $L_\xi \cap R\rho$, then $\gamma\zeta$ belongs to the $\mathscr{D}$-class which contains $R\rho$. Since $\xi\zeta \mathscr{L} \gamma\zeta$ this implies that also $\xi\zeta$ belongs to this $\mathscr{D}$-class. By induction we can then easily show that the subsemigroup of Σ which is generated by the elements of $R\rho \cup (R\rho)'$ is contained in this $\mathscr{D}$-class. If $\alpha \in R\rho$, $\alpha' \in (R\rho)' \cap E_\alpha$ and $\{\alpha''\} = E_{\alpha'}$, then $\alpha'' = \alpha''\alpha'\alpha$, where $\alpha''\alpha' \in E_{\alpha\alpha'}$, and so α'' is a product of elements of $R\rho \cup (R\rho)'$. Hence Σ is generated by the elements of $R\rho \cup (R\rho)'$, and so Σ is bisimple.

EXAMPLE. Let A be a right concellative semigroup with an identity e, and let us suppose that $\leqq$ is a total order on the set A where for any $a \in A$, $Aa = \{x \in A\, a \leqq x\}$. Let B be a semigroup which is isomorphic to A, and let $\rho\colon A \to B$ be an isomorphism of A onto B. We shall suppose that there exists a $k \in A$ such that $x\varphi = x$ for all $x \in Ak$, and that $A \cap B = Ak$. On $R = A \cup B$ we define a multiplication which extends the separations on A and on B by

$$\begin{aligned} ab &= (a\varphi)b \quad \text{if} \quad a \in A \quad \text{and} \quad b \in B \\ &= (a\varphi^{-1})b \quad \text{if} \quad a \in B \quad \text{and} \quad b \in A\,. \end{aligned}$$

It is easy to check that R is a right skew semigroup which satisfies the conditions of Theorem 4.2.

ACKNOWLEDGMENTS. The author wishes to express his sincere thanks to Professor Francis Pastijn for his constructive and creative criticism that have led to considerable improvement of the paper, and his invaluable suggestions regarding style, diction and presentation of the paper. The author also thanks Dr. B. R. Srinivasan for his valuable suggestion and his work on the definition of natural partial order of weakly inverse semigroups.

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Received April 8, 1977 and in revised form August 29, 1979.

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