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HÖLDER ESTIMATES FOR THE $\bar{\partial}$ EQUATION WITH A SUPPORT CONDITION

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A Hölder estimate for solutions to $\bar{\partial}u=\alpha$ in weakly pseudoconvex domains is obtained when the restriction of α to the boundary vanishes near the set of degeneracy of the Levi form. Applications are given to holomorphic approximation and division problems.

1. Introduction. Hölder estimates for the equation $\bar{\partial}u = \alpha$ in a strictly pseudoconvex domain D were first obtained by Kerzman [8] and Lieb [11], and were later sharpened by Romanov and Henkin [16]. In the weakly pseudoconvex case, analogous results were obtained by Range [15] by assuming that the support of is α bounded away from the set of boundary points were the Levi form degenerates. The argument used there requires the additional hypothesis of existence of a Stein neighborhood basis for $\bar{D}$, which in general may not exist (see Diederich and Fornaess [3]). The hypothesis of existence of Stein neighborhoods was removed by the author [1] by using the results of Kohn [10] concerning boundary regularity of $\bar{\partial}$ in weakly pseudoconvex domains to construct a global kernel of the Grauert-Lieb type (see [4]). In the present work this approach is refined in order to relax the support condition on α. In particular, we obtain a Hölder estimate for solutions to $\bar{\partial}u = \alpha$ whenever $\alpha|_{\partial D}$ vanishes near the set of degeneracy of the Levi-form. We remark that a simpler proof based on local solution operators is possible if one imposes a more stringent support condition on α (see Beatrous and Range [2]).

To facilitate the formulation of the main theorem, we introduce the following notation. If D is a smooth, bounded, pseudoconvex domain, we denote the set of strictly pseudoconvex boundary points by $S(D)$, and we set $W(D) = \partial D \backslash S(D)$. If N is a neighborhood of $W(D)$, let $Z_N^q(D)$ denote the set of $\bar{\partial}$ closed $(0, q)$ forms α of class C^1 on D which extend continuously to $\bar{D}$ with $\alpha|_{\partial D \cap N} = 0$. Set $Z^q(D) = \cup Z_N^q(D)$ where N runs over all neighborhoods of $W(D)$. Our main result is the following.

THEOREM 1.1. *Let D be a bounded, pseudoconvex domain in $\boldsymbol{C}^n$ with a smooth boundary. Then for each $q \geqq 1$ there is an operator $E_q: Z^q(D) \rightarrow Z^{q-1}(D)$ with $\bar{\partial}(E_q\alpha) = \alpha$. Moreover, for any neighborhood N of $W(D)$ there is a constant C_N such that the following estimate holds for $\alpha \in Z_N^q(D)$:*

$$|(E_q\alpha)(z') - (E_q\alpha)(z'')| \leqq C_N \|\alpha\|_D |z' - z''|^{1/2}$$

for all $z', z'' \in D$.

Here $\| \ \|_D$ denotes the sup norm on D.

The theorem will be proved by constructing an integral solution operator in § 2 and then estimating the kernel in § 3. In § 4 we give some applications concerning holomorphic functions with continuous boundary values.

2. The solution operator. The construction of a solution operator will require a special defining function for the hypersurface $S(D)$, which we now construct. Let τ be an arbitrary defining function for the domain D. The special defining function will be of the form

$$\rho(z) = \tau(z) \exp(\Phi(z)\tau(z))$$

where Φ is a smooth, positive function in a neighborhood of $S(D)$. Direct computation shows that for $z \in S(D)$ the Levi form of ρ has the form

$$\mathscr{L}_\rho(z; t) = \mathscr{L}_\tau(z; t) + 2\Phi(z)\left|\sum \frac{\partial\tau}{\partial z_j}(z)t_j\right|^2 . \tag{1}$$

Choose Φ to be a smooth, positive function on $S(D)$ which increases so rapidly as z approaches $W(D)$ that $\mathscr{L}_\rho(z; t) > 0$ for every $z \in S(D)$ and every $t \in C^n \backslash \{0\}$. This is possible since the first term in (1) is strictly positive where the second term vanishes. Now extend Φ to be a smooth, positive function in a neighborhood of $S(D)$. Then by continuity there is a neighborhood U of $S(D)$ on which ρ is strictly plurisubharmonic and $\nabla\rho \neq 0$.

The defining function ρ will now be used to construct a smooth family of holomorphic support functions for $S(D)$ (cf. Henkin [6], Lemma 2.4). Let $F(\zeta, z)$ denote the Levi polynomial associated with ρ, i.e.,

$$F(\zeta, z) = -2\sum_i \frac{\partial\rho}{\partial z_j}(\zeta)(z_i - \zeta_i)$$
$$- \sum_{i,j} \frac{\partial^2\rho}{\partial z_i \partial z_j}(\zeta)(z_i - \zeta_i)(z_j - \zeta_j) .$$

THEOREM 2.1. *Let k be a positive integer. There are a neighborhood U of $S(D)$, a smooth, positive function r on U, and a $\mathscr{C}^k$ function Φ on $U \times \bar{D}$ with the following properties:*

(i) *For each $\zeta \in U$, $\Phi(\zeta, \cdot) \in A^k(D) = \mathscr{C}^k(\bar{D}) \cap \mathscr{O}(D)$;*

(ii) $G(\zeta, z) = \Phi(\zeta, z)/F(\zeta, z)$ *is a nonvanishing* $\mathscr{C}^k$ *function on* $\{(\zeta, z) \in U \times \bar{D}: |\zeta - z| \leqq r(\zeta)\}$;

(iii) $\Phi(\zeta, z) \neq 0$ *if* $|\zeta - z| \geqq r(\zeta)$;

(iv) $\operatorname{Re} F(\zeta, z) > \rho(\zeta) - \rho(z) + r(\zeta)|\zeta - z|^2$ *if* $|\zeta - z| \leqq r(\zeta)$.

Proof. By expanding ρ in a Taylor series about $\zeta \in U$, and using the fact that ρ is strictly plurisubharmonic, one obtains a smooth, positive function δ on U such that $\operatorname{Re} F(\zeta, z) > \rho(\zeta) - \rho(z) + \delta(\zeta)|\zeta - z|^2$ whenever $\zeta \in U$ and $0 < |\zeta - z| < \delta(\zeta)$. (Here U denotes the neighborhood of $S(D)$ on which ρ is defined.)

Choose a smooth function $\chi(\zeta, z)$ on $U \times C^n$ with $0 \leqq \chi \leqq 1$ such that $\chi(\zeta, \cdot) = 0$ outside of $B_{\delta(\zeta)}(\zeta)$ and $\chi(\zeta, \cdot) = 1$ on $B_{1/2\delta(\zeta)}(\zeta)$. For each $\zeta \in U$, define a $(0, 1)$ form α_ζ on $\bar{D}$ by setting $\alpha_\zeta(z) = \bar\partial_z(\chi(\zeta, z)(F(\zeta, z))^{-1})$ if $F(\zeta, z) \neq 0$, and $\alpha_\zeta(z) = 0$ if $F_\zeta(z) = 0$. Then, after shrinking U if necessary, the map $\zeta \mapsto \alpha_\zeta$ is a $\mathscr{C}^\infty$ map of U into $\mathscr{C}^\infty_{(0,1)}(\bar{D})$, and clearly $\bar\partial_z \alpha_\zeta(z) = 0$ for each $\zeta \in U$. Thus we can find solutions in D to the equation $\bar\partial_z u_\zeta(z) = \alpha_\zeta(z)$. Moreover, by using the solution operator for the $\bar\partial$-Neumann problem (with an appropriate weight function) and the Sobolev estimates of Kohn [10], the solution u_ζ can be chosen so that $u_\zeta \in \mathscr{C}^k(\bar{D})$ for each ζ and the map $(\zeta \mapsto u_\zeta)\colon U \to \mathscr{C}^k(\bar{D})$ is of class $\mathscr{C}^\infty$.

Define meromorphic functions m_ζ on D by setting $m_\zeta(z) = \chi(\zeta, z)F(\zeta, z)^{-1} - u_\zeta(z)$, and choose a smooth, real valued function τ on U such that $\operatorname{Re} m_\zeta(z) > \tau(\zeta)$ for $z \in \bar{D} \backslash B_{\delta(\zeta)/2}(\zeta)$ and $\operatorname{Re}(m_\zeta(z) - F(\zeta, z)^{-1}) > \tau(\zeta)$ for $z \in \bar{D} \cap B_{\delta(\zeta)}(\zeta)$. Set $\Phi(\zeta, z) = (m_\zeta(z) - \tau(\zeta))^{-1}$. Then, after shrinking U once again, Φ is a $\mathscr{C}^k$ function on $U \times \bar{D}$. Moreover, writing

$$\Phi(\zeta, z) = \frac{F(\zeta, z)}{1 + F(\zeta, z)(m_\zeta(z) - F(\zeta, z)^{-1} - \tau(\zeta))}$$

for z near ζ, one sees that Φ satisfies (i)-(iv) for an appropriately chosen positive function r.

For the kernel construction, it will be necessary to express the function Φ in the form

$$\Phi(\zeta, z) = \sum P_j(\zeta, z)(\zeta_j - z_j) \tag{2}$$

where $P_1, \cdots, P_n$ are sufficiently smooth on $U \times \bar{D}$ and holomorphic in z. One first observes that by property (ii) of Theorem (2.1), this division problem can be solved locally in z and ζ. Next, one uses once again the result of Kohn [10] on boundary regularity of $\bar\partial$ to pass from local to global in z. This step is rather technical, and, since the argument is virtually identical to that used by

Øvrelid in [14], it will not be given here. Finally, since we only require smoothness in ζ, we can pass from local to global via a partition of unity.

We are now ready to construct the solution operator. We will use the formalism of Harvey and Polking [5]. Let $B(\zeta, z)$ denote the Bochner-Martinelli kernel:

$$B(\zeta, z) = (2\pi i)^{-n}\frac{(\bar{\zeta} - \bar{z})\cdot d\zeta}{|\zeta - z|^2} \wedge \left[\frac{d(\bar{\zeta} - \bar{z})\cdot d\zeta}{|\zeta - z|^2}\right]^{n-1}$$

where $(\bar{\zeta} - \bar{z})\cdot d\zeta = \sum(\zeta_j - z_j)d\zeta_j$ and $d(\bar{\zeta} - \bar{z})\cdot d\zeta = \sum d(\zeta_j - z_j)\wedge d\zeta_j$. Then for any $(0, q)$ form α of class $\mathscr{C}^1$ on $\bar{D}$ we have (see [5] or [13]) the Bochner-Martinelli formula

$$\alpha(z) = \bar{\partial}\int_D B(\zeta, z)\wedge\alpha(\zeta) + \int_D B(\zeta, z)\wedge\bar{\partial}\alpha(\zeta) + \int_{\partial D} B(\zeta, z)\wedge\alpha(\zeta)\ ,$$

where it is to be understood that all differentials involving the z variable are to be moved to the right before performing the integration.

Set $P = (P_1, \cdots, P_n)$ where $P_1, \cdots, P_n$ are the functions from (2). Then the Henkin kernel (see [5]) is

$$\begin{aligned} H(\zeta, z) = (2\pi i)^{-n}\frac{P\cdot d\zeta}{\Phi} \wedge \frac{(\bar{\zeta} - \bar{z})\cdot d\zeta}{|\zeta - z|^2} \\ \wedge \sum_{p=1}^{n-1}\left[\frac{\bar{\partial}P\cdot d\zeta}{\Phi}\right]^{p-1} \wedge \left[\frac{d(\bar{\zeta} - \bar{z})\cdot d\zeta}{|\zeta - z|^2}\right]^{n-p-1} \end{aligned} \tag{3}$$

where $\bar{\partial}$ acts coordinatewise on $P = (P_1, \cdots, P_n)$. Our solution operator for the $\bar{\partial}$ equation is then defined by

$$(E_q\alpha)(z) = \int_D B(\zeta, z)\wedge\alpha(\zeta) + \int_{\partial D} H(\zeta, z)\wedge\alpha(\zeta) \tag{4}$$

where $\alpha\in Z^q(D)$ for some $q \geqq 1$.

THEOREM 2.2. *For any $\alpha\in Z^q(D)$ with $q\geqq 1$ we have $\bar{\partial}E_q\alpha = \alpha$ in D.*

Proof. By the Bochner-Martinelli formula we have

$$\begin{aligned} \bar{\partial}(E_q\alpha)(z) &= \alpha(z) - \int_{\partial D} B(\zeta, z)\wedge\alpha(\zeta) - \int_{\partial D}\bar{\partial}_z H(\zeta, z)\wedge\alpha(\zeta) \\ &= \alpha(z) - \int_{\partial D} B(\zeta, z)\wedge\alpha(\zeta) - \int_{\partial D}\bar{\partial}H(\zeta, z)\wedge\alpha(\zeta) \\ &\quad + \int_{\partial D}(\bar{\partial}_\zeta H(\zeta, z))\wedge\alpha(\zeta)\ . \end{aligned} \tag{5}$$

For fixed $z \in D$, let $\varphi(\zeta)$ be a cut off function which vanishes near $W(D)$ and near the singularities of $H(\cdot, z)$, and which is identically 1 near $\operatorname{supp}(\alpha|_{\partial D})$. Then for the last term on the right in (5) we have

$$\begin{aligned}\int_{\partial D} \bar\partial_\zeta H(\zeta, z) \wedge \alpha(\zeta) &= \int_{\partial D} \bar\partial_\zeta[\varphi(\zeta) H(\zeta, z) \wedge \alpha(\zeta)] \\ &= 0 .\end{aligned}$$

The first equality follows since $\bar\partial\alpha = 0$, and the second from Stokes' formula. Moreover, one can compute from (3) (see [5]) that

$$\bar\partial H(\zeta, z) = -B(\zeta, z) + (2\pi i)^{-n} \frac{P(\zeta, z) \cdot d\zeta}{\Phi(\zeta, z)} \wedge \left[\frac{\bar\partial P(\zeta, z) \cdot d\zeta}{\Phi(\zeta, z)}\right]^{n-1} .$$

Since P is holomorphic in z, the second term on the right is of type $(n, n-1)$ in ζ. Thus we obtain the formula

$$\int_{\partial D} \bar\partial H(\zeta, z) \wedge \alpha(\zeta) = -\int_{\partial D} B(\zeta, z) \wedge \alpha(\zeta)$$

and it follows from (5) that

$$\bar\partial(E_q\alpha)(z) = \alpha(z) .$$

3. Estimates for the kernel. In this section we fix a neighborhood N of $W(D)$ and we restrict our attention to forms in $Z_N^q(D)$, $q \geqq 1$. First, we remark that Hölder $1-\varepsilon$ estimates for the Bocher-Martinelli kernel are well known (see Kerzman [8]), so in order to complete the proof of Theorem 1.1 it will suffice to estimate

$$v(z) = \int_{\partial D} H(\zeta, z) \wedge \alpha(\zeta) .$$

Thus, by Lemma 4 of Romanov and Henkin [16] it will suffice to obtain the estimate

$$|\nabla v(z)| \leqq C_N \|\alpha\|_D |\tau(z)|^{-1/2} \tag{6}$$

for all $z \in D$, where τ is some fixed defining function for D.

Choose neighborhoods N' and N'' of $W(D)$ with $N'' \subset\subset N' \subset\subset N$, and let τ be a defining function for D which agrees with ρ near $\partial D \backslash N''$ (where ρ is the defining function for $S(D)$ from the preceding section). Let U, r and F be as in Theorem 2.1 and assume, by shrinking U, that $\tau|_{U\backslash N''} = \rho|_{U\backslash N''}$. Set $U' = (U \cap \bar D)\backslash N'$. Then for $\zeta \in U'$, $r(\zeta)$ is bounded below by a positive number ε. Moreover, by shrinking U' and choosing ε sufficiently small we can choose a positive constant γ such that (c.f. [16] Lemma 1)

(7) $|G(\zeta, z)| \geqq \gamma$ for $\zeta \in U'$ and $|\zeta - z| \leqq \varepsilon$;

(8) $|\Phi(\zeta, z)| \geqq \gamma$ for $\zeta \in U'$ and $|\zeta - z| \geqq \varepsilon$;

(9) $\operatorname{Re} F(\zeta, z) \geqq \rho(\zeta) - \rho(z) + \gamma|\zeta - z|^2$ for $\zeta \in U'$ and $|\zeta - z| \leqq \varepsilon$.

The estimate (6) is now easy if z is bounded away from $\partial D \backslash N$.

LEMMA 3.1. *For any $\sigma > 0$ there is a positive constant C_σ such that $|\nabla v(z)| \leqq C_\sigma \|\alpha\|_D$ whenever $z \in N'$ or* $\operatorname{dist}(z, \partial D) \geqq \sigma$.

Proof. It follows from (7), (8) and (9) that $|\Phi(\zeta, z)|$ is bounded below by a positive constant whenever z is as in the statement of the lemma and $\zeta \in \partial D \backslash N$. Thus, since α vanishes on $\partial D \cap N$, the estimate follows by differentiation under the integral sign.

Choose $\sigma \in (0, \varepsilon]$ sufficiently small that $\zeta \in U'$ whenever $\zeta \in D \backslash N'$ and $\operatorname{dist}(\zeta, \partial D) < 2\sigma$. Then by the preceding lemma it suffices to verify (6) when $z \in U_\sigma = \{z \in D \backslash N' : \operatorname{dist}(z, \partial D) < \sigma\}$. Following Romanov and Henkin, we set $\tilde{\Phi}(\zeta, z) = (F(\zeta, z) - 2\rho(\zeta))G(\zeta, z)$ for $z \in U_\sigma$ and $|\zeta - z| \leqq \sigma$. Let $\tilde{H}(\zeta, z)$ be the kernel defined by replacing Φ with $\tilde{\Phi}$ in (3). Using (7)-(9), one easily verifies that for $z \in U_\sigma$

$$|\nabla v(z)| \leqq \text{const.}\ \|\alpha\|_D + |\nabla v_1(z)| \tag{10}$$

where

$$v_1(z) = \int_{\partial(D \cap B_\sigma(z))} \tilde{H}(\zeta, z) \wedge \alpha(\zeta)\ .$$

Thus, to complete the proof, it will suffice to estimate $|\nabla v_1(z)|$. By Stokes' formula we have (since $\bar{\partial}\alpha = 0$)

$$v_1(z) = \int_{D \cap B_\sigma(z)} \bar{\partial}_\zeta \tilde{H}(\zeta, z) \wedge \alpha(\zeta)\ .$$

By direct computation, one finds that each coefficient of $\bar{\partial}_\zeta \tilde{H}$ has one of the following forms:

$$\frac{(\bar{\zeta}_j - \bar{z}_j)a(\zeta, z)}{\tilde{\Phi}(\zeta, z)^p|\zeta - z|^{2n-2p}}\,, \quad \frac{(\bar{\zeta}_j - \bar{z}_j)a(\zeta, z)}{\tilde{\Phi}(\zeta, z)^{p+1}|\zeta - z|^{2n-2p}}\,, \quad \frac{a(\zeta, z)}{\tilde{\Phi}(\zeta, z)^p|\zeta - z|^{2n-2p}}\,,$$

or

$$\frac{(\bar{\zeta}_i - \bar{z}_i)(\zeta_j - z_j)a(\zeta, z)}{\tilde{\Phi}(\zeta, z)^p|\zeta - z|^{2n-2p+2}}\,.$$

Here $1 \leqq i, j \leqq n$, $1 \leqq p \leqq n - 1$, and $a(\zeta, z)$ denotes some smooth function of ζ and z.

Differentiating the coefficients of $\bar{\partial}_\zeta \tilde{H}(\zeta, z)$ with respect to z, one finds that for $z \in U_\sigma$

$$(11) \qquad |\nabla v_1(z)| \leqq \text{const.}\, \|\alpha\|_D \sum_{p=1}^{n-1} I_1^p + I_2^p + I_3^p$$

where

$$I_1^p = \int_{D\cap B_\sigma(z)} \frac{dV}{|\tilde\Phi(\zeta, z)|^p |\zeta - z|^{2n-2p+1}}\,,$$

$$I_2^p = \int_{D\cap B_\sigma(z)} \frac{dV}{|\tilde\Phi(\zeta - z)|^{p+1} |\zeta - z|^{2n-2p}}\,,$$

and

$$I_3^p = \int_{D\cap B_\sigma(z)} \frac{dV}{|\tilde\Phi(\zeta, z)|^{p+2} |\zeta - z|^{2n-2p-1}}\,.$$

The estimate

$$I_j^p \leqq \text{const.}\, |\rho(z)|^{-1/2}$$

can now be obtained from (7)-(9) as in Romanov and Henkin [16]. Thus, combining (10) and (11) we have the estimate (6). This completes the proof of Theorem 1.1.

4. **Applications.** In this section we give two applications of Theorem 1.1 which generalize certain well known results for strictly pseudoconvex domains.

The problem which originally motivated this investigation was that of approximating a given function in $A(D) = \mathscr{C}(\bar D) \cap \mathscr{O}(D)$ by functions which extend holomorphically across the boundary. An example due to Diederich and Fornaess [3] shows that this may not be possible if D is only assumed to be weakly pseudoconvex. However, Theorem 1.1 implies the following generalization of the classical result for strictly pseudoconvex domains.

THEOREM 4.1. *Suppose that D is a smooth, bounded pseudoconvex domain in C^n and that $f \in A(D)$. Then f can be uniformly approximated on D by functions in $A(D)$ which extend holomorphically across $S(D)$.*

This result has appeared previously in Beatrous [1] and Beatrous and Range [2], and the proof will not be repeated here. The same result had been proved earlier by Range [15] under the additional hypothesis of existence of a Stein neighborhood basis for $\bar D$, but this earlier result did not apply to the example of Diederich and Fornaess [3].

For the second application we consider the following division problem. Let D be a domain in C^n and let p be a point in D. Denote by $M_p(D)$ the maximal ideal of $A(D)$ consisting of functions which vanish at p. We wish to show that $M_p(D)$ is generated (algebraically) by $\{z_1 - p_1, \cdots, z_n - p_n\}$. In the strictly psudoconvex case this problem was solved by Kerzman and Nagel when $n = 2$ and by Lieb [12] and Øvrelid [14] in higher dimensions. If D is a weakly pseudoconvex domain in C^2 we obtain the following result.

THEOREM 4.2. *Suppose that D is a smooth, bounded, pseudoconvex domain in C^2 and that p is a point in D. If there is a complex hyperplane through p which does not meet $W(D)$ then $M_p(D)$ is generated by $\{z_1 - p_1, z_2 - p_2\}$.*

Proof. By translation and rotation of coordinates we may assume that $p = 0$ and that $W(D) \cap \{z_1 = 0\} = \varnothing$. Let f be a function in $A(D)$ satisfying $f(0) = 0$. We must construct functions f_1 and f_2 in $A(D)$ such that $f = z_1 f_1 + z_2 f_2$.

Choose a small polydisc U_0 of radius ε about 0 with $\bar{U}_0 \cap W(D) = \varnothing$ and holomorphic functions f_1^0 and f_2^0 on U_0 with $f = z_1 f_1^0 + z_2 f_2^0$ in U_0. For $j = 1, 2$, set $U_j = \{|z_j| > \varepsilon/2\}$, and define $f_i^j = \delta_i^j z_j^{-1}$ on U_j. Then $f = z_1 f_1^j + z_2 f_2^j$ on U_j. For $0 \leqq i, j \leqq 2$ we have $(f_1^i - f_1^j) z_1 + (f_2^i + f_2^j) z_2 = 0$ on $U_i \cap U_j$. Set $g_{ij} = (f_2^i - f_2^j) z_1^{-1} = (f_1^j - f_1^i) z_2^{-1}$ on $U_i \cap U_j$.

Choose smooth functions χ_j with compact support in U_j such that $0 \leqq \chi_j \leqq 1$, $\chi_0 + \chi_1 + \chi_2 = 1$ on $\bar{D}$, and $\chi_1 = 1$ in a neighborhood of $W(D)$. Set $g_j = \sum_i \chi_i g_{ji}$ on U_j. Then $g_i - g_j = g_{ij}$ on $U_i \cap U_j$, so we can define a $(0, 1)$ form α on $\bar{D}$ by setting $\alpha = \bar{\partial} g_j$ on U_j. Moreover, since $\chi_1 = 1$ near $W(D)$, the support of α is bounded away from $W(D)$. Thus, by Theorem 1.1, there is a function $u \in \mathscr{C}^\infty(D) \cap \mathscr{C}(\bar{D})$ with $\bar{\partial} u = \alpha$. We can now define the functions f_1 and f_2 in $U_i \cap \bar{D}$ by

$$f_1 = f_1^i + (g_i - u) z_2$$

and

$$f_2 = f_2^i - (g_i - u) z_1 .$$

One checks easily that f_1 and f_2 are well defined functions in $A(D)$ and that $z_1 f_1 + z_2 f_2 = f$.

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