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SPECTRAL ANALYSIS IN SPACES OF VECTOR VALUED FUNCTIONS

YITZHAK WEIT

Spectral analysis properties of $L_1^H(\boldsymbol{R})$, where H is a separable Hilbert space, are investigated. It is proved that spectral analysis holds for $L_1^H(\boldsymbol{R})$ if and only if H is finite-dimensional. The one-sided analogue of Wiener's theorem for some subgroups of the Euclidean motion group, is obtained.

1. Introduction. Let A be a Banach space and F a class of bounded linear transformations of A into itself. Following [2] we say that spectral analysis holds for A if every proper closed subspace of A, invariant under F, is included in a closed maximal invariant subspace of A.

The case where A is the Banach space of sequences summable with weights and F is the class of the translation operators was studied in [2].

We are going to study the problem of spectral analysis with A being the Banach space $L_1^H(\boldsymbol{R})$ of functions defined on $\boldsymbol{R}$, taking values in a separable Hilbert space H, and F is the class of translations by the group $\boldsymbol{R}$.

Wiener's classical theorem states that spectral analysis holds for $L_1^H(\boldsymbol{R})$ where H is one-dimensional.

Our main goal is to show that spectral analysis holds for $L_1^H(\boldsymbol{R})$, if and only if, H is finite-dimensional.

In §2 we characterize the minimal w^*-closed, translation invariant subspaces of $L_\infty^H(\boldsymbol{R})$, the dual space of $L_1^H(\boldsymbol{R})$.

Spectral analysis in the finite-dimensional case is considered in §3. In §4 we construct a w^*-closed invariant subspace of $L_\infty^H(\boldsymbol{R})$ which does not contain a nontrivial, minimal, w^*-closed, invariant subspace. One-sided spectral analysis in subgroups of the motion group, is studied in §5.

For $x \in H$ let $\|x\| = (x, x)^{1/2}$ denote the norm of x. For $f \in L_\infty(\boldsymbol{R})$, let $\mathrm{Sp}(f)$ denote the spectrum of f.

2. Minimal invariant subspaces. The minimal invariant w^*-closed subspace of $L_\infty^H(\boldsymbol{R})$ are characterized as follows:

THEOREM 1. *Let H be a separable Hilbert space with the basis $\{e_n\}_{n=1}^\infty$. Then the function $f \in L_\infty^H(\boldsymbol{R})$, $f \neq 0$ generates a minimal, w^*-closed, invariant subspace, if and only if*

$$(f(x), e_n) = a_n e^{i\lambda x} \qquad (n = 1, 2, \cdots,)$$

for some $\lambda \in \boldsymbol{R}$ *and* $\{a_n\}_{n=1}^{\infty} \in l_2$.

Proof. Let $f_n(x) = (f(x), e_n)$ *for* $n = 1, 2, \cdots,$.

If $f_n(x) = a_n e^{i\lambda x}$ then, obviously, the invariant subspace generated by f is one-dimensional.

To prove the "only if" part, let M denote the w^*-closed, invariant subspace generated by f, $f \in L_\infty^H(\boldsymbol{R})$. Suppose that $\lambda_1 \in \mathrm{Sp}(f_k)$, $\lambda_2 \in \mathrm{Sp}(f_m)$ where $m \neq k$ and $\lambda_1 < \lambda_2$. Let $\phi \in L_1(\boldsymbol{R})$ be such that $\mathrm{Supp}\,\hat{\phi} = [r_1, r_2]$ where $r_1 < \lambda_1 < r_2 < \lambda_2$. Let $g \in L_\infty^H(\boldsymbol{R})$ be the function $g(x) = \int_{-\infty}^{\infty} f(x-\alpha)\phi(\alpha)d\alpha$. Let $h \in L_1(\boldsymbol{R})$ with $\mathrm{Supp}\,\hat{h} \subset (r_2, \infty)$, such that $\int_{-\infty}^{\infty} f_m(x)h(x)dx \neq 0$. Then, for $\psi \in L_1^H(\boldsymbol{R})$, where $(\psi(x), e_m) = h(x)$ and $(\psi(x), e_n) = 0$ for $n \neq m$, we have

$$\int_{-\infty}^{\infty} (g(x-\alpha), \psi(x))dx = \int_{-\infty}^{\infty} g_m(x-\alpha)h(x)dx = 0$$

for all $\alpha \in \boldsymbol{R}$, where $g_m(x) = (g_m(x), e_m)$. On the other hand, we have $\int_{-\infty}^{\infty} (f(x), \psi(x))dx = \int_{-\infty}^{\infty} f_m(x)h(x)dx \neq 0$ which implies that M is not minimal and the result follows.

3. The finite-dimensional case. Spectral analysis holds for $L_1^H(\boldsymbol{R})$, where H is finite-dimensional. By duality, this result is a consequence of the following:

THEOREM 2. *Let H be finite-dimensional Hilbert space. Then every w^*-closed, invariant, nontrivial subspace of $L_\infty^H(\boldsymbol{R})$ contains an one-dimensional invariant subspace.*

Proof. Let $f \in L_\infty^H(\boldsymbol{R})$ and $f_n(x) = (f(x), e_n)$ $(n = 1, 2, \cdots, N)$ where $\{e_n\}_{n=1}^N$ is a basis of H. We may assume that $f_1 \neq 0$ and $0 \in \mathrm{Sp}(f_1)$. Let M denote the w^*-closed, invariant subspace of $L_\infty^H(\boldsymbol{R})$ generated by f. Let $\phi_k \in L_1(\boldsymbol{R})$ where $\mathrm{Supp}\,\hat{\phi}_k = [-1/k, 1/k]$ $\hat{\phi}_k(0) \neq 0$ for $k = 1, 2, \cdots,$. Hence, $g_k(x) = \int_{-\infty}^{\infty} f(x-\alpha)\phi_k(\alpha)d\alpha$ is not identically zero and belongs to $M(k = 1, 2, \cdots,)$. Let $g_{k,n}(x) = (g_k, e_n)$ for $k = 1, 2, \cdots,$ and $n = 1, 2 \cdots, N$.

There exist an integer j, $1 \leqq j \leqq N$, and a subsequence $k_l \to \infty$ such that

$$\max_{1 \leqq n \leqq N} \|g_{k_l,n}\|_{L_\infty} = \|g_{k_l,j}\|_{L_\infty}\,.$$

If $\hat{\phi}_{k_l}$ is multiplied by an appropriate function, it will follow that

$$||g_{k_l,j}||_{L\infty} = 1 \quad \text{and} \quad g_{k_l,j}(0) > 1 - \frac{1}{k_l} .$$

By Bernstein's inequality [5, p. 149] we have

$$||g'_{k_l,j}||_{L_\infty} \leqq \frac{1}{k_l} \qquad (l = 1, 2, \cdots,) .$$

Hence,

$$|g_{k,_l,j}(x) - 1| \leqq \frac{1}{k_l}(|x| + 1) \quad \text{which}$$

implies that $\{g_{k_l,j}\}_{l=1}^{\infty}$ converges uniformly on compact sets to the constant function 1.

By the w^*-compactness of the unit ball in $L_\infty(\boldsymbol{R})$ there exists a subsequence of k_l, which will be denoted again by k_l, such that

$$g_{k_l,n}(x) \xrightarrow[l]{w^*} \psi_n(x) \qquad n = 1, 2, \cdots, N$$

where $\psi_n \in L_\infty(\boldsymbol{R})$ and $\psi_j(x) \equiv 1$.

Obviously, $\mathrm{Sp}\,(\psi_n) \subset \{0\}$ and by an elementary theorem on spectral synthesis (see, for instance, [1] or [4] pp. 151 and 181) we deduce

$$\psi_n(x) = c_n \qquad c_n \in \boldsymbol{C} \qquad (n = 1, 2, \cdots, N) .$$

Hence, the function $\psi \in L_\infty^H(\boldsymbol{R})$, $\psi \neq 0$, where $(\psi(x), e_n) = c_n$ $(n = 1, 2, \cdots, N)$ belongs to M which completes the proof of the theorem.

REMARK 1. We have verified, actually, that the analogue of Beurling's theorem [1] in spectral analysis of bounded functions on the real line, holds for $L_\infty^H(\boldsymbol{R})$ where H is finite-dimensional.

REMARK 2. Theorem 2 may be, similarly, proved for $L_\infty^H(\boldsymbol{R}^n)$ where $n > 1$ and H is finite-dimensional.

4. The infinite-dimensional case. Spectral analysis does not hold for $L_1^H(\boldsymbol{R})$ where H is infinite-dimensional. That is, there exists a proper closed, translation invariant subspace of $L_1^H(\boldsymbol{R})$ which is contained in no maximal, closed, invariant subspace of $L_1^H(\boldsymbol{R})$. We prove the following:

THEOREM 3. *Let H be a separable, infinite-dimentional Hilbert space. There exists a nontrivial, w^*-closed, invariant subspace of $L_\infty^H(\boldsymbol{R})$ which does not contain any one-dimensional, invariant subspace.*

For the proof of Theorem 3 we will need the following lemma:

LEMMA 4. *Let f_1 and f_2 be in $L_\infty(\boldsymbol{R}) \cap L_1(\boldsymbol{R})$ such that $\hat{f}_1$ is a constant d in the interval* $[a, b]$.

If ϕ_τ, $\tau \in \Gamma$, is a net in $L_1(\boldsymbol{R})$ such that

$$(f_i * \phi_\tau)(x) \xrightarrow[\tau]{w^*} a_i e^{i\lambda x} \qquad (i = 1, 2)$$

where $a < \lambda < b$, then we have

$$a_1 \hat{f}_2(\lambda) = a_2 d .$$

Proof. We may assume that $\operatorname{Supp} \hat{\phi}_\tau \subseteqq [a, b]$ for every $\tau \in \Gamma$. Hence $f_1 * \phi_\tau = d\phi_\tau$ for any $\tau \in \Gamma$. Suppose that $d \neq 0$. Then $\phi_\tau \xrightarrow{w^*} (a_1/d)e^{i\lambda x}$ and

$$f_2 * \phi_\tau \xrightarrow{w^*} \frac{a_1}{d} \hat{f}_2(\lambda) e^{i\lambda x} .$$

If $d = 0$, then $f * \phi_\tau = 0$ for any $\tau \in \Gamma$ and we have $a_1 = 0$. This completes the proof of the lemma.

For $h \geqq 0$, $q > p$ let $T_{h,p,q}(x)$ be the function:

$$T_{h,p,q}(x) = \begin{cases} \dfrac{3h}{q-p}(x-p) & p \leqq x < \dfrac{2}{3}p + \dfrac{1}{3}q \\ h & \dfrac{2}{3}p + \dfrac{1}{3}q \leqq x < \dfrac{1}{3}p + \dfrac{2}{3}q \\ \dfrac{3h}{p-q}(x-q) & \dfrac{1}{3}p + \dfrac{2}{3}q \leqq x < q \\ 0 & \text{elsewhere} . \end{cases}$$

The proof of Theorem 3. Let $\chi_n(x) = T_{h_n, p_n, q_n}(x)$ satisfy the following conditions:

(i) $\qquad h_1 = 1 , \quad p_1 = -1$ and $q_1 = 2$.

(ii) $\qquad q_n - p_n = \dfrac{3}{n \lg n}$ and $h_n = \lg n \qquad (n = 2, 3, \cdots,)$.

(iii) For each λ, $0 < \lambda < 1$, there exists a sequence $n_k \to \infty$, such that $\lim_{k \to \infty} \chi_{n_k}(\lambda) = \infty$.

Let g_n^* be the sequence defined by

$$\hat{g}_n^*(x) = \chi_n(x) \qquad (n = 1, 2, \cdots,) .$$

Let $g_n = g_n^* * \psi$ where $\psi \in L_1(\boldsymbol{R})$, $\|\psi\|_{L_1} = 1$ and $\operatorname{Supp} \hat{\psi} \subset [0, 1]$. By condition (ii) we have $\|g_n\|_{L_\infty} \leqq 2/n$ $(n = 2, 3, \cdots)$. Hence there exists a function $f \in L_\infty^H(\boldsymbol{R})$ such that $(f(x), e_n) = g_n(x)$ for $n = 1, 2, \cdots$, where $\{e_n\}_{n=1}^\infty$ is a basis of H.

Suppose that the w^*-closed, invariant subspace generated by f contains an one-dimensional invariant subspace. That is, there exist a net ϕ_τ, $\tau \in \Gamma$, $\phi_\tau \in L_1(\boldsymbol{R})$ and a real number μ such that

$$(1) \qquad (g_n * \phi_\tau)(x) \xrightarrow[\tau]{w^*} a_n e^{i\mu x} \qquad (n = 1, 2, \cdots,)$$

where $\{a_n\}_{n=1}^\infty \in l_2$. For every g_n we have $\operatorname{Sp}(g_n) \subset [0, 1]$. Hence, we may assume that $\mu \in (0, 1)$.

From (1) we have $g_n^* * (\psi * \phi_\tau) \underset{\tau}{\overset{w^*}{\to}} a_n e^{i\mu x}$ $(n = 1, 2, \cdots,)$.

By (iii) there exists a sequence $n_k \to \infty$ such that $\lim_{k\to\infty} \chi_{n_k}(\mu) = \infty$. By Lemma 4 we deduce that $a_n = a_1 \chi_n(\mu)$ $(n = 1, 2, \cdots,)$ which implies that $a_n = 0$ for each n. This completes the proof of the theorem.

5. Spectral analysis in subgroups of the motion group. In [5] it was verified that the one-sided analogue of Wiener's theorem fails to hold for the motion group. However, we will prove that the one-sided Wiener's theorem holds for the subgroup M_K where

$$M_K = \left\{ \begin{pmatrix} e^{ik\theta} & z \\ 0 & 1 \end{pmatrix} : \theta = \frac{2\pi}{K},\ k = 0, 1, 2, \cdots, K-1,\ z \in \boldsymbol{C} \right\}.$$

(See also [3].)

By duality, this result is a consequence of the following:

THEOREM 4. *Every w^*-closed, right invariant, nontrivial subspace of $L_\infty(M_K)$ contains an irreducible (minimal) right invariant, nontrivial subspace.*

Proof. Let $f \in V$, $f \neq 0$, where V is a w^*-closed, right invariant subspace of $L_\infty(M_K)$. The subspace V contains all functions g such that $g(e^{ik\theta}, z) = f(e^{i(k+m)\theta}, z - we^{ik\theta})$ where $m \in \boldsymbol{Z}$ and $w \in \boldsymbol{C}$. For a suitable $r \in \boldsymbol{Z}$ the function

$$(2) \qquad \sum_{m=0}^{K-1} f(e^{i(k+m)\theta}, z) e^{-irm\theta} = e^{irk\theta} \sum_{m=0}^{K-1} f(e^{im\theta}, z) e^{-im\theta} = e^{irk\theta} P(z)$$

is nonzero and belongs to V. Let $P_s(z) = P(e^{is}z)$ for $s = 0, 1, \cdots, K-1$. Then by Theorem 2 and Remark 2 (P_s are looked upon as the coordinates of a function in $L_\infty^H(\boldsymbol{R}^2)$ where H is K-dimensional), there exist $\psi_n \in L_1(\boldsymbol{R}^2)$ $(n = 1, 2, \cdots,)$, $\lambda \in \boldsymbol{C}$ and $a_s \in \boldsymbol{C}$ $(s = 0, 1, \cdots, K-1)$ where $\sum_{s=0}^{K-1} |a_s| > 0$, such that

$$\int_{R^2} P_s(z-\xi)\psi_n(\xi) \xrightarrow{w^*} a_s e^{i(\lambda, z)} . \tag{3}$$

(Here, for $z_1 z_2 \in C$, $(z_1, z_2) = x_1 x_2 + y_1 y_2$ where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$.) Let $\chi_n(\xi) = \sum_{s=0}^{K-1} \psi_n(e^{-is\theta}\xi)$ $n = 1, 2, \cdots,$. Obviously, $\chi_n(\xi) = \chi_n(e^{is\theta}\xi)$ for $s = 0, 1, \cdots, K-1$. Then, by (3), we have

$$\int_{R^2} P(z-\xi)\chi_n(\xi)d\xi \xrightarrow{w^*} \sum_{s=0}^{K-1} a_s e^{i(e^{-is\theta}\lambda, z)} . \tag{4}$$

Hence, by (2), the function

$$e^{irk\theta}\int_{R^2} P(z-\xi e^{ik\theta})\chi_n(\xi)d\xi = e^{irk\theta}\int_{R^2} P(z-\xi)\chi_n(\xi)d\xi$$

belongs to V for each n. Finally, by (4), the function $Q \in L_\infty(M_K)$ where $Q(e^{ik\theta}, z) = e^{irk\theta} \sum_{s=0}^{K-1} a_s e^{i(e^{-is\theta}\lambda, z)}$ belongs to V. Arguing as in [5], it can be verified that the w^*-closed, right invariant subspace generated by Q irreducible. This completes the proof.

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University of Haifa
Haifa, Israel