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COMMUTING HYPONORMAL OPERATORS

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A hyponormal operator is normal if it commutes with a contraction T of a Hilbert space, whose powers go to zero strongly, such that $1-T^*T$ has finite-dimensional range and the coefficients of the characteristic function of T lie in a commutative C^*-algebra. The hyponormal operator is a constant multiple of the identity transformation if the rank of $1-T^*T$ is one.

Introduction. Let T be a completely nonunitary contraction on Hilbert space such that $1 - T^*T$ has closed range. There exists a power series $B(z) = \Sigma B_n z^n$ with operator coefficients which converges and is bounded by one in the unit disk such that T is unitarily equivalent to the difference-quotient transformation in the de Branges-Rovnyak space $\mathscr{D}(B)$ [1, Theorem 4]. The characteristic function $B(z)$ is said to be of scalar type if $\{B_n: n \geqq 0\}$ is a commuting family of normal operators. Inner functions of scalar type were introduced and characterized in [10]. In this paper, it is shown that if $\{B_n: n = 0, \cdots, N\}$ is a commuting family of normal operators, then polynomials $p(T)$ in T of degree at most N (weak limits of polynomials in T if $B(z)$ is of scalar type) which satisfy $||p(T)f|| \geqq ||p(T)^*f||$ for every f in the range of $1 - T^*T$ are restrictions of operators which commute with some completely nonunitary, partially isometric extension of T and which satisfy a corresponding property. The construction is made in the space $\mathscr{D}(z^M B)$ for a given positive integer M, and is a modification of an extension procedure of de Branges [1, Theorem 9].

An operator X on Hilbert space is called hyponormal if $||Xf|| \geqq ||X^*f||$ for every vector f. It is well-known [8] that if X is a hyponormal contraction with no isometric part such that the rank of $1 - X^*X$ is finite, then X must be a normal operator acting on a finite-dimensional space. To ensure normality, the finite-rank hypothesis may not be replaced by a trace-class condition: for $0 < p < \infty$, the weighted shift with weights $\{(1 - \lambda_n)^{1/2}: n \geqq 0\}$ where $\{\lambda_n\}$ is a p-summable sequence of real numbers with the property that $0 < \lambda_n \leqq \lambda_{n-1} \leqq 1 (n = 1, 2, \cdots)$ is a hyponormal, nonnormal contraction X with no isometric part such that $1 - X^*X$ is in the Schatten-von Neumann class $\mathscr{C}_p$.

A consequence of the above result in conjunction with the lifting theorems of Sarason [9] and Sz.-Nagy-Foiaș [11] is that if T is a finite direct sum of K contractions T_j, whose powers tend strongly to zero, such that the rank of $1 - T_j^* T_j$ is one, and if X is any operator which commutes with T and satisfies $\|Xf\| \geqq \|X^*f\|$ for all f in the range of $1 - T^*T$, then X is normal with spectrum consisting of at most K points. In particular, the only hyponormal operators commuting with the restriction of the backward shift to an invariant subspace are scalar multiples of the identity.

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1. Preliminaries. For a fixed Hilbert space $\mathscr{C}$, the space $\mathscr{C}(z)$ is the Hilbert space of power series $f(z) = \Sigma a_n z^n$ with coefficients in $\mathscr{C}$ such that $\|f(z)\|^2 = \Sigma |a_n|^2$ is finite. Let $B(z) = \Sigma B_n z^n$ be a power series whose coefficients are operators on $\mathscr{C}$, and suppose that for each fixed z in the unit disk the series converges, in the strong operator topology, to an operator which is bounded by one. For $f(z) = \Sigma a_n z^n$ in $\mathscr{C}(z)$, the Cauchy product $B(z)f(z) = \Sigma(\sum_{k=0}^{n} B_k a_{n-k})z^n$ is in $\mathscr{C}(z)$ and defines an operator bounded by one, which will be denoted by T_B, on $\mathscr{C}(z)$. The series $B(z)$ is an inner function if T_B is a partial isometry.

The de Branges-Rovnyak space $\mathscr{H}(B)$ is the Hilbert space of series $f(z)$ in $\mathscr{C}(z)$ such that

$$\|f(z)\|_B^2 = \sup\{\|f(z) + B(z)g(z)\|^2 - \|g(z)\|^2\}$$

is finite, where the supremum is taken over all elements $g(z)$ of $\mathscr{C}(z)$ ([1], [2], [3]). The space $\mathscr{H}(B)$ is continuously embedded in $\mathscr{C}(z)$, and is isometrically embedded in $\mathscr{C}(z)$ if and only if $B(z)$ is inner, in which case $\mathscr{C}(z) = \mathscr{H}(B) \oplus (\text{range } T_B)$. If $f(z)$ is in $\mathscr{H}(B)$, then $(f(z)-f(0))/z$ is in $\mathscr{H}(B)$ and $\|(f(z)-f(0))/z\|_B^2 \leqq \|f(z)\|_B^2 - |f(0)|^2$. The difference-quotient transformation

$$R(0)\colon f(z) \longrightarrow \frac{f(z) - f(0)}{z}$$

defined on $\mathscr{H}(B)$ is a canonical model for contractions T on Hilbert space with no isometric part (i.e., there is no nonzero vector f such that $\|T^n f\| = \|f\|$ for every $n = 1, 2, \cdots$).

The operator $R(0)^*$ on $\mathscr{H}(B)$ is related to $R(0)$ on $\mathscr{H}(B^*)$ where $B^*(z) = \Sigma \bar{B}_n z^n$ if $B(z) = \Sigma B_n z^n$ and $\bar{B}_n$ is the adjoint of B_n on $\mathscr{C}$. The space $\mathscr{D}(B)$ is the Hilbert space of pairs $(f(z), g(z))$ with $f(z)$ in $\mathscr{H}(B)$ and $g(z)$ in $\mathscr{H}(B^*)$ such that if $g(z) = \Sigma a_n z^n$ then

$$z^n f(z) - B(z)(a_0 z^{n-1} + \cdots + a_{n-1})$$

belongs to $\mathscr{H}(B)$ for every $n = 1, 2, \cdots$, and

$$\|(f(z), g(z))\|^2_{\mathscr{D}(B)}$$
$$= \sup\{\|z^n f(z) - B(z)(a_0 z^{n-1} + \cdots + a_{n-1})\|^2_B + |a_0|^2 + \cdots + |a_{n-1}|^2 : n \geqq 1\}$$

is finite. If $(f(z), g(z))$ is in $\mathscr{D}(B)$, then $(R(0)f(z), zg(z) - B^*(z)f(0))$ is in $\mathscr{D}(B)$ and

$$\|(R(0)f(z), zg(z) - B^*(z)f(0))\|^2_{\mathscr{D}(B)} = \|(f(z), g(z))\|^2_{\mathscr{D}(B)} - |f(0)|^2 .$$

The difference-quotient transformation

$$D\colon (f(z), g(z)) \longrightarrow (R(0)f(z), zg(z) - B^*(z)f(0))$$

defined on $\mathscr{D}(B)$ is a canonical model for completely nonunitary contractions T on Hilbert space (i.e., there is no nonzero vector f such that $\|T^n f\| = \|f\| = \|T^{*n} f\|$ for every $n = 1, 2, \cdots$). The adjoint of D is given by

$$D^*\colon (f(z), g(z)) \longrightarrow (zf(z) - B(z)g(0), R(0)g(z))$$

and satisfies $\|D^*(f(z), g(z))\|^2_{\mathscr{D}(B)} = \|(f(z), g(z))\|^2_{\mathscr{D}(B)} - |g(0)|^2$ for every $(f(z), g(z))$ in $\mathscr{D}(B)$. If D on $\mathscr{D}(B)$ has no isometric part, then D is unitarily equivalent to $R(0)$ on $\mathscr{H}(B)$.

The space $\mathscr{D}(B)$ is a Hilbert space with a reproducing kernel function: for every c in $\mathscr{C}$ and w in the unit disk, the pairs

$$\left(\frac{[1 - B(z)\bar{B}(w)]c}{1 - z\bar{w}}, \frac{[B^*(z) - \bar{B}(w)]c}{z - \bar{w}}\right)$$

and

$$\left(\frac{[B(z) - B(\bar{w})]c}{z - \bar{w}}, \frac{[1 - B^*(z)B(\bar{w})]c}{1 - z\bar{w}}\right)$$

belong to $\mathscr{D}(B)$, where $\bar{B}(w)$ is the adjoint of $B(w)$ on $\mathscr{C}$, and if $(f(z), g(z))$ is an element of $\mathscr{D}(B)$, then

$$\left\langle (f(z), g(z)) , \left(\frac{[1 - B(z)\bar{B}(w)]c}{1 - z\bar{w}}, \frac{[B^*(z) - \bar{B}(w)]c}{z - \bar{w}}\right)\right\rangle_{\mathscr{D}(B)} = \langle f(w), c\rangle$$

and

$$\left\langle (f(z), g(z)) , \left(\frac{[B(z) - B(\bar{w})]c}{z - \bar{w}}, \frac{[1 - B^*(z)B(\bar{w})]c}{1 - z\bar{w}}\right)\right\rangle_{\mathscr{D}(B)} = \langle g(w), c\rangle .$$

Suppose that $\mathscr{D}(A)$, $\mathscr{D}(B)$ and $\mathscr{D}(C)$ are spaces such that $B(z) = A(z)C(z)$. If $(f(z), g(z))$ is in $\mathscr{D}(A)$ and if $(h(z), k(z))$ is in $\mathscr{D}(C)$, then

$$(u(z), v(z)) = (f(z) + A(z)h(z), C^*(z)g(z) + k(z)) ,$$

is in $\mathscr{D}(B)$, and

$$\|(u(z),\ v(z))\|^2_{\mathscr{D}(B)} \leqq \|(f(z),\ g(z))\|^2_{\mathscr{D}(A)} + \|(h(z),\ k(z))\|^2_{\mathscr{D}(C)} .$$

Moreover, every element $(u(z), v(z))$ in $\mathscr{D}(B)$ has a unique minimal decomposition in terms of $\mathscr{D}(A)$ and $\mathscr{D}(C)$ such that equality holds in the above inequality. Factorizations of $B(z)$ correspond to invariant subspaces of D.

2. The lifting theorem. In the following, $B(z) = \Sigma B_n z^n$ is a power series which converges and is bounded by one in the unit disk, where the coefficients are operators on a fixed Hilbert space $\mathscr{C}$.

LEMMA 1. *If $B(z) = \Sigma B_n z^n$, and if A is an operator on $\mathscr{C}$ which commutes with both B_n and $\bar{B}_n$ for every n, then multiplication by A is an operator on $\mathscr{D}(B)$, bounded by $\|A\|$, whose adjoint is multiplication by $\bar{A}$.*

Proof. By [2, Theorem 4], the set of elements of the form $(1 - T_B T_B^*)f(z)$, for $f(z)$ in $\mathscr{H}(B)$, is dense in $\mathscr{H}(B)$, and moreover

$$\begin{aligned}\|A(1 - T_B T_B^*)f(z)\|_B &= \|(1 - T_B T_B^*)Af(z)\|_B \\ &= \|(1 - T_B T_B^*)^{1/2}Af(z)\| \\ &= \|A(1 - T_B T_B^*)^{1/2}f(z)\| \\ &\leqq \|A\|\,\|(1 - T_B T_B^*)^{1/2}f(z)\| \\ &= \|A\|\,\|(1 - T_B T_B^*)f(z)\|_B .\end{aligned}$$

Multiplication by A is therefore defined on a dense subspace of $\mathscr{H}(B)$ and has a continuous extension to all of $\mathscr{H}(B)$. Furthermore, since $\mathscr{H}(B)$ is continuously embedded in $\mathscr{C}(z)$, the extension coincides with the restriction of T_A to $\mathscr{H}(B)$. Similarly, multiplication by $\bar{A}$ is an operator on $\mathscr{H}(B)$, and is the adjoint of multiplication by A since for every $f(z)$ and $g(z)$ in $\mathscr{H}(B)$,

$$\begin{aligned}\langle A(1 - T_B T_B^*)f(z),\ g(z)\rangle_B &= \langle (1 - T_B T_B^*)Af(z),\ g(z)\rangle_B \\ &= \langle Af(z),\ g(z)\rangle \\ &= \langle f(z),\ \bar{A}g(z)\rangle \\ &= \langle (1 - T_B T_B^*)f(z),\ \bar{A}g(z)\rangle_B .\end{aligned}$$

The lemma now follows from the definition of the norm in $\mathscr{D}(B)$ and the polarization identity.

The following result generalizes a direct consequence of Lemma 1. The convention $\sum_r^s(\cdot) = 0$ when $s < r$ is observed.

LEMMA 2. *Let $B(z) = \Sigma B_n z^n$ and let A be an operator on $\mathscr{C}$ which commutes with both B_n and $\bar{B}_n$ for every $n = 0, \cdots, N$. If X and Y (or X^* and Y^*) are polynomials in the difference-quotient* *transformation D in $\mathscr{D}(B)$ of degrees at most N whose coefficients and their adjoints commute with A and B_n for every n, then*

$$\Big\langle X\Big([1 - B(z)\bar{B}(0)]c\,,\ \frac{[B^*(z) - \bar{B}(0)]c}{z}\Big)\,,$$
$$Y\Big([1 - B(z)\bar{B}(0)]Ad\,,\ \frac{[B^*(z) - \bar{B}(0)]Ad}{z}\Big)\Big\rangle_{\mathscr{D}(B)}$$
$$= \Big\langle X\Big([1 - B(z)\bar{B}(0)]\bar{A}c,\ \frac{[B^*(z) - \bar{B}(0)]\bar{A}c}{z}\Big)\,,$$
$$Y\Big([1 - B(z)\bar{B}(0)]d,\ \frac{[B^*(z) - \bar{B}(0)]d}{z}\Big)\Big\rangle_{\mathscr{D}(B)}$$

for every c and d in $\mathscr{C}$.

Proof. Let $X = \sum_0^N A_n D^n$ and $Y = \sum_0^N C_n D^n$. Let the nth coefficient of the power series for $1 - B(z)\bar{B}(0)$ be denoted by $\hat{B}_n$, and let $K(0, z)c = ([1 - B(z)\bar{B}(0)]c, ([B^*(z) - \bar{B}(0)]c)/z)$ for every c in $\mathscr{C}$. By Lemma 1, multiplication by A_n and by C_n are operators on $\mathscr{D}(B)$ for every n, and by the difference-quotient and polarization identities we have the following:

$$\begin{aligned}
&\langle A_{m+n}D^{m+n}K(0, z)c,\ C_nD^nK(0, z)Ad\rangle_{\mathscr{D}(B)}\\
&= \langle D^nA_{m+n}D^mK(0, z)c,\ D^nK(0, z)C_nAd\rangle_{\mathscr{D}(B)\mathscr{D}(B)}\\
&= \langle A_{m+n}D^mK(0, z)c,\ K(0, z)C_nAd\rangle_{\mathscr{D}(B)}\\
&\quad - \sum_{i=0}^{n-1}\langle A_{m+n}\hat{B}_{m+i}c,\ \hat{B}_iC_nAd\rangle\\
&= \langle A_{m+n}\hat{B}_mc,\ C_nAd\rangle - \sum_{i=0}^{n-1}\langle A_{m+n}\hat{B}_{m+i}\bar{A}c,\ \hat{B}_iC_nd\rangle\\
&= \langle A_{m+n}\hat{B}_m\bar{A}c,\ C_nd\rangle - \sum_{i=0}^{n-1}\langle A_{m+n}\hat{B}_{m+i}\bar{A}c,\ \hat{B}_iC_nd\rangle\\
&= \langle A_{m+n}D^{m+n}K(0, z)\bar{A}c,\ C_nD^nK(0, z)d\rangle_{\mathscr{D}(B)}\ .
\end{aligned}$$

The identity now follows for X and Y by linearity and conjugation of inner products.

Similarly, the identity holds for X^* and Y^* polynomials in D since

$$\begin{aligned}
&\langle D^{*^{m+n}}\bar{A}_{m+n}K(0, z)c,\ D^{*^n}\bar{C}_nK(0, z)Ad\rangle_{\mathscr{D}(B)}\\
&= \langle D^{*^m}\bar{A}_{m+n}K(0, z)c,\ \bar{C}_nK(0, z)Ad\rangle_{\mathscr{D}(B)}\\
&\quad - \sum_{i=1}^{n}\langle \bar{A}_{m+n}\bar{B}_{m+i}c,\ \bar{C}_n\bar{B}_iAd\rangle\\
&= \langle \bar{A}_{m+n}K(0, z)c,\ \bar{C}_nD^mK(0, z)Ad\rangle_{\mathscr{D}(B)}\\
&\quad - \sum_{i=1}^{n}\langle \bar{A}_{m+n}\bar{B}_{m+i}\bar{A}c,\ \bar{C}_n\bar{B}_id\rangle
\end{aligned}$$

$$= \langle \bar{A}_{m+n}K(0, z)\bar{A}c, \bar{C}_n D^m K(0, z)d\rangle_{\mathscr{D}(B)}$$

$$- \sum_{i=1}^{n} \langle \bar{A}_{m+n}\bar{B}_{m+i}\bar{A}c, \bar{C}_n\bar{B}_i d\rangle$$

$$= \langle D^{*^{m+n}}\bar{A}_{m+n}K(0, z)\bar{A}c, D^{*^n}\bar{C}_n K(0, z)d\rangle_{\mathscr{D}(B)} .$$

LEMMA 3. *If $B(z) = \Sigma B_n z^n$ where $B_i\bar{B}_j = \bar{B}_j B_i$ for every $i, j = 0, \cdots, N$, and if X is a polynomial of scalar type in the difference-quotient transformation D in $\mathscr{D}(B)$ of degree at most N whose coefficients commute with B_n for every n, then the following identity holds for every c in $\mathscr{C}$:*

$$\left\| DX\left([1 - B(z)\bar{B}(0)]c, \frac{[B^*(z) - \bar{B}(0)]c}{z}\right)\right\|^2_{\mathscr{D}(B)}$$

$$+ \left\| X^*\left([1 - B(z)\bar{B}(0)]\bar{B}(0)B(0)c, \frac{[B^*(z) - \bar{B}(0)]\bar{B}(0)B(0)c}{z}\right)\right\|^2_{\mathscr{D}(B)}$$

$$= \left\| DX^*\left([1 - B(z)\bar{B}(0)]\bar{B}(0)c, \frac{[B^*(z) - \bar{B}(0)]\bar{B}(0)c}{z}\right)\right\|^2_{\mathscr{D}(B)}$$

$$+ \left\| X\left([1 - B(z)\bar{B}(0)]B(0)c, \frac{[B^*(z) - \bar{B}(0)]B(0)c}{z}\right)\right\|^2_{\mathscr{D}(B)} .$$

Proof. Let $X = \sum_0^N A_n D^n$, and let $\hat{B}_n$ and $K(0, z)c$ be defined as in Lemma 2. Let $\mathscr{F}$ be the family of transformations T in $\mathscr{D}(B)$ which satisfy

$$\|DTK(0, z)c\|^2_{\mathscr{D}(B)} + \|T^*K(0, z)\bar{B}_0 B_0 c\|^2_{\mathscr{D}(B)}$$
$$= \|DT^*K(0, z)\bar{B}_0 c\|^2_{\mathscr{D}(B)} + \|TK(0, z)B_0 c\|^2_{\mathscr{D}(B)}$$

for every c in $\mathscr{C}$.

By Fuglede's theorem [4], A_n commutes with $\bar{B}_m$ for every m, and hence by Lemma 1, multiplication by A_n is a normal operator on $\mathscr{D}(B)$. Moreover, $A_n D^n$ is in $\mathscr{F}$ for every $n = 0, \cdots, N$, since

$$\|D(A_n D^n)K(0, z)c\|^2_{\mathscr{D}(B)} + \|D^{*^n}\bar{A}_n K(0, z)\bar{B}_0 B_0 c\|^2_{\mathscr{D}(B)}$$

$$= \left[\|K(0, z)A_n c\|^2_{\mathscr{D}(B)} - \sum_{i=0}^{n} |\hat{B}_i A_n c|^2\right]$$

$$+ \left[\|K(0, z)\bar{B}_0 B_0 A_n c\|^2_{\mathscr{D}(B)} - \sum_{i=1}^{n} |\bar{\hat{B}}_i \bar{B}_0 A_n c|^2\right]$$

$$= (|A_n c|^2 - |\bar{B}_0 A_n c|^2) - (|A_n c|^2 - 2|\bar{B}_0 A_n c|^2 + |B_0\bar{B}_0 A_n c|^2)$$

$$+ (|\bar{B}_0 B_0 A_n c|^2 - |\bar{B}_0^2 B_0 A_n c|^2) - \sum_{i=1}^{n} (|\hat{B}_i A_n c|^2 + |\hat{B}_i B_0 A_n c|^2)$$

$$= |B_0 A_n c|^2 - |B_0^3 A_n c|^2 - \sum_{i=1}^{n} (|\hat{B}_i A_n c|^2 + |\hat{B}_i B_0 A_n c|^2)$$

and similarly

$$\|D(D^{*^n}\bar{A}_n)K(0, z)\bar{B}_0 c\|^2_{\mathscr{D}(B)} + \|A_n D^n K(0, z)B_0 c\|^2_{\mathscr{D}(B)}$$

$$= [\|D^{*n}K(0, z)\bar{B}_0A_nc\|^2_{\mathscr{D}(B)} - |\bar{\hat{B}}_n\bar{B}_0A_nc|^2]$$
$$+ \left[\|K(0, z)B_0A_nc\|^2_{\mathscr{D}(B)} - \sum_{i=0}^{n-1}|\hat{B}_iB_0A_nc|^2\right]$$
$$= \left(|\bar{B}_0A_nc|^2 - |\bar{B}_0^2A_nc|^2 - \sum_{i=1}^{n}|\hat{B}_iA_nc|^2\right) + (|B_0A_nc|^2 - |\bar{B}_0B_0A_nc|^2)$$
$$- (|B_0A_nc|^2 - 2|\bar{B}_0B_0A_nc|^2 + |B_0\bar{B}_0B_0A_nc|^2) - \sum_{i=1}^{n}|\hat{B}_iB_0A_nc|^2$$
$$= |B_0A_nc|^2 - |B_0^3A_nc|^2 - \sum_{i=1}^{n}(|\hat{B}_iA_nc|^2 + |\hat{B}_iB_0A_nc|^2) .$$

Next, observe that if S and T belong to $\mathscr{T}$, then $S + T$ belongs to $\mathscr{T}$ if and only if

$$\begin{aligned} \operatorname{Re}[\langle TK(0, z)B_0c, SK(0, z)B_0c\rangle_{\mathscr{D}(B)} \\ - \langle DTK(0, z)c, DSK(0, z)c\rangle_{\mathscr{D}(B)}] \\ = \operatorname{Re}[\langle T^*K(0, z)\bar{B}_0B_0c, S^*K(0, z)\bar{B}_0B_0c\rangle_{\mathscr{D}(B)} \\ - \langle DT^*K(0, z)\bar{B}_0c, DS^*K(0, z)\bar{B}_0c\rangle_{\mathscr{D}(B)}] \end{aligned} \tag{2.1}$$

for every c in $\mathscr{C}$. For $m \geqq 1$, let $S = A_nD^n$ and $T = A_{m+n}D^{m+n}$. By the difference-quotient identity and polarization,

$$\langle TK(0, z)B_0c, SK(0, z)B_0c\rangle_{\mathscr{D}(B)} - \langle DTK(0, z)c, DSK(0, z)c\rangle_{\mathscr{D}(B)}$$
$$= \langle D^nD^mA_{m+n}K(0, z)B_0c, D^nK(0, z)A_nB_0c\rangle_{\mathscr{D}(B)}$$
$$- \langle D^nD^{m+1}A_{m+n}K(0, z)c, D^nDA_nK(0, z)c\rangle_{\mathscr{D}(B)}$$
$$= [\langle D^mA_{m+n}K(0, z)B_0c, K(0, z)A_nB_0c\rangle_{\mathscr{D}(B)} - \sum_{i=0}^{n-1}\langle A_{m+n}\hat{B}_{m+i}B_0c, \hat{B}_iA_nB_0c\rangle]$$
$$- [\langle DD^mA_{m+n}K(0, z)c, DK(0, z)A_nc\rangle_{\mathscr{D}(B)} - \sum_{i=1}^{n}\langle A_{m+n}\hat{B}_{m+i}c, \hat{B}_iA_nc\rangle]$$
$$= [\langle A_{m+n}\hat{B}_mB_0c, A_nB_0c\rangle - \langle A_{m+n}\hat{B}_mc, A_nc\rangle + \langle A_{m+n}\hat{B}_mc, \hat{B}_0A_nc\rangle]$$
$$+ \sum_{i=1}^{n}\langle A_{m+n}\hat{B}_{m+i}c, A_n\hat{B}_ic\rangle - \sum_{i=0}^{n-1}\langle A_{m+n}\hat{B}_{m+i}B_0c, A_n\hat{B}_iB_0c\rangle$$
$$= \sum_{i=1}^{n}\langle A_{m+n}\bar{A}_n\hat{B}_{m+i}\bar{\hat{B}}_ic, c\rangle - \sum_{i=0}^{n-1}\langle(A_{m+n}\bar{A}_n\hat{B}_{m+i}\bar{\hat{B}}_i)B_0c, B_0c\rangle .$$

Similarly,

$$\langle T^*K(0, z)\bar{B}_0B_0c, S^*K(0, z)\bar{B}_0B_0c\rangle_{\mathscr{D}(B)}$$
$$- \langle DT^*K(0, z)\bar{B}_0c, DS^*K(0, z)\bar{B}_0c\rangle_{\mathscr{D}(B)}$$
$$= \langle D^{*n}D^{*m}\bar{A}_{m+n}K(0, z)\bar{B}_0B_0c, D^{*n}K(0, z)\bar{A}_n\bar{B}_0B_0c\rangle_{\mathscr{D}(B)}$$
$$- \langle D^{*n}D^{*m}\bar{A}_{m+n}K(0, z)\bar{B}_0c, D^{*n}K(0, z)\bar{A}_n\bar{B}_0c\rangle_{\mathscr{D}(B)}$$
$$+ \langle \bar{A}_{m+n}\bar{\hat{B}}_{m+n}\bar{B}_0c, \bar{A}_n\bar{\hat{B}}_n\bar{B}_0c\rangle$$
$$= [\langle D^{*m}\bar{A}_{m+n}K(0, z)\bar{B}_0B_0c, K(0, z)\bar{A}_n\bar{B}_0B_0c\rangle_{\mathscr{D}(B)}$$
$$- \sum_{i=1}^{n}\langle \bar{A}_{m+n}\bar{\hat{B}}_{m+i}\bar{B}_0c, \bar{A}_n\bar{\hat{B}}_i\bar{B}_0c\rangle]$$

$$
\begin{aligned}
&-\Big[\langle D^{*m}\bar{A}_{m+n}K(0, z)\bar{B}_0c,\ K(0, z)\bar{A}_n\bar{B}_0c\rangle_{\mathscr{D}(B)} - \sum_{i=1}^{n}\langle \bar{A}_{m+n}\bar{\hat{B}}_{m+i}c,\ \bar{A}_n\bar{\hat{B}}_ic\rangle\Big] \\
&\qquad + \langle \bar{A}_{m+n}\bar{\hat{B}}_{m+n}\bar{B}_0c,\ \bar{A}_n\bar{\hat{B}}_n\bar{B}_0c\rangle \\
&= [\langle \bar{A}_{m+n}\bar{\hat{B}}_m\bar{B}_0B_0c,\ \bar{A}_n\bar{B}_0B_0c\rangle - \langle \bar{A}_{m+n}\bar{\hat{B}}_m\bar{B}_0c,\ \bar{A}_n\bar{B}_0c\rangle] \\
&\quad + \sum_{i=1}^{n}\langle \bar{A}_{m+n}\bar{\hat{B}}_{m+i}c,\ \bar{A}_n\bar{\hat{B}}_ic\rangle - \sum_{i=1}^{n-1}\langle \bar{A}_{m+n}\bar{\hat{B}}_{m+i}\bar{B}_0c,\ \bar{A}_n\bar{\hat{B}}_i\bar{B}_0c\rangle \\
&= \sum_{i=1}^{n}\langle c,\ A_{m+n}\bar{A}_n\hat{B}_{m+i}\bar{\hat{B}}_ic\rangle - \sum_{i=0}^{n-1}\langle B_0c,\ (A_{m+n}\bar{A}_n\hat{B}_{m+i}\bar{\hat{B}}_i)B_0c\rangle\ .
\end{aligned}
$$

Taking real parts, we have that $\mathscr{F}$ contains $A_nD^n + A_{m+n}D^{m+n}$, and hence by the linearity of the inner products in (2.1), $\mathscr{F}$ contains X.

LEMMA 4. *If $B(z)$ is of scalar type, then the identity in Lemma 3 holds for weak limits X of sequences of polynomials in the difference-quotient transformation D whose coefficients lie in a (fixed) commutative C^*-algebra containing the coefficients of $B(z)$.*

Proof. As in the proof of Lemma 3, the identity (2.1) holds whenever S and T are polynomials of scalar type in D whose coefficients commute with the coefficients of $B(z)$. It follows that (2.1) holds for S an arbitrary such polynomial in D and $T = X$, and subsequently for $S = T = X$. Therefore X satisfies the identity of Lemma 3.

REMARK 1. By Lemma 4 and Sarason's theorem [9], if the coefficient space $\mathscr{C}$ is one-dimensional and $B(z)$ is inner, then the identity in Lemma 3 holds for arbitrary operators X commuting with D. This is false for spaces $\mathscr{C}$ of higher dimension, as the following example shows.

EXAMPLE. Let $B(z) = \begin{pmatrix} b(z) & 0 \\ 0 & b(z) \end{pmatrix}$ where $b(z) = \Sigma b_nz^n$ is a scalar inner function, and let $X = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} D$. Then the identity in Lemma 3 holds for $c = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ only if either $b_0 = 0$ or $|b_1| = 1 - |b_0|^2$.

THEOREM 1. *Let D be the difference-quotient transformation in a space $\mathscr{D}(B)$, and suppose that $1 - D^*D$ has closed range. Let X be an operator on $\mathscr{D}(B)$ which satisfies*

$$\|X(f(z), g(z))\|_{\mathscr{D}(B)} \geqq \|X^*(f(z), g(z))\|_{\mathscr{D}(B)}$$

for every $(f(z), g(z))$ in the range of $1 - D^*D$. If $B(z) = \Sigma B_nz^n$ where either $B_i\bar{B}_j = \bar{B}_jB_i$ for every $i, j = 0, \cdots, N$ and X is a polynomial of scalar type in D of degree at most N whose coefficients *commute with B_n for every n, or, $B(z)$ is of scalar type and X is the limit, in the weak operator topology, of a sequence of polynomials in D whose coefficients lie in a commutative C^*-algebra containing B_n for every n, then X is unitarily equivalent to the restriction to an invariant subspace of an operator $Y = Y_M$ on $\mathscr{D}(z^M B)$ $(M = 1, 2, \cdots)$ which commutes with the partially isometric difference-quotient transformation $V = V_M$ in $\mathscr{D}(z^M B)$ and which satisfies*

$$\| Y(u(z), v(z))\|_{\mathscr{D}(z^M B)} \geqq \| Y^*(u(z), v(z))\|_{\mathscr{D}(z^M B)}$$

for every $(u(z), v(z))$ in the kernel of V. Moreover, $V = (\sum_1^M \oplus S_j^*) \oplus \hat{V}$ where S_j is a truncated shift of index j and the first M powers of $\hat{V}$ are partial isometries such that the kernel of $\hat{V}^*$ has trivial intersection with the subspace $\sum_1^M \oplus \hat{V}^{*j-1} \ker \hat{V}$. If the dimension of $\mathscr{C}$ is finite, then $Y = (\sum_1^M \oplus Y_j) \oplus \hat{Y}$ where Y_j and $\hat{Y}$ commute with S_j^* and $\hat{V}$, respectively, and Y_j is normal for every j. In this case, $Y = (\sum_1^M \oplus Z_j) \oplus Z$ where Z_j is a normal operator on the space $V^{*j-1} \ker V$, and $p(Y \ominus Z) = 0$ for some nonzero (scalar) polynomial $p(z)$ of degree not exceeding the dimension of $\mathscr{C}$.

Proof. Since $\|(1 - DD^*)^{1/2} D(f(z), g(z))\|_{\mathscr{D}(B)} = |\bar{B}(0) f(0)|$ for every $(f(z), g(z))$ in $\mathscr{D}(B)$ and $(1 - DD^*)^{1/2} D = D(1 - D^*D)^{1/2}$, with analogous identities for $1 - D^*D$, it follows that the restriction of $1 - D^*D$ to the closure of its range is unitarily equivalent to the restriction of $1 - B_0\bar{B}_0$ to the closure of its range. Therefore, since $1 - D^*D$ has closed range, so does $1 - B_0\bar{B}_0$.

Let $K(0, z)c = ([1 - B(z)\bar{B}(0)]c, [B^*(z) - \bar{B}(0)]c/z)$ for every c in $\mathscr{C}$. Define a transformation $\hat{\lambda}$ on $\mathscr{C}$ as follows: if $c = (1 - B_0\bar{B}_0)d$ for some (uniquely determined) vector d in the range of $1 - B_0\bar{B}_0$, then $\hat{\lambda}c$ is the unique vector which satisfies

$$\langle \hat{\lambda} c, a\rangle = \langle XK(0, z)B_0 d, K(0, z)a\rangle_{\mathscr{D}(B)}$$

for every a in $\mathscr{C}$; if $(1 - B_0\bar{B}_0)c = 0$, define $\hat{\lambda}c$ to be the zero vector. Since $1 - B_0\bar{B}_0$ has closed range, it follows that $\hat{\lambda}$ is continuous.

To compute $\hat{\lambda}^*$, observe that the range of $1 - B_0\bar{B}_0$ reduces $\hat{\lambda}$: let b be in the kernel of $1 - B_0\bar{B}_0$. Since B_0 is normal, $|\bar{B}_0 b| = |b| = |B_0 b|$, and hence $([B^*(z) - \bar{B}(0)]b)z = ([B(z) - B(0)]b)z = 0$. Moreover, the kernel of $1 - B_0\bar{B}_0$ reduces $\bar{B}_0$, so that $K(0, z)b = (0, 0)$. It follows that b is orthogonal to $\hat{\lambda}(1 - B_0\bar{B}_0)d$ for every vector d, and thus, since b was arbitrary, the range of $1 - B_0\bar{B}_0$ reduces $\hat{\lambda}$. Therefore, if $c = (1 - B_0\bar{B}_0)d$ for some vector d in the range of $1 - B_0\bar{B}_0$, then by Lemmas 1 and 2, $\hat{\lambda}^* c$ is the unique vector which satisfies

$$\langle \hat{\lambda}^* c, a\rangle = \langle X^* K(0, z)\bar{B}_0 d, K(0, z)a\rangle_{\mathscr{D}(B)}$$

for every a if $\mathscr{C}$; in $(1 - B_0\bar{B}_0)c = 0$, then clearly $\hat{\lambda}^* c = 0$.

By the definitions of the norms in $\mathscr{H}(B)$ and $\mathscr{D}(B)$, it follows that the transformation

$$W: (f(z), g(z)) \longrightarrow (z^M f(z), g(z))$$

takes $\mathscr{D}(B)$ isometrically into $\mathscr{D}(z^M B)$.

Let $(u(z), v(z))$ be in $\mathscr{D}(z^M B)$. The minimal decomposition of $(u(z), v(z))$ with respect to $\mathscr{D}(B)$ and $\mathscr{D}(z^M)$ is of the form

$$(u(z), v(z)) = \left(f(z) + B(z)\left(\sum_0^{M-1} c_j z^j\right),\ z^M g(z) + \sum_0^{M-1} c_{M-1-j} z^j\right)$$

with $(f(z), g(z))$ in $\mathscr{D}(B)$ and $(\sum_0^{M-1} c_j z^j, \sum_0^{M-1} c_{M-1-j} z^j)$ in $\mathscr{D}(z^M)$ for some vectors c_j in $\mathscr{C}$. Define a transformation Y in $\mathscr{D}(z^M B)$ as follows:

$$\begin{aligned} Y(u(z), v(z)) &= V^M W X(f(z), g(z)) \\ &+ \sum_0^{M-1} V^j W X\left(\frac{[B(z) - B(0)]c_{M-1-j}}{z},\ [1 - B^*(z)B(0)]c_{M-1-j}\right) \\ &+ \left(\sum_0^{M-1} (\hat{\lambda} c_j) z^j,\ B^*(z)\left(\sum_0^{M-1} (\hat{\lambda} c_{M-1-j}) z^j\right)\right). \end{aligned}$$

Since V, W, X, $\hat{\lambda}$, and minimal decompositions are linear, it follows that Y is linear. Moreover, Y is continuous since V, W, X, and $\hat{\lambda}$ are continuous and

$$\|(u(z), v(z))\|^2_{\mathscr{D}(z^M B)} = \|(f(z), g(z))\|^2_{\mathscr{D}(B)} + \sum_0^{M-1} |c_j|^2 .$$

By a straightforward computation,

$$\begin{aligned} VY(u(z), v(z)) &= V^M W D X(f(z), g(z)) \\ &+ \sum_0^{M-1} V^{j+1} W X\left(\frac{[B(z) - B(0)]c_{M-1-j}}{z},\ [1 - B^*(z)B(0)]c_{M-1-j}\right) \\ &+ \left(\sum_0^{M-2} (\hat{\lambda} c_{j+1}) z^j,\ B^*(z)\left(\sum_0^{M-2} (\hat{\lambda} c_{M-1-j}) z^{j+1}\right)\right). \end{aligned}$$

Also by [1, Theorem 5(D)], the minimal decomposition of $V(u(z), v(z))$ in $\mathscr{D}(z^M B)$ is obtained with

$$(f_1(z), g_1(z)) = D(f(z), g(z)) + \left(\frac{[B(z) - B(0)]c_0}{z},\ [1 - B^*(z)B(0)]c_0\right)$$

in $\mathscr{D}(B)$ and

$$(h_1(z), k_1(z)) = \left(\sum_0^{M-2} c_{j+1} z^j,\ \sum_0^{M-2} c_{M-1-j} z^{j+1}\right)$$

in $\mathscr{D}(z^M)$. Therefore $YV(uz), v(z)) = VY(u(z), v(z))$ since X commutes with D.

Since

$$(f(z), z^M g(z)) = (f(z) + B(z)\cdot 0, z^M g(z) + 0)$$

is minimal in $\mathscr{D}(z^M B)$ with $(f(z), g(z))$ in $\mathscr{D}(B)$ and $(0, 0)$ in $\mathscr{D}(z^M)$, we have that X is unitarily equivalent to the restriction of Y to the subspace $V^M W\mathscr{D}(B)$.

The kernel of V consists of all elements of the form $(c, z^{M-1}B^*(z)c)$ for c in $\mathscr{C}$. The minimal decomposition of $(c, z^{M-1}B^*(z)c)$ in $\mathscr{D}(z^M B)$ is obtained with $K(0, z)c$ in $\mathscr{D}(B)$ and $(\bar{B}(0)c, z^{M-1}\bar{B}(0)c)$ in $\mathscr{D}(z^M)$. Therefore, since $VY(c, z^{M-1}B^*(z)c) = YV(c, z^{M-1}B^*(z)c) = (0, 0)$, it follows that $Y(c, z^{M-1}B^*(z)c) = (d, z^{M-1}B^*(z)d)$ where d is the unique vector which satisfies

$$\langle d, a\rangle = \langle XK(0, z)c, K(0, z)a\rangle_{\mathscr{D}(B)} + \langle \hat{\lambda}\bar{B}(0)c, a\rangle \tag{2.2}$$

for every a in $\mathscr{C}$.

To compute the action of Y^* on $(c, z^{M-1}B^*(z)c)$, let $(u(z), v(z))$ be in $\mathscr{D}(z^M B)$ and write

$$(u(z), v(z)) = \left(f(z) + B(z)\left(\sum_0^{M-1} c_j z^j\right),\ z^M g(z) + \sum_0^{M-1} c_{M-1-j} z^j \right)$$

minimally with $(f(z), g(z))$ in $\mathscr{D}(B)$ and $(\sum_0^{M-1} c_j z^j, \sum_0^{M-1} c_{M-1-j} z^j)$ in $\mathscr{D}(z^M)$. Then

$$\begin{aligned}\langle Y^*(c, z^{M-1}B^*(z)c), (u(z), v(z))\rangle_{\mathscr{D}(z^M B)} \\ &= \langle K(0, z)c, X(f(z), g(z))\rangle_{\mathscr{D}(B)} + \langle c, \hat{\lambda} c_0\rangle \\ &= \langle (f_2(z) + B(z)\hat{\lambda}^* c, z^M g_2(z) + z^{M-1}\hat{\lambda}^* c), (u(z), v(z))\rangle_{\mathscr{D}(z^M B)}\end{aligned}$$

where $(f_2(z), g_2(z)) = X^*K(0, z)c$. Since $(u(z), v(z))$ was arbitrary, it follows that

$$Y^*(c, z^{M-1}B^*(z)c) = (f_2(z) + B(z)\hat{\lambda}^* c, z^M g_2(z) + z^{M-1}\hat{\lambda}^* c)\ .$$

Since

$$\|Y^*(c, z^{M-1}B^*(z)c)\|^2_{\mathscr{D}(z^M B)} \leqq \|X^*K(0, z)c\|^2_{\mathscr{D}(B)} + |\hat{\lambda}^* c|^2$$

and

$$\|Y(c, z^{M-1}B^*(z)c)\|^2_{\mathscr{D}(z^M B)} = \|(d, z^{M-1}B^*(z)d)\|^2_{\mathscr{D}(z^M B)} = |d|^2\ ,$$

it is sufficient to show

$$\|X^*K(0, z)c\|^2_{\mathscr{D}(B)} \leqq |d|^2 - |\hat{\lambda}^* c|^2$$

for all c in $\mathscr{C}$, where $d = d(c)$ is defined by (2.2).

Let c be in $\mathscr{C}$. Write $c = (1 - B_0\bar{B}_0)a + b$ where a is in the range of $1 - B_0\bar{B}_0$ and $(1 - B_0\bar{B}_0)b = 0$. As above, $\hat{\lambda}^*b = 0 = \hat{\lambda}\bar{B}(0)b$ and $K(0, z)b = (0, 0)$. Thus, we may assume $b = 0$, and $c = (1 - B_0\bar{B}_0)a$. In this case, by Lemmas 1 and 2, and the normality of B_0,

$$\begin{aligned}
&\|X^*K(0, z)c\|^2_{\mathscr{D}(B)} \\
&\quad = \langle X^*K(0, z)(1 - B_0\bar{B}_0)a, X^*K(0, z)(1 - B_0\bar{B}_0)a\rangle_{\mathscr{D}(B)} \\
&\quad = \langle X^*K(0, z)a, X^*K(0, z)(1 - B_0\bar{B}_0)a\rangle_{\mathscr{D}(B)} \\
&\qquad - \langle X^*K(0, z)B_0\bar{B}_0a, X^*K(0, z)a\rangle_{\mathscr{D}(B)} + \|X^*K(0, z)B_0\bar{B}_0a\|^2_{\mathscr{D}(B)} \\
&\quad = \|X^*K(0, z)(1 - \bar{B}_0B_0)^{1/2}a\|^2_{\mathscr{D}(B)} - \|X^*K(0, z)\bar{B}_0a\|^2_{\mathscr{D}(B)} \\
&\qquad\qquad + \|X^*K(0, z)\bar{B}_0B_0a\|^2_{\mathscr{D}(B)} .
\end{aligned}$$

Therefore by hypothesis and Lemmas 1 and 2,

$$\begin{aligned}
&\|X^*K(0, z)c\|^2_{\mathscr{D}(B)} \\
&\quad \leqq \|XK(0, z)(1 - \bar{B}_0B_0)^{1/2}a\|^2_{\mathscr{D}(B)} - \|X^*K(0, z)\bar{B}_0a\|^2_{\mathscr{D}(B)} \\
&\qquad\qquad + \|X^*K(0, z)\bar{B}_0B_0a\|^2_{\mathscr{D}(B)} \\
&\quad = \|XK(0, z)a\|^2_{\mathscr{D}(B)} - \|XK(0, z)B_0a\|^2_{\mathscr{D}(B)} - \|X^*K(0, z)\bar{B}_0a\|^2_{\mathscr{D}(B)} \\
&\qquad\qquad + \|X^*K(0, z)\bar{B}_0B_0a\|^2_{\mathscr{D}(B)} \\
&\quad = [\|DXK(0, z)a\|^2_{\mathscr{D}(B)} + |d|^2] - \|XK(0, z)B_0a\|^2_{\mathscr{D}(B)} \\
&\qquad - [\|DX^*K(0, z)\bar{B}_0a\|^2_{\mathscr{D}(B)} + |\hat{\lambda}^*c|^2] + \|X^*K(0, z)\bar{B}_0B_0a\|^2_{\mathscr{D}(B)}
\end{aligned}$$

since $a = c + B_0\bar{B}_0a$. Hence by Lemmas 3 and 4,

$$\|X^*K(0, z)c\|^2_{\mathscr{D}(B)} \leqq |d|^2 - |\hat{\lambda}^*c|^2$$

and therefore

$$\|Y(u(z), v(z))\|_{\mathscr{D}(zM_B)} \geqq \|Y^*(u(z), v(z))\|_{\mathscr{D}(zM_B)}$$

for every $(u(z), v(z))$ in the kernel of V.

By [6, Lemma 2.2], $V, \cdots, V^M$ are partial isometries and hence so are their adjoints. The form of V then follows from a slight modification of [5, Theorem 4.1]. In particular, S_j is the restriction of V^* to the space $\mathscr{H}_j = v$ (span) $\{V^i\mathscr{C}_j: i = 0, \cdots, j - 1\}$ where $\mathscr{C}_j = \ker V^* \cap V^{*^{j-1}} \ker V (j = 1, \cdots, M)$.

Suppose that $\mathscr{C}$ is finite-dimensional. Since $YV = VY$, the kernel of V is invariant under Y, and since it is finite-dimensional, the restriction Z_1 of Y to the kernel of V has an eigenvector, say $(e_1(z), e_2(z))$. Since

$$\|Y(e_1(z), e_2(z))\|_{\mathscr{D}(zM_B)} \geqq \|Y^*(e_1(z), e_2(z))\|_{\mathscr{D}(zM_B)}$$

it follows that $(e_1(z), e_2(z))$ is a reducing eigenvector for Y. By considering the restriction of Y to $\ker V \ominus \{(e_1(z), e_2(z))\}$ and proceeding by induction, we have that the kernel of V reduces Y, and Z_1 is normal. If $\lambda_1, \cdots, \lambda_K$ are the eigenvalues of Z_1 repeated according to multiplicity, then $p(Z_1) = 0$ where $p(z) = \prod_1^K(z - \lambda_j)$. Also note that $\mathscr{H}_1 = \ker V^* \cap \ker V$ is a finite-dimensional invariant subspace of the normal operator $Z_1^* = Y^*|_{\ker V}$ and hence $\mathscr{H}_1$ reduces Y, and the restriction Y_1 of Y to $\mathscr{H}_1$ is normal.

For the induction step, assume that $\mathscr{C}_j$, $\mathscr{H}_j$, and $V^{*^{j-1}}\ker V$ $(j = 1, \cdots, J-1;\ 2 \leqq J \leqq M)$ reduce Y, and the restriction of Y to each of these subspaces is normal. Since the range of V^* reduces Y, if $(r(z), s(z))$ is in the range of V^*, then $Y(r(z), s(z)) = V^*VY(r(z), s(z)) = V^*YV(r(z), s(z))$ and $Y^*(r(z), s(z)) = V^*Y^*V(r(z), s(z))$.

Let $(u(z), v(z))$ be in $V^{*^{J-1}}\ker V$. Since $Y^*Y = YY^*$ on the space $V^{*^{J-2}}\ker V$, $VY = YV$, and $V(u(z), v(z))$ is in $V^{*^{J-2}}\ker V$, it follows that

$$\begin{aligned} Y^*Y(u(z), v(z)) &= V^*Y^*(VV^*)YV(u(z), v(z)) \\ &= V^*Y^*YV(u(z), v(z)) \\ &= V^*YY^*V(u(z), v(z)) \\ &= V^*Y[(1 - VV^*) + VV^*]Y^*V(u(z), v(z)) . \end{aligned}$$

Now $(1 - VV^*)Y^*V(u(z), v(z))$ belongs to $(1 - VV^*)V^{*^{J-2}}\ker V$ which in turn is contained in $\ker V^* \cap V^{*^{J-2}}\ker V = \mathscr{C}_{J-1}$. By the induction hypothesis, $\mathscr{C}_{J-1}$ reduces Y. Therefore,

$$\begin{aligned} Y^*Y(u(z), v(z)) &= V^*YVV^*Y^*V(u(z), v(z)) \\ &= YY^*(u(z), v(z)) . \end{aligned}$$

It follows that

$$\|Y(u(z), v(z))\|_{\mathscr{L}(z^{1/2}B)} = \|Y^*(u(z), v(z))\|_{\mathscr{L}(z^{1/2}B)}$$

for all $(u(z), v(z))$ in $V^{*^{J-1}}\ker V$, and, since $V^{*^{J-1}}\ker V$ is a finite-dimensional invariant subspace for Y^*, we have that $V^{*^{J-1}}\ker V$ reduces Y as above, and the restriction Z_J of Y to $V^{*^{J-1}}\ker V$ is normal. Clearly, $p(\sum_1^J \oplus Z_j) = 0$ since $\bar{p}(Y^*)(V^{*^{j-1}}\ker V) = V^{*^{j-1}}\bar{p}(Z_1^*)\ker V = \{0\}$ for every $j = 1, \cdots, J$.

Next, $\mathscr{C}_J$ reduces Y and $Y|_{\mathscr{C}_J}$ is normal since $\mathscr{C}_J = \ker V^* \cap V^{*^{J-1}}\ker V$ is a finite-dimensional invariant subspace of Y^*, and the restriction of Y^* to $V^{*^{J-1}}\ker V$ is normal.

Finally, $\mathscr{H}_J$ reduces Y and $Y|_{\mathscr{H}_J}$ is normal since $V^i\mathscr{C}_J (i = 0, \cdots, J-1)$ is a finite-dimensional invariant subspace of Y which is contained in $V^{*^{J-i-1}}\ker V$, and the restriction of Y to $V^{*^{J-i-1}}\ker V$ is normal.

COROLLARY 1. *Let D be the difference-quotient transformation in a space $\mathscr{D}(B)$ with a finite-dimensional coefficient space $\mathscr{C}$, and suppose that D has no isometric part. Let X be an operator on*

$\mathscr{D}(B)$ which satisfies

$$\|X(f(z), g(z))\|_{\mathscr{D}(B)} \geqq \|X^*(f(z), g(z))\|_{\mathscr{D}(B)}$$

for every $(f(z), g(z))$ in the range of $1 - D^*D$. If $B(z) = \sum B_n z^n$ where $B_i\bar{B}_j = \bar{B}_j B_i$ for every $i, j = 0, \cdots, N$, and X is a polynomial of scalar type in D of degree at most N whose coefficients commute with B_n for every n, then either X is multiplication by an operator on $\mathscr{C}$ or the dimension of $\mathscr{D}(B)$ is finite $[\leqq N \times (\dim \mathscr{C})^2]$. Moreover, if $B(z)$ is of scalar type, and X is the limit, in the weak operator topology, of a sequence of polynomials in D whose coefficients lie in a commutative C^*-algebra containing B_n for every n, then $p(X) = 0$ for some nonzero (scalar) polynomial $p(z)$ of degree not exceeding the dimension of $\mathscr{C}$.

Proof. Since D has no isometric part, $B(z)c$ is in $\mathscr{H}(B)$ for some vector c only if $c = 0$, and by [2, Lemma 4], $\mathscr{H}(B)$ contains no nonzero element of the form $B(z)c$. Therefore by the minimal decomposition of an element of $\mathscr{D}(zB)$ in terms of $\mathscr{D}(B)$ and $\mathscr{D}(z)$, it follows that the difference-quotient transformation V on $\mathscr{D}(zB)$ has no isometric part. Moreover, as in the proof of Theorem 1, since $1 - B_0\bar{B}_0$ has closed range, so does $1 - D^*D$.

By Theorem 1, X is unitarily equivalent to a part of an operator Y on $\mathscr{D}(zB)$ which commutes with V and satisfies

$$\|Y(u(z), v(z))\|_{\mathscr{D}(zB)} \geqq \|Y^*(u(z), v(z))\|_{\mathscr{D}(zB)}$$

for all $(u(z), v(z))$ in the kernel of V. Moreover, the kernel of V reduces Y and $p(Y)\ker V = \{0\}$ for some nonzero polynomial $p(z)$ of degree at most the dimension of $\mathscr{C}$. Since V has no isometric part, $\mathscr{D}(zB)$ is the closed span of the subspaces $V^{*n}\ker V$ $(n = 0, 1, \cdots)$. Therefore, since $\bar{p}(Y^*)$ commutes with V^{*n}, $p(Y) = 0$ and hence $P(X) = 0$.

Suppose that X is a nonconstant, scalar type polynomial in D of degree at most N. By the above, $q(D) = 0$ for some scalar type polynomial $q(z)$ of degree at most $N \times \dim \mathscr{C}$. Since D has no isometric part, D is unitarily equivalent to $R(0)$ on $\mathscr{H}(B)$. Since any countable family of commuting normal operators on a finite-dimensional space has a common eigenvector, $q(R(0))(=0)$ is the restriction of an operator on $\mathscr{C}(z)$ of the form $\sum_1^{\dim \mathscr{C}} \oplus q_i(R(0)_i)$ where $q_i(z)$ is a scalar polynomial of degree at most $N \times \dim \mathscr{C}$ and $R(0)_i$ is the difference-quotient transformation on $\mathscr{C}_i(z)$ where $\mathscr{C}_i$ is one-dimensional. Since the eigenspace corresponding to an eigenvalue of $R(0)_i$ is one-dimensional, and since the dimension of the kernel of a finite product of operators does not exceed the sum of the dimensions of the kernels of the factors, it follows that the dimension of

$\mathscr{H}(B)$ (and hence of $\mathscr{D}(B)$) does not exceed $N\times(\dim \mathscr{C})^2$.

3. Applications. The following result extends [3, Problem 110] and [7, Corollary 1].

THEOREM 2. *Let T be a contraction on Hilbert space such that rank $(1 - TT^*) \leqq$ rank $(1 - T^*T) = 1$, and suppose that T has no isometric part. If X is the weak limit of a sequence of polynomials in T, and if f is a nonzero vector in the range of $1 - T^*T$, then $\|Xf\| \leqq \|X^*f\|$ with equality holding only if X is a scalar multiple of the identity.*

Proof. By [2, Theorem 1] and [3, Theorem 15], T is unitarily equivalent to the difference-quotient transformation in a space $\mathscr{D}(B)$ where the coefficient space is one-dimensional. The theorem now follows by applying Corollary 1.

THEOREM 3. *Let T be a contraction on Hilbert space such that $T^n (n = 1, 2, \cdots)$ tends strongly to zero, and suppose that $T = \sum_1^K \oplus T_j$ where the rank of $1 - T_j^* T_j$ is one for every j. If X is an operator which commutes with T and satisfies $\|Xf\| \geqq \|X^*f\|$ for every vector f in the range of $1 - T^*T$, then X is normal with spectrum consisting of at most K points.*

Proof. By [3, Theorem 12], there exist scalar inner functions $b_j(z)$ $(j = 1, \cdots, K)$ such that T is unitarily equivalent to the difference-quotient transformation $R(0)$ in $\mathscr{H}(B)$ where

$$B(z) = \begin{pmatrix} b_1(z) & & 0 \\ & \ddots & \\ 0 & & b_K(z) \end{pmatrix}$$

is an inner function of scalar type. The proof proceeds by induction on K.

If $K = 1$, then by Sarason's theorem [9], X is the weak limit of a sequence of polynomials in $R(0)$; and hence by [3, Theorem 13] and Theorem 2, X is a scalar multiple of the identity.

Assume that the theorem is true for the difference-quotient transformations in spaces $\mathscr{H}(B)$ of the form $\mathscr{H}(B) = \sum_1^L \oplus \mathscr{H}(b_j)$ for all integers L, $1 \leqq L < K$, where the b_j's are scalar inner functions. Let X commute with $R(0)$ on $\mathscr{H}(B) = \sum_1^K \oplus \mathscr{H}(b_j)$ and satisfy $\|Xf(z)\| \geqq \|X^*f(z)\|$ for every $f(z)$ in the range of $1 - R(0)^*R(0)$, where $b_j(z)$ is a scalar inner function for every j. By the Sz.-Nagy-Foiaș lifting theorem [11], X is the restriction of an operator on $\mathscr{C}(z) = \sum_1^K \oplus \mathscr{C}_j(z)$ of the form $(T^*_{\varphi_{ij}})_{K\times K}$ where $\mathscr{C}_j$ is the space of complex numbers and $\varphi_{ij}(z)$ is a bounded analytic (scalar) function on the unit disk for all i and j. Moreover, since $\mathscr{H}(B)$ is invariant under $(T^*_{\varphi_{ij}})_{K\times K}$, the range of T_B is invariant under $(T_{\varphi_{ji}})_{K\times K}$, and hence for each k, $1 \leqq k \leqq K$, $\varphi_{jk}(z)b_j(z)$ is contained in the range of T_{b_k} for every $j = 1, \cdots, K$.

For a fixed integer j_0 $(1 \leqq j_0 \leqq K)$, consider an element of $\mathscr{H}(B)$ of the form $f(z) = \sum_1^K \oplus [1 - b_j(z)\bar{b}_j(0)]x_j$ where $x_{j_0} = 1$ and $x_j = 0$ for all $j \neq j_0$. Since $f(z)$ is in the range of $1 - R(0)^*R(0)$, we have that

$$(3.1) \qquad \begin{aligned} \|Xf(z)\|^2 &= \sum_{i=1}^K \|T^*_{\varphi_{ij_0}}[1 - b_{j_0}(z)\bar{b}_{j_0}(0)]\|^2 \\ &\geqq \|X^*f(z)\|^2 \\ &= \sum_{i=1}^K \|P_i T_{\varphi_{j_0 i}}[1 - b_{j_0}(z)\bar{b}_{j_0}(0)]\|^2 \\ &= \sum_{i=1}^K \|P_i\varphi_{j_0 i}(z)\|^2 \end{aligned}$$

where P_i is the (orthogonal) projection of $\mathscr{C}_i(z)$ onto $\mathscr{H}(b_i)$. Moreover, by the case $K = 1$,

$$\begin{aligned} \|P_{j_0}\varphi_{ij_0}(z)\| &= \|P_{j_0}T_{\varphi_{ij_0}}[1 - b_{j_0}(z)\bar{b}_{j_0}(0)]\| \\ &\geqq \|T^*_{\varphi_{ij_0}}[1 - b_{j_0}(z)\bar{b}_{j_0}(0)]\| \end{aligned}$$

for every $i = 1, \cdots, K$. Therefore,

$$\begin{aligned} \sum_{\substack{i=1\\ i\neq j_0}}^K \|P_{j_0}\varphi_{ij_0}(z)\|^2 &\geqq \sum_{\substack{i=1\\ i\neq j_0}}^K \|T^*_{\varphi_{ij_0}}[1 - b_{j_0}(z)\bar{b}_{j_0}(0)]\|^2 \\ &\geqq \sum_{\substack{i=1\\ i\neq j_0}}^K \|P_i\varphi_{j_0 i}(z)\|^2 \end{aligned}$$

which holds for all $j_0 = 1, \cdots, K$.

Combining the above inequalities, by induction we have the following:

$$\begin{aligned} &\sum_{i=2}^K \|P_1\varphi_{i1}(z)\|^2 \\ &\quad \geqq \sum_{i=2}^K \|T^*_{\varphi_{i1}}[1 - b_1(z)\bar{b}_1(0)]\|^2 \\ &\quad \geqq \sum_{i=2}^K \|P_i\varphi_{1i}(z)\|^2 \\ &\quad \geqq \sum_{i=2}^K \Big(\sum_{\substack{j=1\\ j\neq i}}^K \|T^*_{\varphi_{ji}}[1 - b_i(z)\bar{b}_i(0)]\|^2 - \sum_{\substack{j=2\\ j\neq i}}^K \|P_i\varphi_{ji}(z)\|^2\Big) \end{aligned}$$

$$\geqq \sum_{i=2}^{K}\left(\sum_{\substack{j=1\\ j\neq i}}^{K} ||P_j\varphi_{ij}(z)||^2 - \sum_{\substack{j=2\\ j\neq i}}^{K} ||P_i\varphi_{ji}(z)||^2\right)$$

$$= \sum_{i=2}^{K} ||P_1\varphi_{i1}(z)||^2 .$$

The above inequalities are therefore equalities and in particular

$$\begin{aligned}\sum_{\substack{j=1\\ j\neq i}}^{K} ||P_iT_{\varphi_{ji}}[1 - b_i(z)\bar{b}_i(0)]||^2 &= \sum_{\substack{j=1\\ j\neq i}}^{K} ||T^*_{\varphi_{ji}}[1 - b_i(z)\bar{b}_i(0)]||^2 \\ &= \sum_{\substack{j=1\\ j\neq i}}^{K} ||P_jT_{\varphi_{ij}}[1 - b_i(z)\bar{b}_i(0)]||^2 \\ &\leqq \sum_{\substack{j=1\\ j\neq i}}^{K} ||T_{\varphi_{ij}}[1 - b_i(z)\bar{b}_i(0)]||^2\end{aligned}$$

for every $i = 1, \cdots, K$. Hence by the case $K = 1$, it follows that the restriction of $T^*_{\varphi_{ji}}$ to $\mathscr{H}(b_i)$ is a scalar λ_{ji} times the identity for all $j \neq i$, and

$$\sum_{\substack{j=1\\ j\neq i}}^{K} |\lambda_{ji}|^2 \leqq \sum_{\substack{j=1\\ j\neq i}}^{K} |\lambda_{ij}|^2 \tag{3.2}$$

for every $i = 1, \cdots, K$. Therefore by (3.1) and the case $K = 1$, the restriction of $T^*_{\varphi_{ii}}$ to $\mathscr{H}(b_i)$ is a scalar λ_{ii} times the identity for every $i = 1, \cdots, K$, and consequently $X = (\lambda_{ij})_{K\times K}$.

Suppose first that $\mathscr{H}(b_i) = \mathscr{H}(b_j)$ for all i and j. In this case, the range of $1 - R(0)^*R(0)$ reduces X, and since it is finite-dimensional and the restriction of X to it is hyponormal, it follows that $XX^* = X^*X$ on the range of $1 - R(0)^*R(0)$.

Let $h(z)$ be an arbitrary element of $\mathscr{H}(B)$. Then $h(z)$ is the limit of a sequence of vectors of the form $\sum_0^n R(0)^{*j}f_j(z)$ where $f_j(z)$ is in the range of $1 - R(0)^*R(0)$ for every j. Since XX^* and X^*X commute with $R(0)^{*j}$, we have that $XX^*h(z) = X^*Xh(z)$. Hence X is normal.

Let $\lambda_1, \cdots, \lambda_K$ be the eigenvalues of the restriction of X to the range of $1 - R(0)^*R(0)$, listed according to multiplicity, and let $\eta_j = \vee\{f(z) \in \mathscr{H}(B): Xf(z) = \lambda_j f(z)\}$. Since X is normal, if $\lambda_i \neq \lambda_j$, then η_i is orthogonal to η_j. Moreover, since $R(0)$ has no isometric part and $XR(0)^* = R(0)^*X$, it follows that $\mathscr{H}(B) = \vee\{\eta_j: j = 1, \cdots, K\}$. Therefore, X is diagonalizable with $\text{sp}(X) = \{\lambda_j: j = 1, \cdots, K\}$.

Finally, suppose that $\mathscr{H}(b_i) \neq \mathscr{H}(b_j)$ for at least one pair (i, j). There exists a space $\mathscr{H}(b_{i_0})$ which is minimal in the sense that for every i either $\mathscr{H}(b_i) = \mathscr{H}(b_{i_0})$ or $\mathscr{H}(b_i)$ is not contained in $\mathscr{H}(b_{i_0})$. Let Ω be the set of indices i such that $\mathscr{H}(b_i) = \mathscr{H}(b_{i_0})$. Then $\Omega \neq \{1, \cdots, K\}$ by assumption, and for every i in Ω and j not in Ω,

$\lambda_{ij} = 0$. By (3.2),

$$\sum_{\substack{i \in \Omega \\ j \notin \Omega}} |\lambda_{ji}|^2 \leqq \sum_{i,j \in \Omega} (|\lambda_{ij}|^2 - |\lambda_{ji}|^2) = 0 .$$

Therefore, $\lambda_{ij} = 0 = \lambda_{ji}$ for every i in Ω and j not in Ω. It follows that the space $\sum_{i \in \Omega} \oplus \mathscr{H}(b_i)$ reduces X, that the restriction of X to this space satisfies the induction hypothesis and hence is normal with spectrum consisting of at most card Ω points. Similarly, the restriction of X to $\sum_{i \notin \Omega} \oplus \mathscr{H}(b_i)$ is normal with spectrum at most $K -$ card Ω points, and consequently X is normal with spectrum at most K points.

COROLLARY 2. *Let X commute with the difference-quotient transformation D in a space $\mathscr{D}(B)$ where $B(z)$ is an inner function of scalar type and the coefficient space $\mathscr{C}$ is finite-dimensional. If*

$$\| X(f(z), g(z)) \|_{\mathscr{D}(B)} \geqq \| X^*(f(z), g(z)) \|_{\mathscr{D}(B)}$$

for every $(f(z), g(z))$ in the range of $1 - D^*D$, then X is a normal operator whose spectrum consists of a finite number ($\leqq \dim \mathscr{C}$) of points.

Proof. Since any countable family of commuting normal operators on a finite-dimensional space has a common eigenvector, it follows that $\mathscr{D}(B) = \sum_1^{\dim \mathscr{C}} \oplus \mathscr{D}(b_j)$ where $b_j(z)$ is a scalar inner function for all j. Corollary 2 is therefore an immediate consequence of Theorem 3.

REMARK 2. The analytic Toeplitz operator T_φ on $\mathscr{C}(z)$ with $\mathscr{C}$ one-dimensional, for the symbol $\varphi(z)$ an inner function, is a universal model for unilateral shifts. Therefore, the restriction of T_φ^* to an arbitrary invariant subspace is a canonical model for contractions whose powers tend strongly to zero. A consequence of Corollary 2 is that the restriction of T_φ^* to an arbitrary invariant subspace of the backward shift T_z^* is never hyponormal (i.e., only if it is a scalar times the identity).

REFERENCES

1. L. de Branges, *Factorization and invariant subspaces,* J. Math. Anal. Appl., **29** (1970), 163-200.

2. L. de Branges and J. Rovnyak, *Canonical models in quantum scattering theory,* Perturbation Theory and its Applications in Quantum Mechanics, pp. 295-391, Wiley, New York, 1966.

3. ———, *Square Summable Power Series,* Holt, Rinehart, and Winston, New York, 1966.

4. B. Fuglede, *A commutativity theorem for normal operators,* Proc. N. A. S., **36** (1950), 35-40.
5. J. Guyker, *A structure theorem for operators with closed range,* Bull. Austral. Math. Soc., **18** (1978), 169-186.
6. ———, *On partial isometries with no isometric part,* Pacific J. Math., **62** (1976), 419-433.
7. ———, *Reducing subspaces of contractions with no isometric part,* Proc. Amer. Math. Soc., **45** (1974), 411-413.
8. C. R. Putnam, *An inequality for the area of hyponormal spectra,* Math. Z., **116** (1970), 323-330.
9. D. Sarason, *Generalized interpolation in* H^∞, Trans. Amer. Math. Soc., **127** (1967), 179-203.
10. M. J. Sherman, *A spectral theory for inner functions,* Trans. Amer. Math. Soc., **135** (1969), 387-398.
11. B. Sz.-Nagy and C. Foiaş, *Dilatation des commutants d'opérateurs,* C. R. Acad. Sci. Paris, **266** (1968), 493-495.

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