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ON BERRY-ESSEEN APPROXIMATION AND A FUNCTIONAL LIL FOR A CLASS OF DEPENDENT RANDOM FIELDS

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In this paper we derive a Berry-Esseen type approximation for a class of dependent random fields and use it to obtain a functional law of the iterated logarithm.

1. Introduction. In recent years there has been considerable interest in multiparameter stochastic collections or the so-called random fields. In this note we deal with stationary, dependent discrete-parameter random field. In [3] a concept of ϕ-mixing was introduced for such random fields and a functional central limit theorem was proved for them. Here we obtain a Berry-Esseen type approximation for such random fields and use it to prove a functional law of the iterated logarithm.

The set-up and the basic notation is as in [3]. Z^q is the set of all q-tuples of integers ($q \geqq 1$). We denote the points in Z^q by $\boldsymbol{i}$, $\boldsymbol{n}$ etc. or sometimes explicitly by $(i_1, i_2, \cdots, i_q)$, $(n_1, n_2, \cdots, n_q)$ etc. Let $\{\xi_{\boldsymbol{n}}: \boldsymbol{n} \in Z^q\}$ be a stationary, ϕ-mixing random field as defined in [3]. We denote the partial sums of this random field by $S_{\boldsymbol{n}}$ or $S_{n_1,n_2,\cdots,n_q}$ i.e.,

$$S_{n_1,n_2,\cdots,n_q} = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_q=1}^{n_q} \xi_{i_1,i_2,\cdots,i_q}$$

where $n_i \geqq 1$. If some n_i are zero and others $\geqq 1$ then it is convenient to set $S_{n_1,n_2,\cdots,n_q} = 0$.

Let $T^q = [0, 1]^q$ be the q-fold Cartesian product of the unit interval, and let D_q be the Skorohod function space on T^q. We use the uniform metric d on D_q i.e., if $x, y \in D_q$ then $d(x, y) = \sup_t |x(t) - y(t)|$.

A block B in T^q is a product of half-closed intervals i.e., a set of the form $\prod_{i=1}^q (s_i, t_i]$. If x is a function on T^q then $x(B)$ denotes increment of x around B.

We will assume throughout that:

$$(1) \qquad E(\xi_{\boldsymbol{n}}) = 0 \quad \text{and} \quad E|\xi_{\boldsymbol{n}}|^{2+\eta} < \infty \quad \text{for some} \quad \eta > 0\,.$$

We will also assume the following condition in [3] on the rate of ϕ-mixing:

$$(2) \qquad \sum_{q=1}^{\infty} r^{q-1} \phi^{1/2}(r) < \infty\,.$$

It is proved in [3] that under these conditions: $\lim_{n\to\infty} n^{-q}\mathrm{Var}(S_{n,n,\cdots,n}) = \sigma^2(<\infty)$ where

$$\sigma^2 = \sum_{j\in Z^q} E(\xi_0\xi_j) . \quad (\text{Here } \xi_0 = \xi_{0,0,\cdots,0}) .$$

To avoid trivial complications we will assume $\sigma^2 > 0$.

We denote by K_σ the Strassen's set of continuous functions on T^q:

$$K_\sigma = \left\{x\colon x(t_1, t_2, \cdots, t_q) = \int_0^{t_1}\int_0^{t_2}\cdots\int_0^{t_q} y(u_1, u_2, \cdots, u_q)du_1du_2\cdots du_q \right.$$
$$\left. \text{where } \int_0^1\cdots\int_0^1 y^2(u_1, u_2, \cdots, u_q)du_1\cdots du_q \leqq \sigma^2\right\} .$$

Theorem 1 below is a Berry-Esseen type theorem dealing with the speed of convergence of (normalized) $S_{n,n,\cdots,n}$ to normality. Theorem 2 is a functional LIL for these partial sums.

Denote by $(H_n\colon n \geqq 1)$ the sequence of random functions in D_q defined by

$$H_n(t) = (2n^q \operatorname{loglog} n)^{-1/2} S_{[nt_1][nt_2],\cdots,[nt_q]}$$

where $t = (t_1, t_2, \cdots, t_q) \in T^q$ and $[\cdot]$ is the usual greatest-integer function.

2. Theorems and proofs.

THEOREM 1. *Let Φ be the standard normal distribution function. Then under* (1) *and* (2) *there exists $C > 0$, $\alpha > 0$ such that*

$$\sup_x |P\{\sigma^{-1}n^{-q/2}S_{n,n,\cdots,n} < t\} - \Phi(t)| < Cn^{-\alpha} , \quad \textit{for all } n .$$

Proof. For simplicity suppose $q = 2$.

For given integers n, $a = a(n)$ and $b = b(n)$, let μ be the largest integer such that $\mu(a + b) \leqq n$. Then subdivide the square $(0, n]\times(0, n]$ into blocks by taking the product of 2 copies of the partition $0 < a < a + b < 2a + b < \cdots < \mu(a + b) < n$. If $1 \leqq m \leqq \mu$, denote by I_{ma} the interval $((m - 1)(a + b), (m - 1)(a + b) + a]$, by I_{mb} the interval $((m - 1)(a + b) + a, m(a + b)]$ and $I_{(\mu+1)a} = (\mu(a + b), n]$. Set

$$\alpha_m(n) = \sum_{j\in I_{m_1a}\times I_{m_2a}} \xi_j \quad (1, 1) \leqq m \leqq ((\mu + 1), \mu + 1)$$

$$\beta'_m(n) = \sum_{j\in I_{m_1}\times I_{m_2b}} \xi_j \quad (1, 1) \leqq m \leqq (\mu + \mu)$$

$$\beta''_m(n) = \sum_{j\in I_{m_1a}\times I_{m_2b}} \xi_j \quad (1, 1) \leqq m \leqq (\mu + 1, \mu)$$

$$\beta'''_m(n) = \sum_{j \in I_{m_1 b} \times I_{m_2 a}} \xi_j \quad (1, 1) \leqq \boldsymbol{m} \leqq (\mu, \mu + 1)$$

$$u_n = \sum_{m_i=1}^{\mu+1} \alpha_m(n) \quad v'_n = \sum_{m_i=1}^{\mu} \beta'_m(n) \quad v''_n = \sum_m \beta''_m(n) \quad v'''_n = \sum_m \beta'''_m(n) .$$

Then $S_{n,n} = u_n + v'_n + v''_n + v'''_n$.

Because of condition (2), by Proposition 1.1.20 of [6], we have $E(\gamma^2_m(n)) = \#(\gamma_m)(\alpha^2 + \rho_{\#(\gamma_m)})$ where γ^2_m stands for one of the α_m or β''_m, β'''_m, β''_m and $\#(\gamma_m)$ is the "size" of the block γ_m and $\rho_{\#(\gamma_m)} \to 0$ if $\#(\gamma_m) \to \infty$.

Furthermore, as in Theorem 1.1.22 of [6] we get:

(i) $Ev'^2_n \leqq [1 + 4\mu^2\phi^{1/2}(a)][\mu^2 b^2(\alpha^2 + \rho_b)]$

(ii) $Ev''^2_n \leqq [1 + 4(\mu + 1)^2\phi^{1/2}(b)][\mu^2 ab(\sigma^2 + \rho_{ab}) + \mu b(n - \mu(a + b))(\sigma^2 + \rho_{n-\mu(a+b)})]$.

(iii) $Ev'''^2_n \leqq [1 + 4(\mu + 1)^2\phi^{1/2}(b)][\mu^2 ab(\alpha^2 + \rho_{ab}) + \mu b(n - \mu(a + b))(\sigma^2 + \rho_{n-\mu(a+b)})]$.

For $(1, 1) \leqq \boldsymbol{m} \leqq (\mu, \mu)$ define $\alpha'_m(n)$ to be independent random variables having the same law as $\alpha_{(1,1)}(n)$; then as in Theorem 1.1.22 of [6] we get:

$$\text{(iv)} \qquad \left| P\left(\frac{u_n}{\sigma\sqrt{n^2}} < t\right) - P\left(\frac{\sum_{m_i=1}^{\mu} \alpha'_m(n)}{\sigma\sqrt{n^2}} < t\right)\right| \leqq (\mu + 1)^2\phi(b) .$$

For this computation, it is easy to show that the "end blocks" $\alpha_m(n)$(with m_1 or m_2 equal to $\mu + 1$) become negligible for large n.

$$\text{(v)} \qquad \left| P\left(\frac{\sum_{m=1}^{\mu}\alpha'_m(n)}{E^{1/2}(\alpha'_m(n))^2} < \frac{t\sigma\sqrt{n^2}}{E^{1/2}(\sum \alpha'_m)^2}\right) - \Phi\left(\frac{t\sigma\sqrt{n^2}}{E^{1/2}(\sum\alpha'_m(n))^2}\right)\right|$$
$$\leqq \frac{c_\delta(\mu + 1)^2 E|\alpha'_{(1,1)}|^{2+\delta}}{[(\mu + 1)^2 E(\alpha^2_{(1,1)})]|^{1+\delta/2}} \leqq Ac_\delta(1 + \mu)^{-\delta}$$

because by [4, Lemma 7]. $E(|\alpha_{(1,1)}|^{2+\delta}) \leqq A(E(\alpha'^2_{(1,1)}))^{1+\delta/2}$

$$\text{(vi)} \quad \left| \Phi\left(\frac{\sqrt{n^2}}{\sqrt{(\mu + 1)^2 a(\sigma^2 + \rho_a)}} \cdot t\sigma\right) - \Phi(t)\right|$$
$$\leqq \frac{1}{2\pi e}\max\left(1, \sqrt{\frac{(\mu + 1)^2 a^2}{n^2}}\left(1 + \frac{\rho_a}{\sigma^2}\right)\right| \sqrt{\frac{n^2\sigma^2}{(\mu + 1)^2 a^2(\sigma^2 + \rho_a)}} - 1\Bigg)$$
$$= \psi(n) .$$

From (i)-(vi) and using a similar argument as in Theorem 1.1.22 of [6], for $\tau > 0$

$$\left| P\left(\frac{S_{n,n}}{\sigma\sqrt{n^2}} < t\right) - \Phi(t)\right| \leqq (\mu + 1)^2\phi(b) + Ac_\delta(\mu + 1)^{-\delta} + \psi(n)$$
$$+ \frac{\tau}{\sigma\sqrt{n^2}} + \frac{Ev'^2_n}{\tau^2/9} + \frac{Ev''^2_n}{\tau^2/9} + \frac{Ev'''^2_n}{\tau^2/9} .$$

If we choose $a = [n^{.6}]$ $b = [n^{.4}]$ $\tau = [n^{1-\varepsilon}]$ $0 < \varepsilon < 1$ then $\mu = O(n^{.4})$ and $n - \mu(a + b) = O(n^{.6})$. Since condition (2) implies that $r^q\phi^{1/2}(r) \to 0$; then

$$(\mu + 1)^2\phi(b) = ((\mu + 1)^4\phi(b))(\mu + 1)^{-2} = O(n^{-.8}) .$$
$$Ac_\delta(\mu + 1)^{-\delta} = O(n^{-\delta(.4)}) = O(n^{-.4\delta}) .$$

$$\psi(n) = O\left(\left|\sqrt{\frac{n^2\sigma^2}{(\mu + 1)^2a^2(\sigma^2 + \rho_a)}} - 1\right|\right) = O\left(\left|\sqrt{\frac{n^2}{(\mu + 1)^2a^2}} - 1\right|\right)$$
$$= O\left(\frac{b}{a}\right) = O(n^{-.2})$$

$$\frac{\tau}{\sigma\sqrt{n^2}} = O(n^{-\varepsilon}) .$$

$$\frac{Ev_n'^2}{\tau^2/9} \leqq (\text{constant})\frac{\mu^2b^2(\sigma^2 + \rho_b)}{\tau^2/9} \text{ since } \mu^2\phi^{1/2}(a) \longrightarrow 0 .$$
$$= O\left(\frac{n^{2-.4}}{n^{2-2\varepsilon}}\right) = O(n^{2\varepsilon-.4}) .$$

$$\frac{Ev_n''^2}{\tau^2/9} \leqq \frac{(\text{constant})}{\tau^2/9}[\mu^2ab(\sigma^2 + \rho_{ab}) + \mu b(n - \mu(a + b))(\sigma^2 + \rho_{n-\mu(a+b)})]$$
$$= O(n^{2\varepsilon-.2}) .$$

Similarly

$$\frac{Ev_n'''^2}{\tau^2/9} = O(n^{2\varepsilon-.2}) .$$

Then

$$\left|P\left(\frac{S_{n,n}}{\sigma\sqrt{n^2}} < t\right) - \Phi(t)\right| \leqq Cn^{-\alpha}$$

if we set: $\varepsilon = \alpha = \dfrac{1}{15}$ whenever $\delta \geqq \dfrac{1}{6}$.

$\varepsilon = \alpha = .48$ whenever $0 < \delta < \dfrac{1}{6}$.

An analogous proof is valid for the $q > 2$, in that case take

$$\varepsilon = \alpha = \frac{1}{15} \text{ if } \delta \geqq \frac{1}{3q}$$
$$\varepsilon = \alpha = (.2q\delta) \text{ if not .}$$

REMARK. From the proof, it can be seen that a more general theorem can be obtained if we replace $S_{n,n,\cdots,n}$ by S_n where $n' = (n\theta_1, n\theta_2, \cdots, n\theta_q)$ $0 < \theta_i \leqq 1$. Then we have

$$\sup_n |P\{\sigma^{-1}n^{-q/2}(\theta_1 \cdots \theta_q)^{-1/2}S_{n'} < t\} - \Phi(t)| < Cn^{-\alpha} \ \forall n \ .$$

In fact it is in this stronger form that we will use it in the proof of Theorem 2.

THEOREM 2. *Let* (1) *and* (2) *be satisfied. Then*

(a) $$P\{\limsup_{n\to\infty} d(H_n, K_\sigma) = 0\} = 1 \ ,$$

and

(b) $$P\{\bigcap_{x\in K_\sigma} [\liminf_{n\to\infty} d(H_n, x) = 0]\} = 1 \ .$$

Proof. We will give only a very brief sketch of the proof since the arguments used are fairly standard and can be found e.g., in Chover (1967) and Wichura (1973). Take $q = 2$ for simplicity and $\sigma = 1$ without loss of generality.

We begin by showing a kind of asymptotic equi-continuity in the following form: Let $B = \prod_i (s_i, t_i]$ be a block in T^q; write $m(B) = \min_{1\leq i\leq q}(t_i - s_i)$. Then

LEMMA 1. *Given* $\varepsilon > 0$, $\exists\delta > 0$ *such that if* B *is any block with* $m(B) < \delta$ *then the event* $\{|H_n(A)| > \varepsilon\}$ *occurs only finitely often wp.*1.

Proof. Standard arguments (using the triangle inequality) such as those appearing on pp. 56-59 of Billingsley (1968) show that it suffices to prove the following: Given $\varepsilon > 0$, $\exists\delta > 0$ such that

$$\sum_n [P\{\max_{\substack{1\leq i\leq n\delta \\ 1\leq j\leq n}} |S_{i,j}| > \varepsilon\sqrt{2n^2 \operatorname{loglog} n}\} + P\{\max_{\substack{1\leq i\leq n \\ 1\leq j\leq n\delta}} |S_{i,j}| > \varepsilon\sqrt{2n^2 \operatorname{loglog} n}\}] < \infty \ .$$

But this can be proved in a straightforward manner using the maximal inequality developed on pp. 713-714 of [3], Theorem 1 above and the arguments in § 3 of Chover (1967). We omit the details.

Let now m be a positive integer. Consider a partition of the unit square (T^2) into $m\times m$ squares with corners $(i/m, j/m)$, $0 \leqq i, j \leqq m$. We enumerate these squares (blocks) arbitrarily as B_{im}, $1 \leqq i \leqq m^2$. Let $\gamma > 0$ be a small positive number and denote by $B^*_{im} = B^*_{im}(\gamma)$ the square which is concentric with B_{im} (and is contained in B_{im}) with each side being equal to $(1 - 2\gamma)/m$.

If x is a function on T^2 we denote by $\pi_m x$ the function on T^2 defined by

$$(\pi_m x)(t_1, t_2) = \int_0^{t_1}\int_0^{t_2} \sum_{i=1}^{m_2} m^2 x(B_{im}) I_{B_{im}}(u_1, u_2) du_1 du_2$$

where I_B stands for the indicator of the block B.

Lemmas 2 and 3 below follow easily from the arguments used in proving Corollaries 1 and 2 in Chover (1967). Lemma 4 is immediate from Lemma 1.

LEMMA 2. *Given* $\varepsilon > 0$, $\exists m$ *such that*

$$P\{d(\pi_m H_n, H_n) > \varepsilon \text{ only finitely often (in } n)\} = 1 .$$

LEMMA 3. *Given* $\varepsilon > 0$, $\exists c > 1$ *such that, wp.* 1, $\max_{c^n \leqq m \leqq c^{n+1}} d(H_m, H_{[c^n]}) > \varepsilon$ *for only finitely many* n.

LEMMA 4. *Given* $\varepsilon > 0$, $\exists \gamma > 0$ *such that for each* m *and* i $(1 \leqq i \leqq m^2)$,

$$P\{|H_n(B_{im}) - H_n(B^*_{im})| > \varepsilon \text{ for only finitely many } n\} = 1 .$$

We now proceed to prove (a) of the theorem. Let $\{\theta_i: 1 \leqq i \leqq m^2\}$ be real numbers such that $\sum_{i=1}^{m^2} \theta_i^2 = 1$. To prove (a) it suffices to show that for each m,

$$P\left\{\sum_{i=1}^{m^2} \theta_i (\pi_m H_n)(B_{im}) < (1 + \varepsilon) \quad \text{for all large } n \right\} = 1 .$$

In view of the preceding lemmas it thus suffices to prove (with $c > 1$ sufficiently close to 1 and $\gamma > 0$ sufficiently small)

$$\sum_{n=1}^{\infty} P\left\{m \sum_{i=1}^{m^2} \theta_i H_{[c^n]}(B^*_{im}) > (1 + \varepsilon)\right\} < \infty .$$

But the proof of this is essentially the same as given in § 4 of Chover (1967). The only complication here is that the m^2 random variables $\{H_{[c^n]}(B^*_{im}): 1 \leqq i \leqq m^2\}$ are not independent. But there is enough separation among these and it suffices to apply Lemma 1.1.5 in Iosifescu and Theodorescu (1969).

To prove (b) take $x \in K$ with $\int_0^1\int_0^1 (\partial^2 x/\partial t_1 \partial t_2)^2 dt_1 dt_2 < 1$. We need to show that $\forall \varepsilon > 0$, $P(\liminf d(H_n, x) < \varepsilon) = 1$. Again in view of the preceding lemmas and the arguments in Sec.5 of Chover (1967) it is enough to prove for sufficiently small $\delta > 0$, $\gamma > 0$

$P(\lim_{n\to\infty} \sup F_n) = 1$ where

$$F_n = \{|H_{[c^n]}(B^*_{im}) - x(B_{im})| < \delta , \quad \text{all } i, \quad 1 \leqq i \leqq m^2\} .$$

[It might be noted here that (35) in [2] is insufficient; it should be strengthened to $P(\lim_{r\to\infty} \sup \bigcap_\nu C_r^{(\nu)}) = 1$.] Now if the probability of F_n in computed on the assumption that the m^2 random variables $\{H_{[c^n]}(B^*_{im}): 1 \leqq i \leqq m^2\}$ are independent then the error committed is at most $m^2\phi_{[\nu c^n]}$ which forms a term of a convergent series in n. Hence using part (a) of the lemma on page 142 of [5] it is enough to show $\sum_n P(F_n) = \infty$. But given Theorem 1 this follows from computations which are standard in the proof of Strassen's theorem. This completes the proof of Theorem 2.

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