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TAUBERIAN THEOREMS FOR MATRICES GENERATED BY ANALYTIC FUNCTIONS

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Several classes of summability matrices are determined by the coefficients of Maclaurin series of the products of certain analytic functions. These matrices include generalizations of the transforms of Lototsky, Taylor, and others. It is proved that under rather weak restrictions on the analytic functions, $x_k - x_{k+1} = o(k^{-1})$ is a Tauberian condition for the resulting matrix transformations.

1. Introduction. Several classes of summability transforms are generated by products of analytic functions. The matrix $(a_{n,k})$ of such a transform is given by

$$(1) \qquad \prod_{k=0}^{n} f_k(z) = \sum_{k=0}^{\infty} a_{n,k} z^k ,$$

where $f_k(z)$ is analytic at $z = 0$ $(k = 0, 1, 2, \cdots)$. This class of transforms includes, for example, the well-known Euler-Knopp means [6, pp. 56–60] and the Taylor transforms [6, pp. 60–64]. In addition to these two special cases, the transforms of this class for which we shall prove Tauberian theorems are the following: the Karamata transform [8, 9], the generalized Lototsky transform [4], and the $\mathscr{T}(r_n)$ transform [7]. We also give a Tauberian theorem for the $T(r_n)$ transform [5] which, although not a member of this class, is very similar to the others.

In this paper we shall state the Tauberian theorems in sequence-to-sequence form; thus, a typical Tauberian condition for a sequence x is $(\Delta x)_k = o(k^{-1})$, where Δx is given by $(\Delta x)_k = x_k - x_{k+1}$. Our proofs will use recently developed techniques [1, 2] that are based on the concept of a "block-dominated" matrix. For each n, let $\{a_{n,k}\}_{k=1+\mu(n)}^{\nu(n)}$ be a block of consecutive terms of nth row of the matrix A; then A is *dominated by the sequence of blocks* $\{a_{n,k}\}_{k=1+\mu(n)}^{\nu(n)}$ $(n = 0, 1, \cdots)$ provided that

$$(2) \qquad \liminf_n \left\{ \left| \sum_{k=1+\mu(n)}^{\nu(n)} a_{n,k} \right| - \sum_{k \le \mu(n)} |a_{n,k}| - \sum_{k > \nu(n)} |a_{n,k}| \right\} > 0 .$$

Then $L_n \equiv \nu(n) - \mu(n)$ is called the length of the block in the nth row. The results from [1, 2] that we shall use are stated here for convenience.

THEOREM A. *Let A be a regular matrix that is dominated by*

$\{a_{n,k}\}_{k=1+\mu(n)}^{\nu(n)}$; *if* x *is a bounded sequence that is A-summable and*

$$\max_{\mu(n)<k\leqq\nu(n)} |(\Delta x)_k| = o(L_n^{-1}) , \tag{3}$$

then x *is convergent.*

THEOREM B. *Let* A *be a regular matrix that is dominated by* $\{a_{n,k}\}_{k=1+\mu(n)}^{\nu(n)}$ *and* $a_{n,k} = 0$ *whenever* $k > \nu(n)$; *if* x *is A-summable and* (3) *holds, then* x *is convergent.*

LEMMA C. *If* x *is a sequence such that* $(\Delta x)_k = o(k^{-1})$ *and the index sequences* μ *and* ν *are chosen so that*

$$\nu(n) = O(\mu(n)) \quad and \quad \lim_n \mu(n) = \infty , \tag{4}$$

then (3) *is satisfied.*

Thus, for a given matrix A, our method will be to show that the sequences μ and ν can be chosen so that (2) and (4) are satisfied; from this we can then conclude that $(\Delta x)_k = o(L_n^{-1})$ is a Tauberian condition for A.

2. Principle lemmas. Since we shall be considering regular matrices, our general task will be to show that the index sequences μ and ν can be chosen so that for each n,

$$\sum_{k\leqq\mu(n)} |a_{n,k}| \leqq \rho < 1/4 \tag{5}$$

and

$$\sum_{k>\nu(n)} |a_{n,k}| \leqq \rho < 1/4 . \tag{6}$$

These inequalities, coupled with the Silverman-Toeplitz conditions for regularity, guarantee that A is dominated by $\{a_{n,k}\}_{k=1+\mu(n)}^{\nu(n)}$ ($n = 0, 1, 2, \cdots$). In order to establish (4), our method will be to show that μ and ν can be chosen so that each is given (approximately) by a linear expression in n. This is stated precisely using the greatest integer function in the following two lemmas.

LEMMA 1. *Suppose* A *is a matrix given by* (1), *where each* f_k *is analytic on* $\{z\colon |z| \leqq R, R > 1\}$ *and there is a number* M *such that for every* k,

$$\sup_{|z|=R} |f_k(z)| \leqq M ;$$

if $\rho < 1/4$, *then there exist numbers* a *and* b *such that* $\nu(n) =$

$[an + b] + 1$ *and* (6) *holds.*

Proof. Let $\Gamma = \{z : |z| = R\}$ and $g_n(z) = \prod_{k=0}^{n} f_k(z)$. Then we have

$$\begin{aligned}\sum_{k>\nu} |a_{n,k}| &= \sum_{k>\nu} \left| \frac{1}{2\pi i} \int_\Gamma \frac{g_n(t)}{t^{k+1}} dt \right| \\ &\leqq \frac{1}{2\pi} \sup_{t \in \Gamma} |g_n(t)| \sum_{k=\nu+1}^{\infty} \frac{1}{R^{k+1}} (2\pi R) \\ &\leqq \frac{1}{2\pi} M^{n+1} \frac{1}{R^{\nu+2}} \frac{1}{1 - \dfrac{1}{R}} (2\pi R) \\ &= \frac{1}{R-1} \frac{1}{R^\nu} M^{n+1} .\end{aligned}$$

For a given $\rho < 1/4$, $\sum_{k>\nu} |a_{n,k}| \leqq \rho$ will hold provided

$$\frac{1}{R-1} \frac{M^{n+1}}{R^\nu} \leqq \rho ,$$

which is equivalent to

$$R^\nu \geqq \frac{M^{n+1}}{\rho(R-1)} ,$$

or

$$\nu \geqq \frac{(n+1) \ln M - \ln (\rho(R-1))}{\ln R} .$$

Hence, a and b can be chosen so that $\nu(n) = [an + b] + 1$ and (6) holds.

LEMMA 2. *Suppose A is a matrix given by* (1), *where each f_k is analytic on $\{z : |z| \leqq R < 1\}$ and there is a number $M < 1$ such that for every k,*

$$\sup_{|z|=R} |f_k(z)| \leqq M ;$$

if $\rho < 1/4$, then there exist numbers a and b such that $a > 0$, $\mu(n) = [an + b]$, and (5) *holds.*

Proof. Let $\Gamma = \{z : |z| = R < 1\}$ and $g_n(z) = \prod_{k=0}^{n} f_k(z)$. Then we have

$$\sum_{k \leqq \mu} |a_{n,k}| = \sum_{k \leqq \mu} \left| \frac{1}{2\pi i} \int_\Gamma \frac{g_n(t)}{t^{k+1}} dt \right|$$

$$\leqq \frac{1}{2\pi} \sup_{t \in \Gamma} |g_n(t)| \sum_{k=0}^{\mu} \frac{1}{R^{k+1}} (2\pi R)$$

$$\leqq \frac{1}{2\pi} M^{n+1} \frac{1}{R^2} \frac{\dfrac{1}{R^{\mu+1}} - 1}{\dfrac{1}{R} - 1} (2\pi R)$$

$$\leqq \frac{1}{1-R} M^{n+1} \frac{1}{R^{\mu+1}} .$$

For $\rho < 1/4$, $\sum_{k \leqq \mu} |a_{n,k}| \leqq \rho$ will hold provided

$$\frac{1}{1-R} M^{n+1} \frac{1}{R^{\mu+1}} \leqq \rho ,$$

which is equivalent to

$$R^{\mu} \geqq \frac{M^{n+1}}{\rho(1-R)R} ,$$

that is,

$$\mu \leqq \frac{(n+1)\ln M - \ln[\rho(1-R)R]}{\ln R} .$$

Hence, a and b can be chosen so that $\mu(n) = [an + b]$, $a > 0$, and (5) holds.

3. Applications. The first Tauberian theorem that we shall prove concerns the $\mathscr{T}(r_n)$ transform [7]. This generalization of the Taylor transform is defined by

$$\prod_{k=0}^{n} \frac{1 - r_k}{1 - r_k z} = \sum_{k=0}^{\infty} a_{n,n+k} z^k$$

where $a_{n,k} = 0$ for $k < n$ and $\{r_k\}_0^\infty$ is a sequence.

THEOREM 3. *Suppose that* $0 \leqq r_n \leqq \beta < 1$ *for* $n = 0, 1, 2, \cdots$, *and* x *is a bounded sequence that is* $\mathscr{T}(r_n)$*-summable and satisfies* $(\Delta x)_k = o(k^{-1})$; *then* x *is convergent.*

Proof. Since $0 \leqq r_n \leqq \beta < 1$ we have that $\mathscr{T}(r_n)$ is a regular matrix [7, Theorem 3.6]. Choose $\mu(n) = n$. Since $f_k(z) = (1 - r_k)/(1 - r_k z)$, each f_k is analytic on $\{z: |z| \leqq 2/(1+\beta)\}$, and $2/(1+\beta) > 1$. For $R = 2/(1+\beta)$ we have

$$\sup_{|z|=R} |f_k(z)| \leqq \frac{1 - r_k}{|1 - r_k R|} \leqq \frac{1}{1 - \beta\left(\dfrac{2}{1-\beta}\right)} = \frac{1-\beta}{1+\beta} .$$

Thus, the result follows by Lemma 1, Lemma C, and Theorem A.

The Taylor transform $T(r)$ is a special case of the $\mathscr{T}(r_n)$ transform, where $r_n = r$ $(n = 0, 1, 2, \cdots)$.

COROLLARY 4. *If $0 \leqq r < 1$ and x is a bounded sequence that is $T(r)$-summable and satisfies $(\Delta x)_k = o(k^{-1})$, then x is convergent.*

Proof. Choose $\beta = r$ in Theorem 3.

This corollary is an extension of the result contained in [1, Theorem 9].

The Karamata transform $K[a, b]$ [8, 9] is generated by (1) using

$$f_k(z) = f(z) = \frac{a + (1 - a - b)z}{1 - bz} \quad k = 0, 1, 2, \cdots .$$

Then $K[a, b]$ is regular [8, Theorem 3] provided $-1 < -b < a < 1$. Each f_k is analytic on $\{z\colon |z| < 1/|b|\}$ and

$$|f_k(z)| \leqq \frac{|a| + |1 - a - b||z|}{|1 - |b||z||} ;$$

so

$$\sup_{|z|=R} |f_k(z)| \leqq \frac{|a| + |1 - a - b|R}{||b|R - 1|} .$$

THEOREM 5. *If $-1 < -b < a < 1$, and x is a bounded sequence that is $K[a, b]$-summable and satisfies $(\Delta x)_k = o(k^{-1})$, then x is convergent.*

Proof. Suppose $b = 0$. Thus

$$\sup_{|z|=R} |f_k(z)| \leqq |a| + |1 - a|R .$$

To apply Lemma 1, choose $R = 2$ and $M = |a| + 2|1 - a|$. To apply Lemma 2, choose $R = (1 - |a|)/2|1 - a|$. Thus $R < 1$ and

$$\sup_{|z|=R} |f_k(z)| \leqq |a| + |1 - a|\frac{1 - |a|}{2|1 - a|} = \frac{1 + |a|}{2} < 1 .$$

Suppose $b \neq 0$. To apply Lemma 1 choose $R = (|b| + 1)/2|b|$. Then $1 < R < 1/|b|$ and f_k is analytic on $\{z\colon |z| \leqq R\}$. Moreover,

$$\sup_{|z|=R} |f_k(z)| \leqq \frac{|a| + |1 - a - b|\dfrac{|b| + 1}{2|b|}}{\left||b|\dfrac{|b + 1|}{2|b|} - 1\right|} .$$

Now choose M (of Lemma 1) to be the right-hand member of the preceding inequality. To apply Lemma 2, choose

$$R = \frac{1 - |a|}{2(|1 - a - b| + |b|)} .$$

Then $R < 1$ and f_k is analytic on $\{z: |z| \leqq R\}$. Moreover, in this case,

$$\sup_{|z|=R} |f_k(z)| \leqq \frac{|a| + |1 - a - b| \dfrac{1 - |a|}{2(|1 - a - b| + |b|)}}{\left| |a| \dfrac{1 - |a|}{2(|1 - a - b| + |b|)} - 1 \right|} < 1 .$$

Therefore the result follows by Lemma 1, Lemma 2, Lemma C, and Theorem A.

The generalized Lototsky transform $[L, d_n]$ [4] is generated by (1) using $f_0(z) \equiv 1$ and

$$f_k(z) = \frac{z + d_k}{1 + d_k} , \qquad k = 1, 2, \cdots .$$

Then $[L, d_n]$ is a lower triangular matrix and, for $h_k = (1 + d_k)^{-1}$ where $0 < \alpha \leqq h_k \leqq 1$, the transform is regular [4, Theorems 3.1 and 3.2]. Also, the generating functions become $f_k(z) = 1 - h_k + h_k z$ for $k \geqq 1$.

THEOREM 6. *If* $0 < \alpha \leqq (1 + d_k)^{-1} \leqq 1$, *and the sequence* x *is* $[L, d_n]$*-summable and satisfies* $(\varDelta x)_k = o(k^{-1})$, *then* x *is convergent.*

Proof. Let $\nu(n) = n$, and note that each f_k is an entire function. Choose $R = \alpha/2 < 1$. Substituting $h_k = (1 + d_k)^{-1}$, we have

$$|f_k(z)| = |1 - h_k + h_k z| \leqq |1 - h_k| + |h_k||z| .$$

Hence,

$$\sup_{|z|=R} |f_k(z)| \leqq 1 - \alpha + (1)\frac{\alpha}{2} < 1 .$$

The result now follows from Lemma 2, Lemma C, and Theorem B.

In the special case where $h_k \equiv r$, the $[L, d_n]$-transform becomes the Euler-Knopp transform $E(r)$. Thus, the result in Theorem 6 holds for $E(r)$ when $0 < r \leqq 1$. This is a weaker result, however, than the Hardy-Littlewood result [3] for $E(r)$ which uses the Tauberian condition $(\varDelta x)_k = O(k^{-1/2})$.

The $T(r_n)$-transform [5] is defined by $a_{n,k} = 0$ for $k < n$ and

$$[f_n(z)]^n = \sum_{k=0}^{\infty} a_{n,n+k} z^k ,$$

where $f_n(z) = (1 - r_n)/(1 - r_n z)$. This form is slightly different than (1), but with minor modifications in Lemmas 1 and 2, the following Tauberian theorem can be proved for the $T(r_n)$-transform.

THEOREM 7. *If $0 \leqq r_n \leqq \beta < 1$, and x is a bounded sequence that is $T(r_n)$-summable and satisfies $(\Delta x)_k = o(k^{-1})$, then x is convergent.*

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