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ON THE SOLVABILITY OF BOUNDARY AND INITIAL-BOUNDARY VALUE PROBLEMS FOR THE NAVIER-STOKES SYSTEM IN DOMAINS WITH NONCOMPACT BOUNDARIES

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In the present paper the solvability of boundary value problems for the Stokes and Navier-Stokes equations is proved for noncompact domains with several "exits" to infinity. In these problems the velocity satisfies usual boundary conditions and has a bounded Dirichlet integral and the pressure has prescribed limiting values at infinity in some "exits".

1. Preface. It was shown by J. Heywood [1] that solutions of the Navier-Stokes system (even linearized) are not uniquely determined by the usual boundary and initial conditions in some domains with noncompact boundaries. It is connected with the possible noncoincidence of some spaces of divergence free vector fields defined in these domains. These spaces and linear sets of vector fields generating them are introduced as follows.

Let Ω be a domain in R^n, $n = 2, 3$, $\mathscr{C}_0^\infty(\Omega)$ — the set of all infinitely differentiable functions with compact supports contained in Ω, $\mathscr{J}_0^\infty(\Omega)$— the set of all divergence-free vector fields $\vec{u} \in \mathscr{C}_0^\infty(\Omega)$ (i.e., vector fields satisfying the equation $\nabla \cdot \vec{u} = \sum_{i=1}^n (\partial u_i/\partial x_i) = 0$), and $\overset{\circ}{W}{}_2^1(\Omega)$ and $\overset{\circ}{\mathscr{D}}(\Omega)$ — the completions of $\mathscr{C}_0^\infty(\Omega)$ in the norms $\|\vec{u}\|_{W_2^1(\Omega)} = \sqrt{(\vec{u}, \vec{u})^{(1)}}$ and $\|\vec{u}\|_{\mathscr{D}(\Omega)} = \sqrt{[\vec{u}, \vec{u}]}$ respectively, where $(\vec{u}, \vec{v})^{(1)} = \int_\Omega (\vec{u}\cdot\vec{v} + \vec{u}_x\cdot\vec{v}_x)\,dx$, $[\vec{u}, \vec{v}] = \int_\Omega \vec{u}_x\cdot\vec{v}_x\,dx$, $\vec{u}\cdot\vec{v} = \sum_{i=1}^n u_i v_i$, $\vec{u}_x\cdot\vec{v}_x = \sum_{i,j=1}^n (\partial u_i/\partial x_j)(\partial v_i/\partial x_j)$. Let $\mathscr{J}(\Omega)$ and $H(\Omega)$ be completions of $\mathscr{J}_0^\infty(\Omega)$ in these norms and $\hat{\mathscr{J}}(\Omega)$, $\hat{H}(\Omega)$ — the subspaces of all divegence-free vector fields in $\overset{\circ}{W}{}_2^1(\Omega)$ and $\overset{\circ}{\mathscr{D}}(\Omega)$. Clearly, $\hat{\mathscr{J}}(\Omega) \supset \mathscr{J}(\Omega)$ and $\hat{H}(\Omega) \supset H(\Omega)$. In [1] it is shown there are domains for which the quotient spaces $\hat{\mathscr{J}}(\Omega)/\mathscr{J}(\Omega)$, $\hat{H}(\Omega)/H(\Omega)$ are finite-dimensional, i.e., nontrivial (for instance, the domain $\Omega^0 = R^3\backslash S$, $S = \{x \in R^3\colon x_3 = 0,\ x_1^2 + x_2^2 \geqq 1\}$ possesses this property). A large class of such domains is found by O. Ladyzhenskaya, K. Piletskas and the author in [2, 3]. To describe the domains Ω considered in this paper, we define a standard domain $G \subset R^n$ given by the inequality

$$(1)\qquad |z'| < g(z_n)\,, \quad z_n \geqq 0\,,$$

where $|z'| = |z_1|$ for $n = 2$, $|z'| = \sqrt{z_1^2 + z_2^2}$ for $n = 3$ and the function $g(t)$ satisfies the conditions

(2) $g(t) \geqq g_0 > 0\ ,\quad |g(t) - g(t_1)| \leqq M|t - t_1|\ ,\quad \forall t, t_1 > 0\ .$

We impose the following requirements on Ω:

(1) Ω is an open connected set; $\Omega = \Omega_0 \cup (\bigcup_{i=1}^m \omega_i)$, Ω_0 is a bounded domain, the ω_i are unbounded and $\omega_i \cap \omega_j = \varnothing$ for $i \neq j$.

(2) $G_i \subset \omega_i \subset G_i^a$, where G_i and G_i^a are domains defined by inequalities of the form (I) in a certain cartesian coordinate system $\{z^{(i)}\}$, more precisely, by inequalities

$$(3)\qquad |z'^{(i)}| < g_i(z_n^{(i)})\ ,\quad |z^{(i)}| < ag_i(z_n^{(i)})\ ,$$

with $a > 1$, and functions g_i satisfying (2) and

$$\int_0^\infty g_i^{-n-1}(t)dt < \infty \qquad \text{for}\quad i = 1, \cdots, r\ ,\quad 1 \leqq r \leqq m\ ,$$

$$\int_0^\infty g_i^{-n-1}(t)dt = \infty \qquad \text{for}\quad i = r+1, \cdots, m\ .$$

To formulate further restrictions we introduce the following notations: $\omega_i(t)$ is the subdomain of ω_i where $0 < z_n^{(i)} < t$, $\omega_i'(t) = \omega_i \backslash \overline{\omega_i(t)}$, $\Sigma_i(t)$ is the intersection of ω_i with the plane (the straight line for $n = 2$) $z_n^{(i)} = t$; and $\Omega_t = \Omega \backslash \bigcup_{i=1}^m \omega_i'(t)$. We assume:

(3) $H(\Omega_t) = \hat{H}(\Omega_t)$ for all $t \geqq 0$.

(4) Every function $q(x) \in L_2(B_i(t))$ satisfying in the domain $B_i(t) = \omega_i(t + g_i(t)) \backslash \omega_i(t)$ the condition $\int_{B_i} q dx = 0$ can be represented in the form $q = \nabla \cdot \vec{u}(x)$ where $\vec{u} \in \overset{\circ}{\mathscr{D}}(B_i(t))$ (see [2], Lemma 2.5) and $\|\vec{u}\|_{\mathscr{D}(B_i(t))} \leqq c\|q\|_{L_2(B_i(t))}$, the constant c being independent of q, i, t.

(5) The domain Ω_{t_0} with some fixed $t_0 > 0$ possesses the same property.

Sometimes we shall replace (2) by

(2′) $G_i \subset \omega_i \subset G_i^a$ where G_i, G_i^a are domains defined by (3) and

$$\int_0^\infty g_i^{-n-1+2\alpha}(t)dt = \infty\ ,\qquad i = 1, \cdots, r\ ;$$

$$\int_0^\infty g_i^{-n-1+2\alpha}(t)dt < \infty\ ,\qquad i = r+1, \cdots, m\ ;\quad \alpha \in [0, 1]\ .$$

The conditions (1)-(5) determine a somewhat more general class of domains than considered in [3]. On the other hand, the condition $\omega_i \subset G_i^a$ is not satisfied for the domain Ω^0 mentioned above. This condition is also not satisfied for domains considered in [2], for which ω_i may contain unbounded cones (i.e., for which $m = r$ and $g_i(t) = \lambda_i(t + b_i)$, $\lambda_i, b_i > 0$). For such domains the conditions (2) should be replaced by the restrictions formulated in §4 of the paper [2].

THEOREM 1. *If* (1)-(5) *hold, then* $\dim \hat{H}(\Omega)/H(\Omega) = r - 1$; *if the* *conditions* (1), (3)-(5) *and* (2′) *with* $\alpha = 1$ *are fulfilled, then* $\dim \hat{\mathscr{J}}(\Omega)/\mathscr{J}(\Omega) = r - 1$. *In* $\hat{H}(\Omega)/H(\Omega)$ *and in* $\hat{\mathscr{J}}(\Omega)/\mathscr{J}(\Omega)$ *there exist* $r - 1$ *linearly independent vector fields* $\vec{a}_i(x)$ *which are infinitely differentiable in each* ω_j, *which vanish in a neighborhood of* $\partial\Omega \cap \partial\omega_j$, *for each* ω_j, *and for* $|x| \gg 1$, $x \in \omega_j$, $j = r + 1, \cdots, m$, *and which satisfy the inequalities*

$$(4) \qquad |\vec{a}_i(x)| \leqq \frac{C_0}{g_j^{n-1}(x)}, \quad \left|\frac{\partial \vec{a}_i(x)}{\partial x_k}\right| \leqq \frac{C_1}{g_j^n(x)}, \quad x \in \omega_j, \quad j = 1, \cdots, r.$$

This theorem can be proved in the same way as Theorem 4.2 [2] or Theorem 4 [3].

If $H(\Omega) \neq \hat{H}(\Omega)$, the boundary value problem for stationary Navier-Stokes system must contain, beyond the usual boundary conditions at $\partial\Omega$ and at infinity, some additional conditions. One can prescribe the flows of the velocity vector across sections of some ω_i. Boundary value problems of this type are studied in the papers [1, 3, 4]. On the other hand, in [1] another form of additional condition is found. It is shown that the assignment of the difference of limiting values of the pressure for $|x| \to \infty$, $x \in \omega_i$, $i = 1, 2$ also determines uniquely the solution of the boundary value problem for the Stokes system in the domain Ω^0.

2. Preliminaries. We begin with the construction of an auxiliary divergence-free vector field in the domain (1) which is necessary for subsequent considerations and which can be used also for the construction of a basis in $\hat{H}(\Omega)/H(\Omega)$ and $\hat{\mathscr{J}}(\Omega)/\mathscr{J}(\Omega)$. At first let $n = 3$ and define the vector

$$(5) \qquad \vec{a}(z) = \nabla \times \zeta(z)\vec{b}(z') = \nabla\zeta(z) \times \vec{b}(z'),$$

where $\vec{b} = (2\pi)^{-1}(-z_2|z'|^{-2}, z_1|z'|^{-2}, 0)$, $z' = (z_1, z_2)$, and $\zeta(z) \in \mathscr{C}^\infty(G)$ is a function which equals one in a neighbourhood of the surface $\Gamma\colon |z'| = g(z_3)$ and vanishes for small $|z'|$. Consequently, $\vec{a} \in \mathscr{C}^\infty(G)$, $\vec{a} = 0$ near Γ and for small $|z'|$, $\nabla \cdot \vec{a} = 0$ and

$$\int_{\sigma(t)} a_3 dz' = \int_{\partial\sigma(t)} \zeta\vec{b} \cdot d\vec{l} = \frac{1}{2\pi} \int_{\partial\sigma(t)} \left(-\frac{z_2}{|z'|^2} dz_1 + \frac{z_1}{|z'|^2} dz_2\right) = 1$$

($\sigma(t)$ is the intersection of G with the plane $z_3 = t$). In the case $n = 2$ the vector

$$(6) \qquad \vec{a}(z) = \frac{1}{2}\left(-\frac{\partial\tilde{\zeta}(z)}{\partial z_2}, \frac{\partial\tilde{\zeta}(z)}{\partial z_1}\right),$$

where $\tilde{\zeta} \in \mathscr{C}^\infty(G)$, $\tilde{\zeta} = 0$ for small $|z_1|$, $\tilde{\zeta} = \pm 1$ near $\Gamma^\pm\colon z_1 = \pm g(z_2)$, possesses all these properties.

It is convenient to choose the function ζ in a special way. For $n=3$ take

$$\zeta(z) = \psi\left(\varepsilon \ln \frac{\rho(|z'|)}{\Delta(z)}\right) \tag{7}$$

where $\rho, \psi \in \mathscr{C}^\infty(R^1)$, $\psi(t) = 0$ for $t < 0$, $\psi(t) = 1$ for $t > 1$, $\rho(t) = t$ for $t > d > 0$, $\rho(t) = \rho_0 > 0$ for $t < (d/2)$, $\rho(t) \geqq t$, $\rho'(t) \geqq 0$, ρ_0, d, ε are positive constants, and $\Delta(z)$ is a regularized distance from z to Γ (see [5], Ch. VI). In the case $n=2$, take $\tilde{\zeta} = \zeta$ for $z_1 > 0$, and $\tilde{\zeta} = -\zeta$ for $z_1 < 0$. It is easy to see that $\zeta(z) = 0$ for $|z'| \leqq \rho_1$, $\rho_1 > 0$, provided ρ_0 is sufficiently small.

LEMMA 1. *For the vector $\vec{a}$ defined by* (5) *or* (6) *the inequalities*

$$|\vec{a}(z)| \leqq \frac{C_0}{g^{n-1}(z_n)}, \quad \left|\frac{\partial \vec{a}(z)}{\partial z_k}\right| \geqq \frac{C_1}{g^n(z_n)} \tag{8}$$

hold.

Proof. To be definite consider the three-dimensional case. The support of $\vec{a}$ is contained in the domain $\Delta(z) \leqq \rho(|z'|) \leqq e^{1/\varepsilon}\Delta(z)$. As the function g satisfies the Lipshitz condition (2), the regularized distance Δ is a quantity of the same order as the distance from z to $\partial\sigma(z_3)$, i.e., $C_2\Delta(z) \leqq g(z_3) - |z'| \leqq C_3\Delta(z)$, $C_2, C_3 > 0$. Thus for $z \in \operatorname{supp} \vec{a}$ we have $e^{1/\varepsilon}\Delta(z) \geqq \rho(|z'|) \geqq (C_3\Delta(z) + |z'|)(C_3+1)^{-1} \geqq (C_3+1)^{-1}g(z_3)$. In particular, $|z'| = \rho(|z'|) \geqq (C_3+1)^{-1}g(z_3)$ for $|z'| \geqq d$. For $|z'| \leqq d$, $z \in \operatorname{supp} \vec{a}$ the inequalities $g(z_3) \leqq (g(z_3) - |z'|) + |z'| \leqq C_3\Delta(z) + d \leqq C_3\rho(|z'|) + d \leqq C_3\rho(d) + d$ hold and consequently $|z'| \geqq \rho_1 \geqq \rho_1 g(z_3)(C_3\rho(d) + d)^{-1}$. So for all $z \in \operatorname{supp} \vec{a}$ we have $e^{1/\varepsilon}\Delta(z) \geqq \rho(|z'|) \geqq (C_3+1)^{-1}g(z_3)$, $|z'| \geqq C_4 g(z_3)$. Differentiating ζ and taking into account the fact that $|\mathscr{D}^\alpha\Delta(z)| \leqq C_\alpha \Delta^{-|\alpha|+1}(z)$, see [5], we obtain $|\mathscr{D}^\alpha\zeta(z)| \leqq C'_\alpha g^{-|\alpha|}(z_3)$. The same inequality holds for the function $\tilde{\zeta}$ in the case $n=2$. The estimates (8) follow from these inequalities. The lemma is proved.

Let Ω satisfy the conditions (I)-(5). Consider the operator which assigns the function $q = \nabla \cdot \vec{u}$ to every vector $\vec{u} \in \mathring{\mathscr{D}}(\Omega)$. Denote by $\mathscr{M}(\Omega)$ the range of this operator and define in $\mathscr{M}(\Omega)$ the norm

$$||q||_{\mathscr{M}(\Omega)} = \inf_{\substack{\vec{v} \in \mathring{\mathscr{D}}(\Omega) \\ \nabla \cdot \vec{v} = q}} ||\vec{v}||_{\mathscr{D}(\Omega)} = ||P\vec{u}||_{\mathscr{D}(\Omega)} ;$$

here P is a projection on the space $\mathring{\mathscr{D}}(\Omega) \ominus \hat{H}(\Omega)$. Clearly, $\mathscr{M}(\Omega) \subset L_2(\Omega)$. Let $\mathscr{M}^*(\Omega)$ be the dual space to $\mathscr{M}(\Omega)$ with respect to the bilinear form $(p, q) = \int_\Omega pq\,dx$, so that

$$||p||_{\mathscr{M}^*(\Omega)} = \inf_{q \in \mathscr{M}(\Omega)} \frac{\left|\int_\Omega pq dx\right|}{||q||_{\mathscr{M}(\Omega)}} .$$

We investigate below the behavior of $p(x) \in \mathscr{M}^*(\Omega)$ for $|x| \to \infty$ and show that in some sense $p(x) \to 0$ when $|x| \to \infty$, $x \in \omega_i$, $i = 1, \cdots, r$.

Let ω be one of the ω_i, $i = 1, \cdots, m$, $\gamma = \partial\omega \backslash \Sigma(0)$ (γ is the "lateral surface" of ω), and $\mathscr{C}_\gamma^\infty(\Omega)$—the set of all infinitely differentiable functions vanishing near γ and for $|z| \geqq 0$. Define $\overset{\circ}{\mathscr{D}}_\gamma(\omega)$ as the closure of $\mathscr{C}_\gamma^\infty(\Omega)$ in the norm $\mathscr{D}(\omega)$ and $\mathscr{M}(\omega)$ as the closure of $\mathscr{C}_\gamma^\infty(\Omega)$ in the norm $|||f|||_\omega$ corresponding to the scalar product

$$(9) \qquad \langle f, h\rangle_\omega = \int_\omega f(z)h(z)dz + \int_0^\infty F(t)H(t)g^{-n-1}(t)dt$$

where $F(t) = \int_{\omega(t)} f(z)dz$ provided $\int_0^\infty g^{-n-1}(t)dt < \infty$ and $F(t) = -\int_{\omega'(t)} f(z)dz$ in the opposite case. The formula (9) has a sense for all $f, h \in \widetilde{\mathscr{M}}(\omega)$, $F(t)$ being the primitive function for $\int_{\Sigma(t)} f dz'$ vanishing at infinity (or, more exactly, having the finite integral $\int_0^\infty F^2(t)g^{-n-1}(t)dt$) in the case $\int_0^\infty g^{-n-1}(t)dt = \infty$.

THEOREM 2. *If $\vec{u} \in \overset{\circ}{\mathscr{D}}_\gamma(\omega)$, then $f = \nabla \cdot \vec{u} \in \widetilde{\mathscr{M}}(\omega)$ and*

$$(10) \qquad |||f|||_\omega \leqq C_1 ||\vec{u}||_{\mathscr{D}(\omega)} .$$

For any function $f \in \widetilde{\mathscr{M}}(\omega)$ there exists a vector $\vec{u} \in \overset{\circ}{\mathscr{D}}_\gamma(\omega)$ such that $f = \nabla \cdot \vec{u}$ and

$$(11) \qquad ||\vec{u}||_{\mathscr{D}(\omega)} \leqq C_2 |||f|||_\omega .$$

The constants C_1 and C_2 do not depend on $\vec{u}$ and f.

Proof. Let $\vec{u} \in \mathscr{C}_\gamma^\infty(\omega)$, $f = \nabla \cdot \vec{u}$. Clearly,

$$(12) \qquad ||f||_{L_2(\omega)} \leqq C_3 ||\vec{u}||_{\mathscr{D}(\omega)} .$$

It follows from the relations

$$-\int_{\omega'(t)} f(z)dz = \int_{\Sigma(t)} u_n dz' ,$$
$$\int_{\omega(t)} f(z)dz = \int_{\Sigma(t)} u_n dz' - \int_{\Sigma(0)} u_n dz' ,$$

that

$$\int_0^\infty g^{-n-1}(t)F^2(t)\,dt \leqq 2\int_0^\infty g^{-n-1}(t)dt\left|\int_{\Sigma(t)} u_n dz'\right|^2 + C_4\left|\int_{\Sigma(0)} u_n dz'\right|^2 \leqq C_5\|\vec{u}\|^2_{\mathscr{D}(\omega)}\,,$$

which with (12) proves the estimate (10).

To prove the second part of the theorem, take an arbitrary function $f \in \mathscr{C}_\gamma^\infty(w)$ and define the vector $\vec{w}(z) = F(z_n)\vec{a}(z)$, where $\vec{a}(z)$ is given by (5) or (6) for $z \in G$, $\vec{a} = 0$ for $z \in \omega\backslash G$ and F is the same as in (9). In virtue of (8)

$$\left|\frac{\partial \vec{w}(z)}{\partial z_k}\right| \leqq C_6\Big(|F(z_n)|g^{-n}(z_n) + \delta_{kn}g^{-n+1}(z_n)\left|\int_{\Sigma(z_n)} f dz'\right|\Big)\,,$$

so that

$$\|\vec{w}\|^2_{\mathscr{D}(\omega)} \leqq C_7\Big(\int_\omega g^{-2n}(z_n)F^2(z_n)dz + \int_\omega \frac{dz}{g^{2(n-1)}(z_n)}\left|\int_{\Sigma(z_n)} f dz'\right|^2\Big) \leqq C_8|||f|||^2_\omega\,.$$

Now consider the function $h = f - \nabla\cdot\vec{w} = f - a_n(z)\int_{\Sigma(z_n)} f dz'$. It is easy to see that $\int_{\Sigma(z_n)} h dz' = 0$ and hence $\int_{B(t)} h dz = 0$ for all $t > 0$ (we recall that $B(t) = \omega(t + g(t))\backslash\omega(t)$). Split ω into layers B_j by planes (straight lines) $z_n = t_j$ where $t_j = t_{j-1} + g(t_{j-1})$, $t_0 = 0$. In virtue of the property 4) of Ω, in every B_j one can represent h in the form $h = \nabla\cdot\vec{v}^{(j)}$ where $\vec{v}^{(j)} \in \mathring{\mathscr{D}}(B_j)$ and $\|\vec{v}^{(j)}\|_{\mathscr{D}(B_j)} \leqq C_9\|h\|_{L_2(B)j}$. Consequently the vector $\vec{v} \in \mathring{\mathscr{D}}(\omega)$ which equals $\vec{v}^{(j)}(z)$ for $z \in B_j$ satisfies the equation $\nabla\cdot\vec{v} = h$ and

$$\|\vec{v}\|^2_{\mathscr{D}(\omega)} = \sum_j \|\vec{v}^{(j)}\|^2_{\mathscr{D}(B_j)} \leqq C_9^2\sum_j \|h\|^2_{L_2(B_j)} = C_9^2\|h\|^2_{L_2(\omega)} \leqq C_{10}\|f\|^2_{L_2(\omega)}\,.$$

Clearly, the vector $\vec{w} + \vec{v} = \vec{u}$ is that which is sought. The theorem is proved.

REMARK 1. For $g(t) = \lambda(t + b)$, $b > 0$, we have

$$\int_0^\infty g^{-n-1}(t)dt\left|\int_{\omega(t)} f dz\right|^2 \leqq C\|f\|^2_{L_2(\omega)}\,,$$

so that $\tilde{\mathscr{M}}(\omega) = L_2(\omega)$.

REMARK 2. If $\int_0^\infty g^{-n-1}(t)dt < \infty$, then $\vec{w} \in \mathring{\mathscr{D}}(\omega)$ and hence $\vec{u} \in \mathring{\mathscr{D}}(\omega)$.

Now define the space $\tilde{\mathscr{M}}(\Omega)$ as the completion of $\mathscr{C}_0^\infty(\Omega)$ in the norm $|||f|||_\Omega$ which corresponds to the scalar product

$$\langle f, h\rangle = \int_\Omega fh\,dx + \sum_{i=1}^m \int_0^\infty g_i^{-n-1}(t)F_i(t)H_i(t)dt\,, \tag{13}$$

where $F_i(t) = \int_{\omega_i(t)} f dx$ for $i = 1, \cdots, r$, and F_i is a primitive function for $\int_{\Sigma_i(t)} f dz'$ vanishing at infinity if $i = r + 1, \cdots, m$.

THEOREM 3. *If $\vec{u} \in \mathring{\mathscr{D}}(\Omega)$, then $f = \nabla \cdot \vec{u} \in \widetilde{\mathscr{M}}(\Omega)$ and*

$$|||f|||_\Omega \leqq C||\vec{u}||_{\mathscr{D}(\Omega)} . \tag{14}$$

For every function $f \in \widetilde{\mathscr{M}}(\Omega)$ one can find a vector $\vec{u} \in \mathring{\mathscr{D}}(\Omega)$ such that $f = \nabla \cdot \vec{u}$ and

$$||\vec{u}||_{\mathscr{D}(\Omega)} \leqq C_1|||f|||_\Omega . \tag{15}$$

Proof. The first statement is a consequence of the corresponding statement of Theorem 2. We now prove the second part of the theorem. If $f \in \widetilde{\mathscr{M}}(\Omega)$, then $f|_{\omega_i} \in \widetilde{\mathscr{M}}(\omega_i)$ and by Theorem 2 there exist vectors $\vec{u}^{(i)} \in \mathring{\mathscr{D}}_{\Gamma_i}(\omega_i)$ in domains ω_i, $i = r + 1, \cdots, m$, such that $f = \nabla \cdot \vec{u}^{(i)}$ and $||\vec{u}^{(i)}||_{\mathscr{D}(\omega_i)} \leqq C_2|||f|||_{\omega_i}$. Let $\mu \in C^1(\Omega)$ be a function which is equal to 1 in Ω_0 and to zero in $\Omega \backslash \Omega_{t_0}$ (Ω_{t_0} is just the same as in condition (5), §I) and $0 \leqq \mu \leqq 1$. The vectors $\vec{v}^{(i)} = \vec{u}^{(i)}(1 - \mu)$ belong to $\mathring{\mathscr{D}}(\omega_i)$, satisfy the equation $\nabla \cdot \vec{v}^{(i)} = f(1 - \mu) - \vec{u}^{(i)} \cdot \nabla \mu$ and the inequality $||\vec{v}^{(i)}||_{\mathscr{D}(\omega_i)} \leqq C_3||\vec{u}^{(i)}||_{\mathscr{D}(\omega_i)} \leqq C_2C_3|||f|||_{\omega_i}$. Further, let $h \in L_2(\Omega_{t_0})$ be a function which is equal to zero in Ω_0, to $\vec{u}^{(i)} \cdot \nabla \mu$ in ω_i, $i = r + 1, \cdots, m$ and to $h_0\mu(1 - \mu)$ in ω_i, $i = 1, \cdots, r$, the constant h_0 being chosen in such a way that $\int_{\Omega_{t_0}} h(x)dx = -\int_{\Omega_{t_0}} f\mu dx$ (since $r > 1$, h_0 is determined uniquelly).

It is clear that

$$||h||^2_{L_2(\Omega_{t_0})} \leqq C_4\Big(||f||^2_{L_2(\Omega_{t_0})} + \sum_{i=r+1}^{m} ||\vec{u}^{(i)}||^2_{\mathscr{D}(\omega_i)}\Big) \leqq C_5|||f|||^2_\Omega .$$

By the condition (5) §1, there exists a vector $\vec{w} \in \mathring{\mathscr{D}}(\Omega_{t_0})$ such that $\nabla \cdot \vec{w} = f\mu + h$ and $||\vec{w}||_{\mathscr{D}(\Omega_{t_0})} \leqq C_6(||f||_{L_2(\Omega_{t_0})} + ||h||_{L_2(\Omega_{t_0})}) \leqq C_7|||f|||_\Omega$. Setting $\vec{w} = 0$ in $\Omega \backslash \Omega_{t_0}$, we obtain an element of $\mathring{\mathscr{D}}(\Omega)$.

Finally we find in ω_i, $i = 1, \cdots, r$, vectors $\vec{v}^{(i)} \in \mathring{\mathscr{D}}(\omega_i)$ such that for $x \in \omega_i$, $\nabla \cdot \vec{v}^{(i)} = f(1 - \mu) - h$ and $||\vec{v}^{(i)}||_{\mathscr{D}(\omega_i)} \leqq C_8|||f(1 - \mu) - h|||_{\omega_i} \leqq C_9|||f|||_\Omega$. Their existence is a consequence of Theorem 2 and Remark 2. The vector $\vec{u} = \vec{w} + \vec{v} \in \mathring{\mathscr{D}}(\Omega)$, where $\vec{v} = \vec{v}^{(i)}$ for $x \in \omega_i$ and $\vec{v} = 0$ for $x \in \Omega_0$, satisfies the equation $\nabla \cdot \vec{u} = f$ and the inequality (15). The theorem is proved.

COROLLARY. *$\mathscr{M}(\Omega) = \widetilde{\mathscr{M}}(\Omega)$ and the norms $||f||_{\mathscr{M}(\Omega)}$ and $|||f|||_\Omega$ are equivalent.*

THEOREM 4. *Any function $p(x) \in \mathscr{M}^*(\Omega)$ can be represented in* *the form*

$$(16)\quad p(x) = f(x) + \sum_{i=1}^{r} \chi_i(x)\int_{z_n^{(i)}(x)}^{\infty} F_i(t)\frac{dt}{g_i^{n+1}(t)} + \sum_{i=r+1}^{m} \chi_i(x)\int_0^{z_n^{(i)}(x)} F_i(t)\frac{dt}{g_i^{n+1}(t)}$$

where $f \in \mathscr{M}(\Omega) = \tilde{\mathscr{M}}(\Omega)$ *and* χ_i *is the characteristic function of* ω_i. *The inequality* $c_1 |||f|||_\Omega \leqq ||p||_{\mathscr{M}^*(\Omega)} \leqq C_2 |||f|||_\Omega$ *holds with constants* C_1, C_2 *independent on* p.

Proof. By the Riesz theorem, any linear functional of $h \in \mathscr{M}(\Omega)$ can be represented in the form (13) with $f \in \mathscr{M}(\Omega)$. If $h \in \mathscr{C}_0^\infty(\Omega)$, then, changing the orders of integration in the right-hand side of (13), we obtain the formula $\langle f, h\rangle_\Omega = \int_\Omega phdx$ where p is the function (16). Hence follows the statement of the theorem.

COROLLARY. *Any function* $p(x) \in \mathscr{M}^*(\Omega)$ *tends to zero as* $|x| \to \infty$, $x \in \omega_i$, $i = 1, \cdots, r$.

Indeed, for $x \in \omega_i$, $i \leqq r$,

$$p(x) = f(x) + \int_{z_n^{(i)}(x)}^{\infty} F_i(t)\frac{dt}{g_i^{n+1}(t)}$$

where $f(x) \in L_2(\omega_i)$ and

$$\left|\int_{z_n^{(i)}}^{\infty} F_i(t)\frac{dt}{g_i^{n+1}(t)}\right|^2 \leqq \int_{z_n^{(i)}}^{\infty} F_i^2\frac{dt}{g_i^{n+1}} \int_{z_n^{(i)}}^{\infty} \frac{dt}{g_i^{n+1}} \xrightarrow[z_n^{(i)} \to \infty]{} 0 .$$

THEOREM 5. *Any linear functional* $l(\vec{\varphi})$ *of* $\vec{\varphi} \in \mathring{\mathscr{D}}(\Omega)$ *vanishing for* $\vec{\varphi} \in \hat{H}(\Omega)$ *can be represented in a unique way in the form*

$$l(\vec{\varphi}) = \int_\Omega p\nabla\cdot\vec{\varphi}dx ,$$

where $p \in \mathscr{M}^*(\Omega)$, *and the norm of the functional is equivalent to* $||p||_{\mathscr{M}^*(\Omega)}$.

Proof. By the Riesz theorem, there exists a vector $\vec{w} \in \mathring{\mathscr{D}}(\Omega) \ominus \hat{H}(\Omega)$ such that $l(\vec{\varphi}) = [\vec{w}, \vec{\varphi}] = [\vec{w}, P\vec{\varphi}]$. The right-hand side is a linear bounded functional of $h = \nabla\cdot\vec{\varphi} \in \mathscr{M}(\Omega)$ and from this fact follows the statement of theorem.

An analoguous theory can be developed for weighted spaces. We formulate here the corresponding definitions and results.

Let $\mathring{\mathscr{D}}_\alpha(\Omega)$ and $H_\alpha(\Omega)$ be completions of the sets of vectors $C_0^\infty(\Omega)$ and $\mathscr{J}_0^\infty(\Omega)$ in the norm $||\vec{u}||_{\mathscr{D}_\alpha(\Omega)}$ which is generated by the scalar product

$$[\vec{u}, \vec{v}]_\alpha = \int_{\Omega_0} \vec{u}_x \cdot \vec{v}_x dx + \sum_{j=1}^{m} \int_{\omega_j} \vec{u}_x \cdot \vec{v}_x [1 + g_j^2(z_n^{(j)}(x))]^\alpha dx ,$$

$\hat{H}_\alpha(\Omega)$ is the subspace of divergence-free vectors from $\overset{\circ}{\mathscr{D}}_\alpha(\Omega)$, $\mathscr{M}_\alpha(\Omega)$ is the set of functions $q = \nabla \cdot \vec{u}$, $\vec{u} \in \overset{\circ}{\mathscr{D}}_\alpha(\Omega)$ with the norm $\|q\|_{\mathscr{M}_\alpha(\Omega)} = \inf_{\nabla\cdot\vec{v}=q} \|\vec{v}\|_{\mathscr{D}_\alpha(\Omega)}$, $\mathscr{M}_\alpha^*(\Omega)$ is the space dual to $\mathscr{M}_\alpha(\Omega)$ with respect to the bilinear form $\int_\Omega pq dx$. The following propositions are valid.

(a) If the domain Ω satisfies (1), (2′), (3)–(5), then $\dim \hat{H}_\alpha(\Omega)/H(\Omega) = r-1$ and there exists a basis $\{\vec{a}, (x), \cdots, \vec{a}_{r-1}(x)\}$ in $\hat{H}_\alpha/H_\alpha$, the vectors $\vec{a}_s$ being linearly independent and satisfying the inequalities (4) for $|x| \gg 1$.

(b) The space $\mathscr{M}_\alpha(\Omega)$ consists of functions which can be approximated by functions from $\mathscr{C}_0^\infty(\Omega)$ in the norm $|||f|||_{\alpha,\Omega}$:

$$\begin{aligned} |||f|||_{\alpha,\Omega}^2 = &\int_{\Omega_0} |f|^2 dx + \sum_{j=1}^{m} \int_{\omega_j} |f|^2 [1 + g^2(z_n^{(j)}(x))]^\alpha dx \\ &+ \sum_{j=1}^{r} \int_0^\infty g^{-n-1+2\alpha}(t) dt \left| \int_{\omega_j(t)} f dx \right|^2 \\ &+ \sum_{j=r+1}^{m} \int_0^\infty g^{-n-1+2\alpha}(t) dt \left| \int_{\omega_j'(t)} f dx \right|^2 , \end{aligned}$$

and this norm is equivalent to the norm $\|f\|_{\mathscr{M}_\alpha(\Omega)}$.

(c) Any function from $\mathscr{M}_\alpha^*(\Omega)$ can be represented in the form

$$p(x) = f(x) + \sum_{j=1}^{r} \chi_i(x) \int_{z_n^{(j)}}^\infty F_j(t) \frac{dt}{g_j^{n+1-2\alpha}(t)} + \sum_{j=r+1}^{m} z \int_0^{z_n^{(j)}(x)} F_j(t) \frac{dt}{g_j^{n+1-2\alpha}(t)} , \tag{18}$$

where f and F_j are functions with finite norms

$$\left(\int_{\Omega_0} |f|^2 dx + \sum_{j=1}^{m} \int_{\omega_j} |f|^2 \frac{dx}{[1 + g_j^2(z_n^{(j)}(x))]^\alpha} \right)^{1/2} , \quad \left(\int_0^\infty \frac{F_j^2(t)}{g_j^{n+1-2\alpha}} dt \right)^{1/2} .$$

(d) Any linear functional of $\vec{\varphi} \in \overset{\circ}{\mathscr{D}}_\alpha(\Omega)$ vanishing for $\vec{\varphi} \in \hat{H}_\alpha(\Omega)$ can be represented in a unique way in the form (17) with $p \in \mathscr{M}_\alpha^*(\Omega)$.

All these propositions can be proved in the same way as were Theorems 1–5.

Let $n = 3$ and let Ω satisfy the conditions (1), (2′), (3)–(5) with $\alpha = 1$. Define the space $N(\Omega)$ as the range of the operator $\nabla \cdot \vec{u}$, $\vec{u} \in \overset{\circ}{W}{}_2^1(\Omega)$, and set $\|q\|_{N(\Omega)} = \inf_{\nabla\cdot\vec{v}=q} \|\vec{v}\|_{W_2^1(\Omega)}$.
Denote by $N^*(\Omega)$ its dual space with the norm

$$\|p\|_{N^*(\Omega)} = \inf_{q \in N(\Omega)} \frac{\left| \int_\Omega pq dx \right|}{\|q\|_{N(\Omega)}} .$$

THEOREM 6. *$\overset{\circ}{\mathscr{D}}_1(\Omega) \subset \overset{\circ}{W}{}_2^1(\Omega)$, $N(\Omega) \supset \mathscr{M}_1(\Omega)$, and $N^*(\Omega) \subset \mathscr{M}_1{}^*(\Omega)$.*

Proof. Let $\vec{u} \in \overset{\circ}{\mathscr{D}}_1(\Omega)$. Since $G_j \subset \omega_j \subset G_j^a$, we have $\|\vec{u}\|^2_{L_2(\Sigma_j(t))} \leqq cg_j^2(t_i)\|\vec{u}\|^2_{\mathscr{D}(\Sigma_j(t))}$, $\|\vec{u}_x\|^2_{L_2(\omega_j)} \leqq C_1\int_{\omega_j} |\vec{u}_x|^2 g_j^2(z_n^{(j)}(x))dx$ and consequently $\|\vec{u}\|^2_{W_2^1(\Omega)} \leqq C_2\|\vec{u}\|^2_{\mathscr{D}_1(\Omega)}$, i.e., $\overset{\circ}{\mathscr{D}}_1(\Omega) \subset \overset{\circ}{W}{}_2^1(\Omega)$. Thus, $\mathscr{M}_1(\Omega) \subset N(\Omega)$ and $\mathscr{M}_1{}^*(\Omega) \supset N^*(\Omega)$.

3. Stationary problems. Consider in a domain Ω satisfying conditions (1)-(5) the boundary value problem

$$(19) \qquad -\nabla^2\vec{v} + \nabla p = \vec{f}\,, \quad \nabla\cdot\vec{v} = 0\,, \quad \vec{v}|_{\partial\Omega} = 0\,, \quad \vec{v}|_{|x|=\infty} = 0$$

with additional conditions

$$(20) \qquad p_i - p_r = \beta_i\,, \quad i = 1, \cdots, r-1\,,$$

where $p_i = \lim_{\substack{|x|\to\infty \\ x\in\omega_i}} p(x)$. The constant p_r can be considered as an arbitrary constant in the definition of the function $p(x)$.

Now we give a generalized formulation of the problem (19), (20). If $\vec{v}$, p is its classical solution, then for any $\vec{\varphi} \in \hat{H}(\Omega)$ we have

$$(21) \quad \int_{\Omega_t} \vec{f}\cdot\vec{\varphi}dx = \int_{\Omega_t} \vec{v}_x\cdot\vec{\varphi}_x dx + \sum_{j=1}^{m}\left(\int_{\Sigma_j(t)} p\vec{\varphi}\cdot\vec{n}dS - \int_{\Sigma_j(t)} \frac{\partial\vec{v}}{\partial n}\cdot\vec{\varphi}dS\right),$$

where $\vec{n}$ is the unit normal vector to $\Sigma_j(t)$, directed exterior to Ω_t. Suppose that for $x \in \omega_j$, $|x| \gg 1$, we have $p(x) = q(x) + p_j$ where $q \in \mathscr{M}^*(\Omega)$. Then passing to the limit in (21) as $t \to \infty$ (at least along a certain sequence), we obtain

$$\int_\Omega \vec{v}_x\cdot\vec{\varphi}_x dx + \sum_{j=1}^{r} p_j \int_{\Sigma_j} \vec{\varphi}\cdot\vec{n}dS = \int_\Omega \vec{f}\cdot\vec{\varphi}dx\,.$$

Since $\sum_{j=1}^{r}\int_{\Sigma_j} \vec{\varphi}\cdot\vec{n}dS = 0$ (it follows from Theorem 3 of [3] that $\int_{\Sigma_j} \vec{\varphi}\cdot\vec{n}dS = 0$ for $j = r+1, \cdots, m$, $\vec{\varphi} \in \hat{H}(\Omega)$), the relation

$$\sum_{j=1}^{r} p_j \int_{\Sigma_j} \vec{\varphi}\cdot\vec{n}dS = \sum_{j=1}^{r-1} (p_j - p_r) \int_{\Sigma_j} \vec{\varphi}\cdot\vec{n}dS$$

holds. These arguments give us the motivation for the following definition.

A weak solution of the problem (19), (20) is a vector $\vec{v} \in \hat{H}(\Omega)$ which satisfies for all $\vec{\varphi} \in \hat{H}(\Omega)$ the integral identity

$$(22) \qquad \int_\Omega \vec{v}_x\cdot\vec{\varphi}_x dx + \sum_{j=1}^{r-1} \beta_j \int_{\Sigma_j} \vec{\varphi}\cdot\vec{n}dS - \int_\Omega \vec{f}\cdot\vec{\varphi}dx = 0\,.$$

THEOREM 7. *Let* $\int_\Omega \vec{f}\cdot\vec{\varphi}_x dx$ *be a linear functional of* $\vec{\varphi}\in\mathring{\mathscr{D}}(\Omega)$, i.e., *for all* $\vec{\varphi}\in\mathring{\mathscr{D}}(\Omega)$, $\left|\int_\Omega \vec{f}\cdot\vec{\varphi}dx\right| \leqq C_f\|\vec{\varphi}\|_{\mathscr{D}(\Omega)}$. *Then the problem* (19), (20) *has a unique weak solution. Moreover, there evists a unique, modulo a constant summand, function* $p(x)\in L_{2,\mathrm{loc}}(\Omega)$ *satisfying for all* $\vec{\varphi}\in\mathring{\mathscr{D}}(\Omega')(\bar{\Omega}'\subset\Omega,\ \bar{\Omega}'$ *compact) the relation*

$$\int_{\Omega'}\vec{v}_x\cdot\vec{\varphi}_x dx = \int_{\Omega'}\vec{f}\cdot\vec{\varphi}dx + \int_{\Omega'}p\nabla\cdot\vec{\varphi}dx\ . \tag{23}$$

Proof. The first statement follows from the Riesz theorem on the general form of a functional in a Hilbert space (see [6], Ch. I, §1). To prove the second statement note that for any $\vec{\varphi}\in\hat{H}(\Omega_1) = H(\Omega_1)$ (Ω_1 is a bounded subdomain of Ω with a Lipshitzean boundary) the identity (22) takes the form $\int_{\Omega_1}\vec{v}_x\cdot\vec{\varphi}_x dx = \int_{\Omega_1}\vec{f}\cdot\vec{\varphi}dx$. As is shown in [2], for $\vec{\varphi}\in\mathring{\mathscr{D}}(\Omega_1)$, we then have

$$\int_{\Omega_1}\vec{v}_x\cdot\vec{\varphi}_x dx - \int_{\Omega_1}\vec{f}\cdot\vec{\varphi}dx = \int_{\Omega_1}p_1\nabla\cdot\vec{\varphi}dx\ ,\quad \text{for some } p_1\in L_2(\Omega_1)\ ,$$

and the functions p_1 and p_2 corresponding to two intersecting domains Ω_1 and Ω_2 differ from each other by a constant. Therefore it is possible to define in Ω a function $p\in L_{2,\mathrm{loc}}(\Omega)$ satisfying (23). □

Now let us show that as $|x|\to\infty$, $x\in\omega_i$, $i\leqq r$, the function $p(x)$ tends to a constant and that (20) is satisfied. The expression

$$l(\vec{\varphi}) = \int_\Omega \vec{v}_x\cdot\vec{\varphi}_x dx + \sum_{j=1}^{r-1}\beta_j\int_{\Sigma_j}\vec{\varphi}\cdot\vec{n}dS - \int_\Omega \vec{f}\cdot\vec{\varphi}dx$$

is a linear functional of $\vec{\varphi}\in\mathring{\mathscr{D}}(\Omega)$ vanishing for $\vec{\varphi}\in\hat{H}(\Omega)$, so by virtue of Theorem 6

$$\int_\Omega \vec{v}_x\cdot\vec{\varphi}_x dx + \sum_{j=1}^{r-1}\beta_j\int_{\Sigma_j}\vec{\varphi}\cdot\vec{n}dS - \int_\Omega \vec{f}\cdot\vec{\varphi}dx = \int_\Omega q\nabla\cdot\vec{\varphi}dx\ , \tag{24}$$

where $q\in\mathscr{M}^*(\Omega)$ and $\vec{\varphi}$ is an arbitrary element of $\mathring{\mathscr{D}}(\Omega)$. The sections Σ_j of ω_j in (22) may be chosen arbitrarily but in (24) they should be fixed; the function q depends on Σ_j. Let $\Sigma_j = \Sigma_j(0)$ and take in (24) $\vec{\varphi}\in\mathring{\mathscr{D}}(\Omega')$ where $\Omega'\subset\omega_j$, $j<r$, $\Omega'\cap\Sigma_j = \varnothing$. Then in virtue of (23) we have

$$\int_{\Omega'}(\vec{v}_x\cdot\vec{\varphi}_x - \vec{f}\cdot\vec{\varphi})dx = \int_{\Omega'}p\nabla\cdot\vec{\varphi}dx = \int_{\Omega'}q\nabla\cdot\vec{\varphi}dx \tag{25}$$

and consequently in ω_j, $p = q + p_j$, $p_j = \text{const}$. Analogous arguments show that in $\Omega_0\cup\omega_r$, $p = q + p_r$.

Now let $\Omega' \subset \Omega$ be a bounded domain which is divided by the surface Σ_j into two subdomains, Ω_1 and $\Omega_2 \subset \omega_j$. In this case we have, instead of (25),

$$\int_{\Omega'} p\nabla\cdot\vec{\varphi}dx + \beta_j\int_{\Sigma_j}\vec{\varphi}\cdot\vec{n}dS = \int_{\Omega'} q\nabla\cdot\vec{\varphi}dx = \int_{\Omega_1}(p-p_r)\nabla\cdot\vec{\varphi}dx + \int_{\Omega_2}(p-p_j)\nabla\cdot\vec{\varphi}dx = \int_{\Omega'} p\nabla\cdot\vec{\varphi}dx + (p_j - p_r)\int_{\Sigma_j}\vec{\varphi}\cdot\vec{n}ds\,.$$

Consequently, $\beta_j = p_j - p_r$ and we have justified the above definition of weak solution of the problem (19), (20).

Consider now the nonlinear problem

$$(26)\qquad \begin{aligned} &-\nabla^2\vec{v} + (\vec{v}\cdot\nabla)\vec{v} + \nabla p = \vec{f}\,, \quad \nabla\cdot\vec{v} = 0\,, \\ &\vec{v}|_{\partial\Omega} = 0,\ \vec{v}|_{|x|=\infty} = 0\,, \quad p_j - p_r = \beta_j\,, \quad j = 1, \cdots, r-1\,, \end{aligned}$$

in a domain $\Omega \subset R^3$ satisfying the conditions (1)–(5). Let $\hat{\mathscr{H}}(\Omega)$ be the linear set of vector fields $\vec{\varphi} = \sum_{j=1}^{r-1}\lambda_j\vec{a}_j + \vec{\eta}(x)$ where $\lambda_j \in R^1$, $\vec{\eta} \in \mathscr{J}_0^\infty(\Omega)$ and the $\vec{a}_j(x)$ are vectors forming a basis in $\hat{H}(\Omega)/H(\Omega)$ and satisfying (4). This set is dense in $\hat{H}(\Omega)$.

Denote by $\mathscr{C}_R^\infty(\Omega_R)$ the set of infinitely differentiable vectors defined in the domain Ω_R and vanishing near the surface $S_R = \partial\Omega_R \backslash \bigcup_{i=1}^r \sum_i(R)$, by $\overset{\circ}{\mathscr{D}}_R(\Omega_R)$ the completion of $\mathscr{C}_R^\infty(\Omega_R)$ in the norm of $\mathscr{D}(\Omega_R)$, and by $H'(\Omega_R)$ the set of all divergence-free vectors belonging to $\overset{\circ}{\mathscr{D}}_R(\Omega)$.

Define a weak solution of (26) to be a vector $\vec{v} \in \hat{H}(\Omega)$ satisfying for all $\vec{\varphi} \in \hat{\mathscr{H}}(\Omega)$ the integral identity

$$(27)\qquad \int_\Omega \vec{v}_x\cdot\vec{\varphi}_x dx - \int_\Omega \vec{v}\cdot(\vec{v}\cdot\nabla)\vec{\varphi}dx = \int_\Omega \vec{f}\cdot\vec{\varphi}dx - \sum_{j=1}^{r-1}\beta_j\int_{\Sigma_j}\vec{\varphi}\cdot\vec{n}dS$$

(the convergence of the integral $\int_\Omega \vec{v}\cdot(\vec{v}\cdot\nabla)\vec{\varphi}dx$ with $\vec{v} \in \hat{H}(\Omega)$, $\vec{\varphi} \in \hat{\mathscr{H}}(\Omega)$ follows from the estimates (4)).

THEOREM 8. *Suppose that the domain $\Omega \subset R^3$ satisfies the conditions* (1)–(5), $g_i(t)_{t\to\infty} \to \infty$ *for* $i = 1, \cdots, r$, f *satisfies the conditions of Theorem* 7, *and that for all* $\vec{\varphi} \in H'(\Omega_R)$,

$$\left|\int_{\Omega_R}\vec{f}\cdot\vec{\varphi}dx\right| \leqq C'_f\|\vec{\varphi}\|_{\mathscr{D}(\Omega_R)}$$

(C'_f *does not depend on* R *or* $\vec{\varphi}$). *Then problem* (26) *has at least one weak solution.*

Proof. Consider in Ω_R an auxiliary problem of finding a vector $\vec{v}^R \in H'(\Omega_R)$ which satisfies the integral identity

$$\begin{aligned}\int_{\Omega_R} \vec{v}_x^R \cdot \vec{\varphi}_x dx - \int_{\Omega_R} \vec{v}^R \cdot (\vec{v}^R \cdot \nabla)\vec{\varphi} dx + \frac{1}{2}\sum_{j=1}^{r} \int_{\Sigma_j(R)} (\vec{v}^R \cdot \vec{n})(\vec{v}^R \cdot \vec{\varphi}) dS \\ = \int_{\Omega_R} \vec{f} \cdot \vec{\varphi} dx - \sum_{j=1}^{r-1} \beta_j \int_{\Sigma_j} \vec{\varphi} \cdot \vec{n} dS\end{aligned} \tag{28}$$

for all $\vec{\varphi} \in H'(\Omega_R)$ (we suppose that $\Sigma_j \subset \Omega_R$).

Taking $\vec{\varphi} = \vec{v}^R$ in (28) it is easy to show that for any solution of this problem the estimate

$$\|\vec{v}^R\|_{\mathscr{D}(\Omega_R)} \leqq C'_f + C \sum_{j=1}^{r-1} |\beta_j| \tag{29}$$

holds. Therefore the existence of a solution may be derived from the Leray-Schauder theorem in the same way as in [6], Ch. V, §1.

Moreover, it follows from (29) that there exists a sequence $R_k \to \infty$ such that: (1) the sequence $\vec{V}_i^{R_k} = \partial \vec{v}^{R_k}/\partial x_i$ for $x \in \Omega_{R_k}$, $\vec{V}_i^{R_k} = 0$ for $x \in \Omega \backslash \Omega_{R_k}$ converges weakly in $L_2(\Omega_I$ to $\partial \vec{v}/\partial x_i$, $\vec{v} \in \overset{\circ}{\mathscr{D}}(\Omega)$, 2) the sequence $\vec{v}^{R_k}$ converges in $L_4(\Omega)$ to $\vec{v}$ for any fixed M. Now let $R_k \to \infty$ and pass to the limit in (28). Clearly, for $\vec{\varphi} \in \mathscr{J}_0^\infty(\Omega)$, this passage leads us to (27). The same is true for $\vec{\varphi} = \vec{a}_j$, since

$$\begin{aligned}\left| \int_{\Sigma_i(R_k)} (\vec{v}^{R_k} \cdot \vec{n})(\vec{v}^{R_k} \cdot \vec{a}_j) dS \right| &\leqq C_1 g_i^{-2}(R_\kappa) \int_{\Sigma_i(R_k)} |\vec{v}^{R_k}|^2 dS \\ &\leqq C_2 \|\vec{v}^{R_k}\|^2_{\mathscr{D}(\Omega_{R_k})} g^{-1}(R_k) \longrightarrow 0\end{aligned}$$

and, for $R_k > M$,

$$\begin{aligned}&\left| \int_{\Omega_{R_k}} \vec{v}^{R_k} \cdot (\vec{v}^{R_k} \cdot \nabla) \vec{a}_j dx - \int_\Omega \vec{v} \cdot (\vec{v} \cdot \nabla) \vec{a}_j dx \right| \\ &\leqq \left| \int_{\Omega_M} [\vec{v}^{R_k} \cdot (\vec{v}^{R_k} \cdot \nabla) - \vec{v} \cdot (\vec{v} \cdot \nabla)] \vec{a}_j \cdot dx \right| \\ &\quad + C_3 \left(\sum_{j=1}^{r} \int_{\omega_j(R_k) \backslash \omega_j(M)} |\vec{v}^{R_k}|^2 g_j^{-3}(z_3^{(j)}(x)) dx + \sum_{j=1}^{m} \int_{\omega j \backslash \omega_j(M)} |\vec{v}|^2 g_j^{-3}(z_3^{(j)} dx \right) .\end{aligned} \tag{30}$$

The second term in the right-hand side does not exceed

$$C_4 \left(\sum_{j=1}^{r} \sup_{t > M} g_j^{-1}(t) \|\vec{v}^{R_k}\|^2_{\mathscr{D}(\Omega R_k)} + g_0^{-1} \|\vec{v}\|^2_{\mathscr{D}(\Omega \backslash \Omega_M)} \right) ;$$

consequently, it can be made less than any fixed $\varepsilon > 0$ by an appropriate choice of the number $M \gg 1$. After that we can make the first term less than ε by taking R_k large enough. This shows that

$$\int_{\Omega_{R_k}} \vec{v}^{R_k} \cdot (\vec{v}^{R_k} \cdot \nabla) \vec{a}_j dx \xrightarrow[R_k \to \infty]{} \int_\Omega \vec{v} \cdot (\vec{v} \cdot \nabla) \vec{a}_j dx .$$

Hence, $\vec{v}(x)$ satisfies (27) for any $\vec{\varphi} = \vec{\eta} + \sum_j \lambda_j \vec{a}_j \in \hat{\mathscr{H}}(\Omega)$.

The justification of the above definition of a weak solution can not be carried out in the same way as for the linear problem, since the functional

$$(31)\qquad l((\vec{\varphi}) = \int_\Omega \vec{v}_x\cdot\vec{\varphi}_x dx + \sum_{j=1}^{r-1}\beta_j\int_{\Sigma_j}\vec{\varphi}\cdot\vec{n}dS - \int_\Omega \vec{v}\cdot(\vec{v}\cdot\nabla)\vec{\varphi}dx - \int_\Omega \vec{f}\cdot\vec{\varphi}dx\,,$$

with $\vec{v}\in\hat{H}(\Omega)$, may not be defined for all $\vec{\varphi}\in\overset{\circ}{\mathscr{D}}(\Omega)$ (clearly, it is continuous if $\vec{v}\in\hat{H}(\Omega)\cap L_4(\Omega)$). We carry out the justification with some additional restrictions on Ω.

THEOREM 9. *Let $\int_0^\infty g_i^{-3}(t)\,dt<\infty$ for $i=1,\cdots,r$. Then there exists a unique function $q\in\mathscr{M}^*_{1/2}(\Omega)$ such that $l(\vec{\varphi})=\int_\Omega q\nabla\cdot\vec{\varphi}dx$ for all $\vec{\varphi}\in\overset{\circ}{\mathscr{D}}_{1/2}(\Omega)$.*

Proof. As $\hat{\mathscr{H}}(\Omega)$ is dense in $\hat{H}_{1/2}(\Omega)$ under the conditions of the theorem, it suffices to prove that $l(\vec{\varphi})$ is a continuous functional in $\overset{\circ}{\mathscr{D}}_{1/2}(\Omega)$. This fact is evident for all terms on the right-hand side of (31) except perhaps the integral

$$\mathscr{T}[\vec{\varphi}] = \int_\Omega \vec{v}\cdot(\vec{v}\cdot\nabla)\vec{\varphi}dx = \int_{\Omega 0}\vec{v}\cdot(\vec{v}\cdot\nabla)\vec{\varphi}dx + \sum_{j=1}^{m}\int_{\omega j}\vec{v}\cdot(\vec{v}\cdot\nabla)\vec{\varphi}dx\,.$$

We have

$$\left|\int_{\Omega_0}\vec{v}\cdot(\vec{v}\cdot\nabla)\vec{\varphi}dx\right| \leqq C_1\|\vec{\varphi}\|_{\mathscr{D}(\Omega_0)}\|\vec{v}\|^2_{L_4(\Omega_0)}\,,$$

$$\left|\int_{\omega j}\vec{v}\cdot(\vec{v}\cdot\nabla)\vec{\varphi}dx\right| \leqq C_2(\vec{v})\Big(\int_{\omega j}|\vec{\varphi}_x|^2 g_j(z_3^{(j)}(x))dx\Big)^{1/2}$$

where

$$\begin{aligned} C_2^2(\vec{v}) &= C_3\int_0^\infty \|\vec{v}\|^4_{L_4(\Sigma_j(t))}\frac{dt}{g_j(t)} \\ &\leqq C_3\sup\|\vec{v}\|^2_{L_4(\Sigma_j(t))}\int_0^\infty\|\vec{v}\|^2_{L_4(\Sigma_j(t))}g_j^{-1}(t)dt \\ &\leqq C_4\|\vec{v}\|^2_{\mathscr{D}(\omega_j)}\int_0^\infty\|\vec{v}\|^2_{\mathscr{D}(\Sigma_j(t))}dt = C_4\|\vec{v}\|^4_{\mathscr{D}(\omega_j)}\,. \end{aligned}$$

Consequently, $|\mathscr{T}[\vec{\varphi}]|\leqq C_5\|\vec{v}\|^2_{\mathscr{D}(\Omega)}\|\vec{\varphi}\|_{\mathscr{D}_{1/2}(\Omega)}$ and $|l(\vec{\varphi})|\leqq C_6\|\vec{\varphi}\|_{\mathscr{D}_{1/2}(\Omega)}$. □

It follows from this theorem that the pressure $p(x)$ corresponding to the weak solution $\vec{v}(x)$ of (26) differs from $q(x)$ in every "exit" ω_j by a constant p_j and $p_j-p_r=\beta_j$. It is seen from (18) that any function $q\in\mathscr{M}^*_{1/2}(\Omega)$ in a certain sense tends to zero when $|x|\to\infty$, so that $p_j=\lim_{\substack{|x|\to\infty\\ x\in\omega_j}}p(x)$.

4. Non-stationary problems. If the domain Ω satisfies the conditions (1), (2′), (3)-(5) with $\alpha = 1$, it is possible to prove the solvability of initial-boundary value problems for the non-stationary Navier-Stokes system with additional conditions of the form (20). We restrict overselves to consideration of the linear problem

$$\begin{aligned} &\vec{v}_t - \nabla^2\vec{v} + \nabla p = \vec{f}(x, t)\,, \quad \nabla\cdot\vec{v} = 0 \quad (x \in \Omega,\, t \in (0, T))\,, \\ &\vec{v}|_{t=0} = \vec{v}_0(x)\,, \quad \vec{v}|_{\partial\Omega} = 0\,, \quad \vec{v}|_{|x|\to\infty} = 0\,, \\ &p_i(t) - p_2(t) = \beta_i(t)\,, \quad i = 1, \cdots, r-1 \end{aligned} \tag{32}$$

where $p_i(t) = \lim_{|x|\to\infty, x\in\omega_i} p(x, t)$. Denote by $\mathscr{J}^1(Q_T)$, $Q_T = \Omega\times(0, T)$, the space of divergence-free vectors with a finite norm

$$\left[\int_0^T\int_\Omega(\vec{v}^2 + \vec{v}_t^2 + \vec{v}_x^2)\,dxdt\right]^{1/2}$$

belonging to $\hat{\mathscr{J}}(\Omega)$ for almost all $t \in (0, T)$. Define a weak solution of (32) as a vector $\vec{v} \in \mathscr{J}^1(Q_T)$ satisfying the initial condition $\vec{v}|_{t=0} = \vec{v}_0(x)$ and the integral identity

$$\int_0^T\int_\Omega(\vec{v}_t\cdot\vec{\eta} + \vec{v}_x\cdot\vec{\eta}_x)dxdt = \int_0^T\int_\Omega\vec{f}\cdot\vec{\eta}dxdt - \sum_{j=1}^{r-1}\int_0^T\beta_j(t)dt\int_{\Sigma_j}\vec{\eta}\cdot\vec{n}dS \tag{33}$$

for all $\vec{\eta} \in L_2(0, T; \hat{H}(\Omega))$.

THEOREM 10. *Let the domain $\Omega \subset R^3$ satisfy* (1), (2′), (3)-(5) *with* $\alpha = 1$. *Then for any $f \in L_2(Q_T)$, $\beta_j(t) \in W_2^1(0, T)$, $\vec{v}_0 \in \hat{H}(\Omega)$ the problem* (32) *has a unique weak solution.*

This theorem may be proved by Galerkin's method (see [6], Ch. VI, §6). The proof is based on two estimates for Galerkin approximations. The first estimate is the energy inequality

$$\begin{aligned} &\sup_{\tau\in(0,T)}\int_\Omega|\vec{v}(x, \tau)|^2dx + \int_0^T\int_\Omega|\vec{v}_x|^2dxdt \\ &\qquad \leqq C_1\Big(\int_\Omega|\vec{v}_0(x)|^2dx + \int_0^T\int_\Omega|\vec{f}(x, t)|^2dxdt + \sum_{j=1}^{r-1}\int_0^T|\beta_j|^2dt\Big)\,, \end{aligned}$$

which can be easily obtained from (33) after the substitution $\vec{\eta}(x, t) = \vec{v}(x, t)$ for $0 \leqq t \leqq \tau$, $\vec{\eta} = 0$ for $\tau < t \leqq T$. Taking in (33) $\vec{\eta} = \vec{v}_t$ and making the transformation

$$\begin{aligned} \int_0^T\beta_j(t)dt\int_{\Sigma_j}\vec{v}_t\cdot\vec{n}dS = &-\int_0^T\frac{d\beta_j}{dt}dt\int_{\Sigma_j}\vec{v}\cdot\vec{n}dS \\ &+ \beta_j(T)\int_{\Sigma_j}\vec{v}(x, T)\cdot\vec{n}dS - \beta_j(0)\int_{\Sigma_j}\vec{v}_0\cdot\vec{n}dS\,, \end{aligned}$$

we obtain an estimate for $\int_0^T\int_\Omega\vec{v}_t^2dxdt$ in terms of the data. As the

Galerkin approximations satisfy an equality of the form (33), both estimates are valid for them. The proof of the existence of a weak solution is quite standard and may be omitted. Now, taking in (33) $\vec{\eta}(x, t) = \xi(t)\vec{\varphi}(x)$, $\vec{\varphi} \in \hat{H}(\Omega)$, we see that for almost all $t \in (0, T)$,

$$l(\vec{\varphi}) = \int_{\Omega}(\vec{v}_t \cdot \vec{\varphi} + \vec{v}_x \cdot \vec{\varphi}_x - \vec{f} \cdot \vec{\varphi})dx + \sum_{j=1}^{r-1} \beta_j(t) \int_{\Sigma_j} \vec{\varphi} \cdot \vec{n} dS = 0 ,$$

hence, for $\vec{\varphi} \in \overset{\circ}{W}{}_2^1(\Omega)$, $l(\vec{\varphi}) = \int_{\Omega} q(x, t)\nabla \cdot \vec{\varphi} dx$ and $q \in N^*(\Omega) \subset M_1^*(\Omega)$. From the estimate

$$\|q\|^2_{N^*(\Omega)} \leqq C\Big(\|\vec{v}_t\|^2_{L_2(\Omega)} + \|\vec{v}_x\|^2_{L_2(\Omega)} + \|\vec{f}\|^2_{L_2(\Omega)} + \sum_{j=1}^{r-1} |\beta_j(t)|^2\Big)$$

we deduce that $q(x, t) \in L_2(0, T; N^*(\Omega)) \subset L_2(0, T; M_1^*(\Omega))$ and therefore in a certain sense $q \to 0$, as $|x| \to \infty$. Repeating the arguments of §3, it is easy to prove that in ω_j, $j = 1, \cdots, r-1$,

$$p(x, t) = q(x, t) + p_j(t) , \quad p_j(t) - p_r(t) = \beta_j(t) .$$

Thus, we see that the presence in the integral identity (33) of an additional term $\sum_{j=1}^{r-1} \int_0^T \beta_j(t)dt \int_{\Sigma_j} \vec{\eta} \cdot \vec{n} dS$ does not lead to any essential change in the well-known proof of the solvability of the linear non-stationary problem. The same is true for the non-linear problem with additional conditions of the type (20). As in [6], it is possible to prove that the non-linear problem with these additional conditions is solvable locally with respect to t.

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