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ON THE LOCAL SPECTRUM AND THE ADJOINT

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Let X be a Banach space, X^* the dual space, and suppose that T is a closed linear operator on X. Assume that the domain of T is dense in X, so that the adjoint operator T^* is a closed linear operator on X^*. The local spectrum $\sigma(T, x)$ is defined below. In this paper we investigate some of the relations between $\sigma(T, x)$ and $\sigma(T^*, x^*)$. In particular we show that if $\sigma(T, x)$ and $\sigma(T^*, x^*)$ are both empty, then $x^*x = 0$.

The resolvent function of T is $R_T(\lambda) = (\lambda I - T)^{-1}$; it is an operator valued function, and is defined and analytic for λ not in $\sigma(T)$, the spectrum of T. Setting $f_x(\lambda) = R_T(\lambda)x$, then f_x is analytic and satisfies $(\lambda I - T)f_x(\lambda) = x$ for all λ not in $\sigma(T)$. However, it may be possible to find analytic solutions to $(\lambda I - T)f(\lambda) = x$ for some (or all) values of λ that are in the spectrum of T. So we are led to define a *local resolvent function* of T at x as a vector valued analytic function f which satisfies $(\lambda I - T)f(\lambda) = x$. It is easily shown that for λ not in $\sigma(T)$, the only local resolvent is $f(\lambda) = R_T(\lambda)x$. But for λ in $\sigma(T)$, there may be more than one local resolvent function. The *local resolvent set* is the union of the domains of all the local resolvent functions. The point at infinity is included if there is a local resolvent function which is defined and bounded for $|\lambda| > r$. The *local spectrum* $\sigma(T, x)$ is the complement of the local resolvent set. Clearly $\sigma(T, x)$ is a closed subset of the spectrum $\sigma(T)$; it may be equal to $\sigma(T)$, properly contained in it, or even empty.

The operator T has the *single valued extension property* if there is at most one local resolvent function defined near any λ in $\boldsymbol{C}$; that is, whenever both f and g are local resolvent functions defined near λ, then $f = g$ there. In this case, there is a unique local resolvent with maximal domain. It can be shown that T has the single valued extension property iff $\sigma(T, x)$ is not empty for any $x \neq 0$. (Note that $\sigma(T, 0)$ is always empty; the local resolvent $f(\lambda) = 0$ is defined for all λ.)

If both T and T^* have the single valued extension property, then $\sigma(T, x) \cap \sigma(T^*, x^*) = \varnothing$ implies $x^*x = 0$ (see Lemma 1). This is not true if $\sigma(T, x) = \varnothing$ for some $x \neq 0$. For then $x^*x = 0$ for all x^*, from which it would follow that $x = 0$. Similarly, it is not true if $\sigma(T^*, x^*)$ is empty. But in Theorem 3 we show that if *both* $\sigma(T, x)$ and $\sigma(T^*, x^*)$ are empty, then $x^*x = 0$.

Suppose that T does not have the single valued extension property. Then for some open set $D \subseteq C$ there are distinct local resolvent functions for some x. Subtracting them, we find that there is a nonzero analytic function $f: D \to X$ which satisfies $(\lambda I - T)f(\lambda) = 0$ for λ in D. For a fixed λ_0 in D, f has a series expansion $f(\lambda) = \sum_{n=0}^{\infty} x_n(\lambda - \lambda_0)^n$. It is not hard to show that $(\lambda_0 I - T)x_{n+1} = -x_n$, $(\lambda_0 I - T)x_0 = 0$, and $(\lambda_0 I - T)^{n+1}x_n = 0$. Also, $\sigma(T, x_n)$ is empty for $n = 0, 1, 2, \cdots$. To show this, we give two resolvent functions for x_n, whose domains together cover the extended complex plane. Set

$$g_n(\lambda) = \sum_{k=0}^{n} (-1)k(\lambda_0 I - T)^k x_n/(\lambda - \lambda_0)^{k+1}$$
$$= \sum_{k=0}^{n} x_{n-k}/(\lambda - \lambda_0)^{k+1} \quad \text{for} \quad \lambda \neq \lambda_0 ,$$

and

$$h_n(\lambda) = \sum_{k=0}^{\infty} x_{n+1+k}(\lambda - \lambda_0)^k \quad \text{for} \quad |\lambda - \lambda_0| < r .$$

The series for h converges in the same disk as the series for f. Straightforward calculation shows that $(\lambda I - T)g_n(\lambda) = x_n$ for $\lambda \neq \lambda_0$. To show that $(\lambda I - T)h_n(\lambda) = x_n$ for $|\lambda - \lambda_0| < r$, first consider the finite sum $S_m(\lambda) = \sum_{k=0}^{m-1} x_{n+1+k}(\lambda - \lambda_0)^k$. Then $(\lambda I - T)S_m(\lambda) = x_n - x_{n+m}(\lambda - \lambda_0)^m$, which converges to x_n as $m \to \infty$. But $S_m \to h_n$ as $m \to \infty$. Since T is a closed operator, we have $(\lambda I - T)h_n(\lambda) = x_n$.

From the above remarks we see that when an operator T does not have the single valued extension property, it acts something like a left shift operator. For a specific example, consider an orthonormal basis $\{e_n\}$ for a separable Hilbert space H. Let T be the left shift: $Te_{n+1} = e_n$ and $Te_0 = 0$. Then we have $(\lambda I - T)f(\lambda) = 0$ on $|\lambda| < 1$ for $f(\lambda) = \sum_0^\infty e_n\lambda^n$. And so $\sigma(T, e_n)$ is empty for all n. Actually $\sigma(T, x)$ is contained in the unit circle $\{\lambda: |\lambda| = 1\}$ for all x in H, although $\sigma(T) = \{\lambda: |\lambda| \leqq 1\}$. The function $g_x(\lambda) = -\sum \lambda^n S^{n+1}x$ is a local resolvent defined on $|\lambda| < 1$, where S is the right shift: $Se_n = e_{n+1}$. Further, it is not hard to see that $\sigma(T^*, x^*)$ is the entire unit disk $\{\lambda: |\lambda| \leqq 1\}$ for all x^* in H^*. In Theorem 2 below we show that this behavior is typical. This example was first given by Kakutani; see Dunford and Schwartz, page 1932. Kakutani's formulation is on the Hilbert space of functions analytic on the unit disk.

LEMMA 1 (Bishop 1959). *Suppose f and g are analytic functions with range in X and X^* respectively, and which satisfy $(\lambda I - T)f(\lambda) = x$ and $(\lambda I^* - T^*)g(\lambda) = x^*$. If the union of the domains of f and g is the entire complex plane, and if f or g is bounded in a neighborhood of infinity, then $x^*x = 0$.*

The proof is an application of Liouville's Theorem to the function $h(\lambda) = x^*f(\lambda)$ for λ in the domain of f, and $h(\lambda) = g(\lambda)x$ otherwise.

THEOREM 2. *Let T be a closed linear operator with domain dense in a Banach space X. Suppose that T does not have the single valued extension property, so that there is an analytec function $f: D \subseteq C \to X$ with $(\lambda I - T)f(\lambda) = 0$ for λ in D. Assume that D is a connected open set. Then for each x^* in X^*, either*

(i) x^* is identically zero on the subspace spanned by $\{f(\lambda): \lambda$ in $D\}$, or

(ii) $\sigma(T^*, x^*)$ contains all of D.

Proof. Let Y be the closed linear span of $\{f(\lambda): \lambda \text{ in } D\}$. For any λ_0 in D, expand f in a Taylor series around λ_0, $f(\lambda) = \sum x_n(\lambda - \lambda_0)^n$. Each x_n is in Y, since

$$x_n = (2\pi i)^{-1} \int_\Gamma f(\lambda)(\lambda - \lambda_0)^{-n-1}\, d\lambda\ ,$$

where Γ is a circle around λ_0 contained in D. Moreover, Y is actually spanned by $\{x_n\}$. For if $x^*x_n = 0$ for all n, then $x^*f(\lambda) = 0$ wherever the series converges. But $x^*f(\lambda)$ is analyic on the connected set D, so $x^*f(\lambda) = 0$ for λ in D. It follows that $x^*y = 0$ for all y in Y, and so $\{x_n\}$ spans Y.

Now suppose that λ_0 is in D, but is not in the local spectrum $\sigma(T^*, x^*)$. Then, by definition, there is an analytic function h defined near λ_0 with $(\lambda I^* - T^*)h(\lambda) = x^*$. For each n, $g_n(\lambda) = \sum_{k=0}^{n} (-1)^k(\lambda_0 I - T)^k x_n/(\lambda - \lambda_0)^{k+1}$ satisfies $(\lambda I - T)g_n(\lambda) = x_n$ for $\lambda \neq \lambda_0$. Then from Lemma 1, $x^*x_n = 0$ for all n. Hence x^* is identically zero on Y. □

Suppose that the local spectrums $\sigma^*(T, x)$ the $\sigma(T^*, x^*)$ are both empty. Can we conclude that $x^*x = 0$? The answer is yes, but not directly from Theorem 2, for we do not know that x is in a subspace Y as described there. Instead, we show that x is in the sum of a finite number of such subspaces. Then since x^* is identically zero on each subspace, it is zero on the sum. And then $x^*x = 0$.

Suppose that $\sigma(T, x)$ is empty. Then in particular the point at infinity is not in $\sigma(T, x)$; by definition this means that there is a bounded function $g(\lambda)$ defined on $D = \{\lambda: |\lambda| > r\}$ which satisfies $(\lambda I - T)g(\lambda) = x$. This function g has a special form.

LEMMA 3. *The function $g(\lambda)$ described above has the series representation $g(\lambda) = \sum_{n=0}^{\infty} T^n x/\lambda^{n+1}$.*

Proof. Since g is bounded for λ in D, it must have a series representation $\sum_{n=0}^{\infty} x_n/\lambda^n$. Then $g(\lambda) \to x_0$ as $\lambda \to \infty$, and so $g(\lambda)/\lambda \to 0$. But $T(g(\lambda)/\lambda) = T(g(\lambda))/\lambda = (\lambda g(\lambda) - x)/\lambda \to x_0$. T is a closed operator, hence $x_0 = T(0) = 0$. Now, $\lambda g(\lambda) \to x_1$ as $\lambda \to \infty$, and so $T(g(\lambda)) = \lambda g(\lambda) - x$ converges to $x_1 - x$. But $g(\lambda) \to 0$; thus $0 = T(0) = x_1 - x$, or $x_1 = x$.

We show that $Tx_n = x_{n+1}$ for $n \geqq 1$, and therefore that $x_n = T^n x$, using induction. Let

$$\begin{aligned} h(\lambda) &= x_n + x_{n+1}/\lambda + x_{n+2}/\lambda^2 + \cdots \\ &= \lambda^n g(\lambda) - \lambda^{n-1}x_1 - \cdots - \lambda x_{n-1} . \end{aligned}$$

Then

$$\begin{aligned} Th(\lambda) &= \lambda^n(\lambda g(\lambda) - x) - \lambda^{n-1}x_2 - \cdots - \lambda x_n \\ &= \lambda^{n+1}g(\lambda) - \lambda^n x_1 - \cdots - \lambda x_n \\ &= x_{n+1} + x_{n+2}/\lambda + \cdots . \end{aligned}$$

Thus as $\lambda \to \infty$, $h(\lambda) \to x_n$ and $Th(\lambda) \to x_{n+1}$. Since T is closed, we have $Tx_n = x_{n+1}$. □

THEOREM 4. *Suppose T is a closed linear operator on a Banach space X with domain dense in X. If $\sigma(T, x)$ and $\sigma(T^*, x^*)$ are both empty, then $x^*x = 0$.*

Proof. The idea is to show that if $\sigma(T, x)$ is empty, then x is in a finite sum of subspaces spanned by $f_k(\lambda)$, where $(\lambda I - T)f_k(\lambda) = 0$ for λ in the domain of f_k. It then follows from Theorem 2 that $x^*x = 0$.

If $\sigma(T, x)$ is empty, then for every complex number z, there is an analytic function $g_z(\lambda)$ which satisfies $(\lambda I - T)g_z(\lambda) = x$ and which is defined (at least) on some disk D_z around z. In particular, there is such a function $g_\infty(\lambda)$ defined on $D_\infty = \{\lambda: |\lambda| \geqq r_0\}$. Take B_z an open disk around z contained in D_z; and B_∞ a neighborhood of ∞ contained in D_∞. The B's form an open cover of the extended complex plane, which is compact. Hence a finite number of them, B_∞, $B_1, \cdots, B_n$ also form an open cover. Let $g_0, g_1, \cdots, g_n$ be the corresponding functions. Now the idea is that where the domains of g_i and g_j overlap, the function $f_{jk} = g_j - g_k$ satisfies $(\lambda I - T)f_{jk}(\lambda) = 0$. The problem is to show that x is in the sum of the subspaces generated by all the f_{jk}'s.

We may assume that no B_j is in the union of the previous B's; drop any that is. We may also assume that no B_j is in the union of the closure of the others, by taking the B_j slightly smaller if necessary. Now define closed sets by $C_0 = \text{cl}\,(B_\infty)$ and $C_k = \text{cl}\,(B_k - C_0 - \cdots - C_{k-1})$. We see that $B_\infty \cup B_1 \cup \cdots \cup B_k$ is contained in $C_0 \cup \cdots \cup C_k$ and that C_k is disjoint from $B_\infty, B_1, \cdots, B_{k-1}$. Thus the family C_k is a closed cover of the extended complex plane. The sets C_k may have more than one component, but each component is bounded by a finite number of circular arcs. Let Γ_j be the boundary of C_j. Given a point on Γ_j, any neighborhood of that point contains interior points of C_j and interior points of some other C_k; by the construction, the point must then be on Γ_k. That is, every point on Γ_j is on at least one other Γ_k. Also, a point where three or more of the Γ_k meet must be a point where two or more of the original circular boundaries of the B_k meet. Only a finite number of such points exist, since distinct circles intersect in at most two points.

Note that $2\pi ix = \sum_j \int_{\Gamma_j} g_j(\lambda)\, d\lambda$, where the integral around Γ_0 is in the counterclockwise direction and the others in the clockwise direction. This is true since $\int_{\Gamma_j} g_j = 0$ for $j = 1, \cdots, n$, and $\int_{\Gamma_0} g_0 = 2\pi ix$ from Lemma 3. Now consider an arc γ on Γ_j that extends from one intersection of three or more Γ_k to another, without passing through a third (if there are not two such intersections on Γ_j, let $\gamma = \Gamma_j$). Each interior point of γ is on exactly one other Γ_k; in fact, all interior points are on the same Γ_k since otherwise an intersection of three Γ_k would be an interior point of γ. The paths of integration of $\int_{\Gamma_j} g_j$ and $\int_{\Gamma_k} g_k$ both contain γ, and the integrals "go over" γ in opposite directions. Thus these portions of those integrals may be combined to $\pm\int_\gamma (g_j - g_k)$. Doing this for each arc on each Γ_j, we see that the expression for $2\pi ix$ becomes a linear combination of integrals of the form $\int_\gamma (g_j - g_k)$. Since $(\lambda I - T)(g_j - g_k) = 0$, setting $f_{jk} = g_j - g_k$, we have x in the sum of the spaces spanned by the f_{jk}. □

THEOREM 5. *Suppose that T is as above, and in addition the range of T is closed. If for every $x \neq 0$ the local spectrum $\sigma(T, x)$ contains the disk $D = \{\lambda: |\lambda| \leqq r\}$, then T^* does not have the single valued extension property. In particular, there are elements $x^* \neq 0$ in X^* for which $\sigma(T^*, x^*)$ is empty.*

Proof. First T is one-one. For if $Tx = 0$, then $f(\lambda) = x/\lambda$ is a local resolvent function defined for all $\lambda \neq 0$. And then $\sigma(T, x) \subseteqq \{0\}$, contrary to our assumption. However, 0 is in the spectrum of T. Thus T cannot be onto. Since the range of T is closed, T^* is onto but not one-one. This is sufficient to show that T^* does no have the single valued extension property. For take any x_0^* with $x_0^* = 1$ and $T^*x_0^* = 0$, and form a sequence $x_0^*, x_1^*, x_2^*, \cdots$ with $T^*x_{n+1}^* = x_n^*$.

Because T^* is an open mapping, this can be done so that $\|x^*_{n+1}\| < k\|x^*_n\|$. It is then fairly easy to show that $f^*(\lambda) = \sum x^*_n \lambda^n$ converges for $|\lambda| < 1/k$ and satisfies $(\lambda I^* - T^*)f^*(\lambda) = 0$. □

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